ONE DIMENSIONAL CRITICAL KINETIC FOKKER-PLANCK EQUATIONS, BESSEL AND STABLE PROCESSES

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Abstract. We consider a particle moving in one dimension, its velocity being a reversible diffusion process, with constant diffusion coefficient, of which the invariant measure behaves like $(1+|v|)^{-\beta}$ for some $\beta > 0$. We prove that, under a suitable rescaling, the position process resembles a Brownian motion if $\beta \geq 5$, a stable process if $\beta \in [1, 5)$ and an integrated symmetric Bessel process if $\beta \in (0, 1)$. The critical cases $\beta = 1$ and $\beta = 5$ require special rescalings. We recover some results of [21, 7, 16] and [1] on the kinetic Fokker-Planck equation, with an alternative approach.

1. Introduction and results

We consider a particle moving in one dimension, its velocity $(V_t)_{t\geq 0}$ solving, for some $\beta > 0$,

$$\mathrm{d}V_t = \mathrm{d}B_t - \frac{\beta}{2}\frac{V_t}{1+V_t^2}\mathrm{d}t,$$

or a slightly generalized equation. Its invariant distribution behaves like $(1+|v|)^{-\beta}$. We prove that, under a suitable rescaling, the position process $X_t = X_0 + \int_0^t V_s \mathrm{d}s$ resembles, in large time, a Brownian motion if $\beta \geq 5$, a stable process if $\beta \in [1, 5)$ and an integrated symmetric Bessel process if $\beta \in (0, 1)$. The critical cases $\beta = 1$ and $\beta = 5$ require special rescalings.

1.1. **Introduction.** Consider a one-dimensional particle with position $X_t \in \mathbb{R}$ and velocity $V_t \in \mathbb{R}$, evolving in a force field Φ and undergoing many small random shocks. The smooth force field $\Phi : \mathbb{R} \to \mathbb{R}$ is supposed to depend only on the velocity. The Newton equations describing the evolution of this particle are

$$\mathrm{d}X_t = V_t\mathrm{d}t, \qquad \mathrm{d}V_t = \Phi(V_t)\mathrm{d}t + \mathrm{d}B_t, \tag{1}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion modeling the random shocks. Langevin [15] studied the case where Φ is the restoring/friction force $\Phi(v) = -v$ and showed that the position of the particle behaves, when t tends to infinity, as a Brownian motion.

In the whole paper, we consider a restoring force of the form $\Phi(v) = -\frac{1}{2}U'(v)$ where U is a smooth nonnegative even potential. Hence the velocity process $(V_t)_{t\geq 0}$ is a reversible diffusion process with invariant measure $\exp(-U(v))\mathrm{d}v$. When $\mathcal{Z}_U = \int_{\mathbb{R}} \exp(-U(v))\mathrm{d}v$ is finite, it holds that V_t converges in law to a random variable with density $\exp(-U(v))/\mathcal{Z}_U$, as $t \to \infty$.

Denote, for each $t \geq 0$, by f_t the law of (X_t, V_t), which is a probability measure on $\mathbb{R}\times\mathbb{R}$. Then $(f_t)_{t\geq 0}$ is a weak solution of the *kinetic Fokker-Planck equation*

$$\partial_t f_t(x,v) + v\partial_x f_t(x,v) = \frac{1}{2}\partial_{vv} f_t(x,v) - \partial_v[\Phi(v)f_t(x,v)]. \tag{2}$$

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We say that there is a normal diffusion limit when, for some constant $\sigma > 0$ and in a weak sense,

$$\lim_{\epsilon\to 0} \epsilon^{-1/2} f_{t/\epsilon}(\epsilon^{-1/2}x, v) = \frac{e^{-x^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}} \times \frac{e^{-U(v)}}{\mathcal{Z}_U},$$

meaning that for each $t > 0$, $(\epsilon^{1/2}X_{t/\epsilon}, V_{t/\epsilon})$ converges in law to $(\sigma W_t, \bar{V})$, where $(W_t)_{t\geq 0}$ is a Brownian motion independent of some random variable $\bar{V}$ with density $\exp(-U(v))/\mathcal{Z}_U$.

Roughly, normal diffusion limit occurs when the restoring force field is strong enough, or equivalently when the potential U grows sufficiently fast to infinity. Probabilistic techniques to get such results are described in Pardoux-Veretennikov [22], see also Cattiaux, Chafaï and Guillin [6].

Then it is tempting to study what is going on when the force field is weakly restoring, or equivalently when the potential U grows slowly to infinity or equivalently when the invariant measure is heavy-tailed. For example, choosing drastically $\Phi(v) = 0$, there is no hope to get a diffusion limit for X_t since in that case $X_t = X_0 + \int_0^t B_s \mathrm{d}s$ and it has a scaling in $\epsilon^{3/2}$. Precisely, by the scaling property of Brownian motion it comes that $\epsilon^{3/2}X_{t/\epsilon}$ converges in law to $\int_0^t B_s \mathrm{d}s$ which is a Gaussian process but no longer a Markov process. We say in that case that there is an *anomalous diffusion limit*, and more generally we use this terminology in situations where $\zeta(\epsilon)X_{t/\epsilon}$ converges in law to some non-trivial process, with a scaling function $\zeta(\epsilon)$ different from $\epsilon^{1/2}$.

In fact, and it is the main subject of our article, it is possible to find a family of *critical* forces which give limits with scaling functions that interpolate between the Brownian scale $\epsilon^{1/2}$ and the *integrated* Brownian scale $\epsilon^{3/2}$. If we look at forces such that $\Phi(v) \simeq -\mathrm{sg}(v)|v|^\gamma$ for large values of $|v|$, one can check that if $\gamma > -1$ then the force is restoring enough so that normal diffusion limit occurs, while if $\gamma < -1$ the force is too weakly restoring at infinity and $(X_t)_{t\geq 0}$ behaves as $\int_0^t B_s \mathrm{d}s$, with the scaling function $\epsilon^{3/2}$. Roughly speaking, those critical forces have to be taken such that the term $\Phi(V_s)\mathrm{d}s$, in the dynamics of $(V_t)_{t\geq 0}$, has the same scaling than $\mathrm{d}B_s$. The only way is to choose $\gamma = -1$, *i.e* $\Phi(v) \sim -\beta/v$ for large values of $|v|$ where β is some nonnegative constant.

The answer to this kind of questions, and it was the starting point of our article, can be found in a series of P.D.E. papers by Nasreddine-Puel [21], Cattiaux-Nasreddine-Puel [7] and Lebeau-Puel [16]. They precisely study the family of critical forces

$$\Phi(v) = -\frac{\beta}{2}\frac{v}{1+v^2} \quad i.e. \quad U(v) = \frac{\beta}{2}\log(1+v^2) \tag{3}$$

with $\beta > 0$. The case of large β is treated in [21] ($\beta > 5$) in any dimension and indeed a normal diffusion limit occurs. The case $\beta = 5$ is treated in [7]. Some smaller values of β are explored in [16]: they prove that, in dimension one, anomalous *fractional* diffusion limit occurs for $\beta \in (1, 5) \setminus \{2, 3, 4\}$. The case $\beta \in (0, 1] \cup \{2, 3, 4\}$ is thus left open. We recall the main result in [16].

Theorem (Lebeau-Puel)**.** *Consider a solution $(f_t)_{t\geq 0}$ solution to* (2) *with Φ given by* (3) *for some $\beta \in (1, 5) \setminus \{2, 3, 4\}$ and set $\alpha = (\beta + 1)/3$. Then for $t > 0$, in a weak sense,*

$$\lim_{\epsilon\to 0} \epsilon^{-1/\alpha} f_{t/\epsilon}(\epsilon^{-1/\alpha}x, v) = \rho(t, x)\frac{e^{-U(v)}}{\mathcal{Z}_U},$$

where $\rho(t, x)$ solves the fractional heat equation

$$\partial_t \rho + \kappa(-\Delta)^{\alpha/2}\rho = 0,$$

κ being an explicit constant depending on β.

Note that such fractional (anomalous) diffusion limits are not so unexpected and often arise in physics. Many works show how to modify the collision kernel in some Boltzmann-like linear equations to get some fractional diffusion limit. One can e.g. linearize the Boltzmann equation around a fat tail equilibrium or consider some *ad hoc* cross section. This was initiated by Mischler, Mouhot and Mellet

[19], with close links to the earlier work of Milton, Komorowski and Olla [20] on Markov chains. This was continued by Mellet [18], Ben Abdallah, Mellet and Puel [2, 3] and others.

Nevertheless, in the case of the above *critical* kinetic Fokker Planck model, the anomalous diffusion case $\beta \in (1,5)$ seems rather difficult to treat, in comparison to above cited works [19, 18, 2, 3] on Boltzmann-like equations. In particular, while the stable index α is more or less prescribed from the beginning in [19, 18, 2, 3], it is a rather mysterious function of β in the present case. The paper [16] relies on a deep spectral analysis, making a wide use of special functions, and the result is impressive.

In probabilistic words, the above theorem states that $\epsilon^{1/\alpha} X_{t/\epsilon}$ converges in law to a symmetric α-stable random variable. Since $\beta \in (1,5)$ and $\alpha = (\beta+1)/3$, one observes as expected an interpolation between the scale functions $\epsilon^{1/2}$ and $\epsilon^{3/2}$. This clearly sounds probabilistic, and our main goal is to give a probabilistic proof, establishing a *α-stable* limit theorem for the additive functional $\epsilon^{1/\alpha}\int_0^{t/\epsilon} V_s \mathrm{d}s$, including if possible the cases $\beta \in (0,1] \cup \{2,3,4\}$ previously left open.

1.2. **A first probabilistic approach.** Let us describe a natural probabilistic approach to treat the problem. First recall informally (see Revuz-Yor [23, Chapter XI] and Subsection 2.3 below for more precisions) that a Bessel process $(R_t)_{t\geq 0}$ with dimension δ is a nonnegative diffusion process with the following dynamics

$$\mathrm{d}R_t = \mathrm{d}B_t + \frac{\delta-1}{2}\frac{1}{R_t}\mathrm{d}t. \tag{4}$$

It has the same scaling as the Brownian motion, namely $(\epsilon^{1/2}R_{t/\epsilon})_{t\geq 0} \overset{d}{=} (R_t)_{t\geq 0}$. The subtlety is that (4) makes no clear sense when the process reaches 0, and the following phase transitions occur. When $\delta \geq 2$ it does never reach 0; when $\delta \in (0,2)$ it reaches 0 infinitely often and bounces on it; when $\delta \leq 0$, the process reaches 0 and is then absorbed forever.

When $\Phi(v) = \frac{-\beta v}{2(1+v^2)}$, as in [21, 7, 16], it holds that $\Phi(v) \simeq \frac{-\beta}{2v}$ for large values of $|v|$. hence the velocity process $(V_t)_{t\geq 0}$ should behave, when far away from 0, like a (signed) Bessel process with dimension $\delta = 1-\beta$. But even when $\delta \leq 0$, the velocity process is not stuck when it reaches 0, because Φ is smooth. Hence in any case, it should be possible to approximate the position process $X_t = X_0 + \int_0^t V_s \mathrm{d}s$ by a sum of (signed) i.i.d. areas of excursions (outside of 0) of a Bessel process with dimension $\delta = 1-\beta$. Observe that the definition of such an excursion is not clear, in particular when $\beta \in (0,1]$, since then $\delta \leq 0$, the process started at 0 never leaves 0. However, one can let it start from $r > 0$ and let $r \to 0$ after with a correct rescaling.

Using some explicit computations relying on modified Bessel functions it seems possible to show that the (random) area A of the excursion of a Bessel process with dimension $\delta = 1-\beta$ has a distribution with a fat tail, namely that $\mathbb{P}[A \geq a] \simeq c_\beta a^{-\alpha}$ as $a \to \infty$, where $\alpha = (\beta+1)/3$ and where $c_\beta > 0$ is a constant.

All in all, this Bessel excursion area has a moment of order 2 when $\beta > 5$, so that one expects a classical central limit theorem to hold, yielding normal diffusion for the position process $(X_t)_{t\geq 0}$. But when $\beta \in (1,5)$, this central limit theorem has to be replaced by a stable limit theorem yielding to an anomalous stable diffusion limit. And when $\beta \in (0,1)$, there is one more issue, due to the fact that the duration of the Bessel excursion is no longer integrable.

Surprisingly, while searching for some information about the law of the area of a Bessel excursion, we found the paper by Barkai, Aghion and Kessler [1] in the physics literature. They use exactly the above probabilistic strategy to prove precisely the same kind of results, with the very same critical force field, but motivated by another physical phenomenon.

Actually physicists discovered that atoms, when cooled by a laser, diffuse anomalously, like *Lévy walks*. See Castin, Dalibard and Cohen-Tannoudji [5], Sagi, Brook, Almog and Davidson [24] and Marksteiner, Ellinger and Zoller [17]. A theoretical study has been proposed by Barkai, Aghion and

Kessler [1] (see also and Hirschberg, Mukamel and Schütz [10]). They precisely model the motion of atoms by (2) with the force (3) induced by the laser field. They prove, with quite a high level of rigor, using tedious explicit computations relying on special functions, the results Puel et al. [21, 16], excluding the critical cases and treating also the case where $\beta \in (0,1)$ that they call Obukhov-Richardson phase. Actually, in this last case, as already mentioned, the duration of the Bessel excursion is no longer integrable and Barkai, Aghion and Kessler introduce some Bessel bridges. The result in [1] consists of an explicit expansion formula involving generalized hypergeometric functions.

As we will see, the situation when $\beta \in (0,1)$ is actually rather simple, because at least at the informal level, the Bessel process with dimension $\delta = 1-\beta > 0$ is not stuck when it reaches 0, so that one can simply approximate the velocity process by a *true* (symmetrized) Bessel process with dimension δ.

1.3. **Our strategy.** We found another way, which is more qualitative and even more probabilistic, making use of the connections (or similarities) between Bessel and stable processes, see Section 2. We provide a rather concise proof, that moreover allows us to deal with slightly more general forces of the form (6) below. We also hope that this approach is more robust and may apply to other models.

The core of the paper (when $\beta \le 5$, which is the most interesting case) consists in making precise the following *informal* arguments. For $(W_t)_{t\ge 0}$ a Brownian motion and for τ_t the inverse of the time change $A_t = (\beta+1)^{-2}\int_0^t |W_s|^{-2\beta/(\beta+1)}\mathrm{d}s$, the process $Y_t = W_{\tau_t}$ should classically solve, see e.g. Revuz-Yor [23, Proposition 1.13 page 373], $Y_t = (\beta+1)\int_0^t |Y_s|^{\beta/(\beta+1)}\mathrm{d}B_s$, for some other Brownian motion $(B_t)_{t\ge 0}$. Hence, still informally, $V_t = \mathrm{sgn}(Y_t)|Y_t|^{1/(\beta+1)}$, where sgn is the sign function with the convention that $\mathrm{sgn}(0) = 0$, should solve, by the Itô formula,

$$\mathrm{d}V_t = \mathrm{d}B_t - \frac{\beta}{2}\frac{\mathrm{sgn}(V_s)}{|V_s|}\mathrm{d}s. \tag{5}$$

This is a rough version of the initial equation (1) with $\Phi(v) = -\frac{\beta}{2v}$ and it should describe, as explained in the previous subsection, the large time behavior of the solution to (1) with $\Phi(v) = -\frac{\beta v}{2(1+v^2)}$, after rescaling.

We recognize in (5) the S.D.E. (4) of a (symmetrized) Bessel process of dimension $\delta = 1-\beta$.

If $\beta \in (0,1)$, i.e. $\delta > 0$, such a (symmetric) Bessel process is well-defined and non-trivial, see also Definition 5 below. Thus it is not surprising that we will find that $X_{t/\epsilon}$, rescaled by $\epsilon^{3/2}$ converges to and integrated symmetric Bessel process (Theorem 2-(e) below).

If $\beta \in (1,5)$, i.e. $\delta \le 0$, it is well-known that V_t will remain stuck at 0. But it actually appears that A_t is infinite and, in some sense to be made precise, proportional to the local time L^0_t of $(W_t)_{t\ge 0}$. Hence, up to correct rescaling,

$$\begin{aligned}
X_t =& \int_0^t V_s \mathrm{d}s \\
\simeq& \int_0^t \mathrm{sgn}(W_{\tau_s})|W_{\tau_s}|^{1/(\beta+1)}\mathrm{d}s \\
=& \int_0^{\tau_t} \mathrm{sgn}(W_s)|W_s|^{1/(\beta+1)}\mathrm{d}A_s \\
=&(\beta+1)^{-2}\int_0^{\tau_t} \mathrm{sgn}(W_s)|W_s|^{(1-2\beta)/(\beta+1)}\mathrm{d}s.
\end{aligned}$$

Since $(\tau_t)_{t\ge 0}$ is proportional to the inverse of the local time of $(W_t)_{t\ge 0}$, we know from Biane-Yor [4] that $(X_t)_{t\ge 0}$ is an α-stable Lévy process, with $\alpha = (\beta+1)/3$. See Theorem 4 below for a precise statement and a few explanations. These arguments are completely informal. In particular, we always have $W_{\tau_t} = 0$, so that the equality $\int_0^t \mathrm{sgn}(W_{\tau_s})|W_{\tau_s}|^{1/(\beta+1)}\mathrm{d}s = \int_0^{\tau_t} \mathrm{sgn}(W_s)|W_s|^{1/(\beta+1)}\mathrm{d}A_s$ is far from being fully justified.

All this requires some work to be justified, but does not rely on deep computations involving special functions, unless one wants to know the value of the diffusion constant. This is why we say that our proof is *qualitative*.

The simplicity of our arguments allow us to treat all the values of $\beta > 0$, including the critical cases, and to deal with slightly more general forces. Also, it allows us to treat the multidimensional case, with possible asymmetries in the force, in a (much more technical) companion paper [9].

To summarize, we believe the main interest of the present paper is to provide a qualitative proof of the results of [21, 7, 16] and [1], with sufficiently simple arguments to treat all the values of $\beta > 0$, including the critical cases, and with a possible (tedious) extension to the multidimensional case and maybe to other models (e.g. involving jump processes).

1.4. **Assumptions and notation.** We consider the force field $\Phi = -\frac{\beta}{2}F$, for some $\beta > 0$ on some $F : \mathbb{R} \to \mathbb{R}$ satisfying

$$(6) \qquad F = -\frac{\Theta'}{\Theta}, \text{ for some even } \Theta : \mathbb{R} \to (0,\infty) \text{ of class } C^2 \text{ satisfying } \lim_{|v|\to\infty} |v|\Theta(v) = 1.$$

The typical example we have in mind is $F(v) = v/(1+v^2)$, as mentioned in the previous paragraphs, and corresponds to $\Theta(v) = (1+v^2)^{-1/2}$.

For $\beta > 0$, we introduce the measure μ_β, unique solution (up to multiplicative constants) of the equation $\frac{1}{2}\mu_\beta'' + \frac{\beta}{2}(F\mu_\beta)' = 0$ in the sense of distributions, defined by

$$\mu_\beta(\mathrm{d}v) = c_\beta[\Theta(v)]^\beta \mathrm{d}v,$$

and we choose $c_\beta^{-1} = \int_{\mathbb{R}}[\Theta(v)]^\beta \mathrm{d}v < \infty$ if $\beta > 1$ and $c_\beta = 1$ if $\beta \in (0,1]$.

We finally define, for each $\beta \geq 1$, the diffusion constant $\sigma_\beta > 0$ as follows:

- $\sigma_\beta^2 = 8c_\beta \int_0^\infty \Theta^{-\beta}(v)[\int_v^\infty u\Theta^\beta(u)\mathrm{d}u]^2 \mathrm{d}v$ if $\beta > 5$,
- $\sigma_5^2 = 4c_5/27$,
- $\sigma_\beta^\alpha = 3^{1-2\alpha}2^{\alpha-1}c_\beta\pi/[(\Gamma(\alpha))^2 \sin(\pi\alpha/2)]$, where $\alpha = (\beta+1)/3$, if $\beta \in (1,5)$,
- $\sigma_1^{2/3} = 2^{2/3}3^{-5/6}\pi/[\Gamma(2/3)]^2$.

1.5. **P.D.E. statement.** We consider the following kinetic Fokker-Planck equation

$$(7) \qquad \partial_t f_t(x,v) + v\partial_x f_t(x,v) = \frac{1}{2}\partial_{vv} f_t(x,v) + \frac{\beta}{2}\partial_v[F(v)f_t(x,v)], \qquad t \geq 0,\ x \in \mathbb{R},\ v \in \mathbb{R}.$$

For $E = \mathbb{R}$ or $\mathbb{R}\times\mathbb{R}$, we endow the set of probability measures $\mathcal{P}(E)$ on E with the weak convergence topology, using bounded and continuous functions as test functions.

For $f \in \mathcal{P}(\mathbb{R}\times\mathbb{R})$ and $a, b > 0$, we abusively denote by $(ab)^{-1}f(a^{-1}x, b^{-1}v)$ the probability measure $f_{a,b}$ on $\mathbb{R}\times\mathbb{R}$ defined by

$$f_{a,b}(A) = \int_{\mathbb{R}\times\mathbb{R}} \mathbf{1}_{\{(ax,bv)\in A\}} f(\mathrm{d}x, \mathrm{d}v)$$

for all Borel subset A of $\mathbb{R}\times\mathbb{R}$. Similarly, for $\rho \in \mathcal{P}(\mathbb{R})$ and $a > 0$, we denote by $a^{-1}\rho(a^{-1}x)$ the probability measure ρ_a on $\mathbb{R}$ defined by $\rho_a(A) = \int_{\mathbb{R}} \mathbf{1}_{\{ax\in A\}}\rho(\mathrm{d}x)$ for all Borel subset A of $\mathbb{R}$.

Theorem 1. *Assume* (6)*. For any $\beta > 0$ and any $f_0 \in \mathcal{P}(\mathbb{R}\times\mathbb{R})$, there exists a solution $(f_t)_{t\geq 0} \in C([0,\infty), \mathcal{P}(\mathbb{R}\times\mathbb{R}))$, in the sense of distributions, to* (7) *starting from f_0 and enjoying the following properties.*

(a) If $\beta > 5$, then for all $t > 0$,

$$\lim_{\epsilon\to 0} \epsilon^{-1/2} f_{\epsilon^{-1}t}(\epsilon^{-1/2}x, v) = g_t \otimes \mu_\beta \quad \text{in } \mathcal{P}(\mathbb{R}\times\mathbb{R}),$$

where g_t is the Gaussian density with variance $\sigma_\beta^2 t$, characterized by $\int_{\mathbb{R}} g_t(x)e^{i\xi x}\mathrm{d}x = \exp(-t|\sigma_\beta\xi|^2/2)$.

(b) If $\beta = 5$, then for all $t > 0$,

$$\lim_{\epsilon\to 0} \epsilon^{-1/2}|\log\epsilon|^{1/2} f_{\epsilon^{-1}t}(\epsilon^{-1/2}|\log\epsilon|^{1/2}x, v) = g_t \otimes \mu_\beta \quad \text{in } \mathcal{P}(\mathbb{R}\times\mathbb{R}),$$

where g_t is the Gaussian density with variance $\sigma_5^2 t$ characterized by $\int_{\mathbb{R}} g_t(x)e^{i\xi x}\mathrm{d}x = \exp(-t|\sigma_5\xi|^2/2)$.

(c) If $\beta \in (1, 5)$, setting $\alpha = (\beta+1)/3$, it holds that for all $t > 0$,

$$\lim_{\epsilon\to 0} \epsilon^{-1/\alpha} f_{\epsilon^{-1}t}(\epsilon^{-1/\alpha}x, v) = g_t \otimes \mu_\beta \quad \text{in } \mathcal{P}(\mathbb{R}\times\mathbb{R}),$$

where g_t is the stable law characterized by $\int_{\mathbb{R}} g_t(x)e^{i\xi x}\mathrm{d}x = \exp(-t|\sigma_\beta\xi|^\alpha)$.

(d) If $\beta = 1$, setting $\rho_t(\mathrm{d}x) = \int_{v\in\mathbb{R}} f_t(\mathrm{d}x, \mathrm{d}v)$, it holds that for all $t > 0$,

$$\lim_{\epsilon\to 0} |\epsilon\log\epsilon|^{-3/2} \rho_{\epsilon^{-1}t}(|\epsilon\log\epsilon|^{-3/2}x) = g_t \quad \text{in } \mathcal{P}(\mathbb{R}),$$

where g_t is the symmetric stable law characterized by $\int_{\mathbb{R}} g_t(x)e^{i\xi x}\mathrm{d}x = \exp(-t|\sigma_1\xi|^{2/3})$.

(e) If $\beta \in (0, 1)$, then there is $(h_t)_{t\geq 0} \in C([0,\infty), \mathcal{P}(\mathbb{R}\times\mathbb{R}))$ such that for all $t > 0$,

$$\lim_{\epsilon\to 0} \epsilon^{-2} f_{\epsilon^{-1}t}(\epsilon^{-3/2}x, \epsilon^{-1/2}v) = h_t \quad \text{in } \mathcal{P}(\mathbb{R}\times\mathbb{R}).$$

Moreover, $(h_t)_{t\geq 0}$ is symmetric in the sense that $h_t(-x,-v) = h_t(x,v)$ for all $t \geq 0$, has no trivial part in the sense that $h_t(\mathbb{R}\times\{0\}) = 0$ for all $t > 0$, and solves (7) with $h_0 = \delta_{(0,0)}$ and $F(v) = v^{-1}\mathbf{1}_{\{v\neq 0\}}$ in the following (very) weak sense: for all $\varphi \in C_c^2(\mathbb{R}\times\mathbb{R}_*)$, all $t \geq 0$,

$$\int_{\mathbb{R}\times\mathbb{R}} \varphi(x,v)h_t(\mathrm{d}x,\mathrm{d}v) = \varphi(0,0) + \int_0^t \int_{\mathbb{R}\times\mathbb{R}} \Big[v\partial_x\varphi(x,v) + \frac{1}{2}\partial_{vv}\varphi(x,v) - \frac{\beta}{2v}\partial_v\varphi(x,v)\Big] h_s(\mathrm{d}x,\mathrm{d}v)\mathrm{d}s. \tag{8}$$

It is likely that the weak solution $(f_t)_{t\geq 0}$ to (7), given f_0, is unique. We did not address this question and refer to [21] when f_0 is a L^1-function and $F(v) = \frac{v}{1+v^2}$.

In points (a), (b), (c), we recover and slightly generalize the results of [21, 7, 16]. Observe that in point (d), which is new, it does not seem easy to treat both the position and velocity, because μ_β is not integrable, so that it is not possible to get $g_t \otimes \mu_\beta$ as limiting probability.

Point (e) is very different from the other cases: while in (a)-(b)-(c)-(d), g_t depends only on x and solves a (possibly fractional) autonomous heat equation, there is no autonomous equation for the position process in (e), meaning that this process is not Markov.

We are not certain that the conditions in (e) (symmetry, absence of trivial part and validity of (8) for all $\varphi \in C_c^2(\mathbb{R}\times\mathbb{R}_*)$) are sufficient to characterize uniquely $(h_t)_{t\geq 0}$. This might be a difficult question. But this is not really an issue, since, as we will see, h_t can be characterized as the law of $(\int_0^t U_s^{(1-\beta)}\mathrm{d}s, U_t^{(1-\beta)})$, this last object being uniquely defined in Definition 5 below. In particular, $(h_t)_{t\geq 0}$ does not depend on the initial condition f_0.

Actually, something like (e) holds true for any $\beta > 0$, but we believe that when $\beta \geq 1$, the only possible solution to (8) should be $h_t = \delta_{(0,0)}$ for all $t \geq 0$, so that the scaling is not relevant.

1.6. **Probabilistic statements.** Theorem 1 will be deduced from the study of the following one-dimensional stochastic kinetic model. We assume (6) and consider, for some $\beta > 0$,

$$V_t = V_0 + B_t - \frac{\beta}{2}\int_0^t F(V_s)\mathrm{d}s \quad \text{and} \quad X_t = X_0 + \int_0^t V_s\mathrm{d}s. \tag{9}$$

Here $(B_t)_{t\geq 0}$ is a Brownian motion independent of the initial condition (X_0, V_0). The drift F being C^1, (9) classically has a pathwise unique (possibly local) strong solution, and we will see that it is global. The velocity process $(V_t)_{t\geq 0}$ is Markov and its invariant measure is μ_β, see Subsection 1.4.

For $E = \mathbb{R}$ or $\mathbb{R}\times\mathbb{R}$ and for a family $((Z_t^\epsilon)_{t\geq 0})_{\epsilon\geq 0}$ of E-valued processes, we write

$$(Z_t^\epsilon)_{t\geq 0} \xrightarrow{f.d.} (Z_t^0)_{t\geq 0}$$

if for all finite subsets $S \subset [0,\infty)$, the vector $(Z_t^\epsilon)_{t\in S}$ converges in law to $(Z_t^0)_{t\in S}$, in E^S, as $\epsilon \to 0$. We write

$$(Z_t^\epsilon)_{t\geq 0} \xrightarrow{d} (Z_t^0)_{t\geq 0}$$

if $(Z_t^\epsilon)_{t\geq 0}$ converges in law to $(Z_t^0)_{t\geq 0}$ in $C([0,\infty), E)$, endowed with the uniform convergence on compact time intervals. This second notion of convergence is strictly stronger than the first one. Our main result writes as follows.

Theorem 2. *Assume (6), fix $\beta > 0$ and consider the solution $(X_t, V_t)_{t\geq 0}$ to (9). Let $(W_t)_{t\geq 0}$ be a Brownian motion, let $(S_t^{(\alpha)})_{t\geq 0}$ be a symmetric stable process with index $\alpha \in (0,2)$ such that $\mathbb{E}[\exp(i\xi S_t^{(\alpha)})] = \exp(-t|\xi|^\alpha)$ and let $(U_t^{(\delta)})_{t\geq 0}$ be a symmetric Bessel process of dimension $\delta \in (0,1)$, see Definition 5.*

(a) If $\beta > 5$,

$$(\epsilon^{1/2} X_{t/\epsilon})_{t\geq 0} \xrightarrow{f.d.} (\sigma_\beta W_t)_{t\geq 0}.$$

(b) If $\beta = 5$,

$$(\epsilon^{1/2}|\log\epsilon|^{-1/2} X_{t/\epsilon})_{t\geq 0} \xrightarrow{f.d.} (\sigma_5 W_t)_{t\geq 0}.$$

(c) If $\beta \in (1,5)$, setting $\alpha = (\beta+1)/3$,

$$(\epsilon^{1/\alpha} X_{t/\epsilon})_{t\geq 0} \xrightarrow{f.d.} (\sigma_\beta S_t^{(\alpha)})_{t\geq 0}.$$

(d) If $\beta = 1$,

$$(|\epsilon\log\epsilon|^{3/2} X_{t/\epsilon})_{t\geq 0} \xrightarrow{f.d.} (\sigma_1 S_t^{(2/3)})_{t\geq 0}.$$

(e) If $\beta \in (0,1)$,

$$(\epsilon^{3/2} X_{t/\epsilon}, \epsilon^{1/2} V_{t/\epsilon})_{t\geq 0} \xrightarrow{d} \Big(\int_0^t U_s^{(1-\beta)} \mathrm{d}s, U_t^{(1-\beta)}\Big)_{t\geq 0}.$$

This result provides convergence of processes and in that is slightly stronger than Theorem 1 and than the statements of [21, 7, 16, 1]. The convergence in (e) is stronger than in the other cases. Still in (e), it is natural to state a result for the joint law of the position and the velocity, because in this sole case, the limit position process alone is not Markov, while the limit of the couple (position, velocity) is Markov.

From that result we will deduce the following decoupling between the position and the velocity. We now deal with the convergence in law in $\mathbb{R}\times\mathbb{R}$ of the (rescaled) random variable (X_t, V_t) for large t, and no longer with the convergence of processes.

Theorem 3. *Fix $\beta > 1$, adopt the same notation as in Theorem 2 and consider a μ_β-distributed random variable $\bar{V}$ independent of everything else.*

(a) If $\beta > 5$, for each $t > 0$,

$$(\epsilon^{1/2} X_{t/\epsilon}, V_{t/\epsilon}) \xrightarrow{d} (\sigma_\beta W_t, \bar{V}).$$

(b) If $\beta = 5$, for each $t > 0$,

$$(\epsilon^{1/2}|\log\epsilon|^{-1/2} X_{t/\epsilon}, V_{t/\epsilon}) \xrightarrow{d} (\sigma_5 W_t, \bar{V}).$$

(c) If $\beta \in (1,5)$, for each $t>0$, setting $\alpha = (\beta+1)/3$,

$$(\epsilon^{1/\alpha} X_{t/\epsilon}, V_{t/\epsilon}) \xrightarrow{d} (\sigma_\beta S_t^{(\alpha)}, \bar{V}).$$

We excluded the case $\beta = 1$ because the invariant measure μ_1 is not integrable, so it is not possible to define a μ_1-distributed random variable $\bar{V}$.

1.7. **Plan of the paper.** In the next section, we recall some facts about the family of local times of a Brownian motion and give explicit expressions, in terms of a Brownian motion, of the symmetric stable process and of what we call a symmetric Bessel process. Section 3, which starts with a detailed plan of the strategy, is devoted to the proof of Theorems 2, 3 and 1.

2. Brownian motion's local times, Stable and Bessel processes

In this section, we first recall the definition and elementary properties of the family of local times of the Brownian motion. We next provide an explicit formulation of the symmetric α-stable process in terms of a Brownian motion. Finally, we explain what we call a symmetric Bessel process of dimension $\delta \in (0,2)$. We recall that we denote by $\mathrm{sgn} : \mathbb{R} \to \{-1,0,1\}$ the function $\mathrm{sgn}(x) = \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x<0\}}$.

2.1. **Local times.** Local times are discussed in details in Revuz-Yor [23] for general semimartingales. Let us summarize the results of [23, Chapter VI, Section 1] that we will use. For $(W_t)_{t\geq 0}$ a Brownian motion and for each $x \in \mathbb{R}$, each $t \geq 0$, we introduce

$$L_t^x = |W_t - x| - |x| - \int_0^t \mathrm{sgn}(W_s - x)\mathrm{d}W_s.$$

Since the second derivative of $|\cdot|$ is twice the Dirac mass, the Itô formula tells us that, informally, $L_t^x = \int_0^t \delta_{W_s = x}\mathrm{d}s$. It indeed holds true that for any $t \geq 0$, any $x \in \mathbb{R}$,

$$L_t^x = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t \mathbf{1}_{\{|W_s - x|<\epsilon\}}\mathrm{d}s \quad \text{a.s.}$$

The process $(L_t^x)_{t\geq 0}$ is a.s. continuous and nondecreasing and is called the local time of W at x. Let us mention that the (random) nonnegative measure $\mathrm{d}L_t^x$ on $[0,\infty)$ is a.s. carried by the set $\{t \geq 0 : W_t = x\}$. We will use the famous occupation times formula, see [23, Corollary 1.6 page 224], that asserts that for any $t \geq 0$, any Borel function $\varphi : \mathbb{R} \to \mathbb{R}_+$, a.s.,

$$\int_0^t \varphi(W_s)\mathrm{d}s = \int_{\mathbb{R}} \varphi(x) L_t^x \mathrm{d}x.$$

We will finally use [23, Corollary 1.8 page 226]: the map $x \mapsto L_t^x$ is a.s. Hölder continuous of order θ for any $\theta \in (0,1/2)$, uniformly on every compact time interval.

2.2. **Stable processes.** The following representation theorem of Biane-Yor [4] is crucial for our study. Similar results were already present in Itô-McKean [11, page 226] and Jeulin-Yor [13] when $\alpha \in (0,1)$.

For any $\alpha \in (0,2)$, it provides an explicit expression, in terms of a Brownian motion and its local time at 0, of the symmetric α-stable process. We recall that a Lévy process $(S_t)_{t\geq 0}$ is said to be a symmetric α-stable process if there is some constant $\kappa > 0$ such that $\mathbb{E}[\exp(i\xi S_t)] = \exp(-\kappa t|\xi|^\alpha)$ for all $t \geq 0$ and all $\xi \in \mathbb{R}$. The value of $\kappa > 0$ is not very important, since we can modify it by multiplying $(S_t)_{t\geq 0}$ by some positive deterministic constant.

Theorem 4 (Biane-Yor)**.** *Fix $\alpha \in (0,2)$. Consider a Brownian motion $(W_t)_{t\geq 0}$, its local time $(L_t^0)_{t\geq 0}$ at 0 and its right-continuous generalized inverse $\tau_t = \inf\{u \geq 0 : L_u^0 > t\}$. For $\eta > 0$, let*

$$K_t^\eta = \int_0^t \mathrm{sgn}(W_s)|W_s|^{1/\alpha - 2}\mathbf{1}_{\{|W_s|\geq\eta\}}\mathrm{d}s.$$

Then $(K_t^\eta)_{t\geq 0}$ a.s. converges, uniformly on compact time intervals, to some process $(K_t)_{t\geq 0}$, as $\eta \to 0$. Moreover, $(K_{\tau_t})_{t\geq 0}$ is a symmetric α-stable process such that for all $t \geq 0$, all $\xi \in \mathbb{R}$,

$$\mathbb{E}[\exp(i\xi K_{\tau_t})] = \exp(-\kappa_\alpha t|\xi|^\alpha), \quad \text{where} \quad \kappa_\alpha = \frac{2^\alpha \pi \alpha^{2\alpha}}{2\alpha[\Gamma(\alpha)]^2 \sin(\pi\alpha/2)}.$$

Observe that when $\alpha \in (0,1)$, we simply have $K_t = \int_0^t \mathrm{sgn}(W_s)|W_s|^{1/\alpha-2}\mathrm{d}s$, since this integral is a.s. absolutely convergent. Observe also that since $(L_t^0)_{t\geq 0}$ is a.s. constant on many time intervals, its generalized inverse $(\tau_t)_{t\geq 0}$ has many jumps. Actually, $(\tau_t)_{t\geq 0}$ is itself a Lévy process (more precisely, it is a $1/2$-stable nondecreasing process, see Revuz-Yor [23, p 240]).

Let us mention that Theorem 4 is very natural and easy to verify, so long as we are not interested in the exact value of κ_α. Indeed, τ_t is a stopping-time, with $W_{\tau_t} = 0$, for each $t \geq 0$. Hence the strong Markov property implies that the process $Z_t^\varphi = \int_0^{\tau_t} \varphi(W_s)\mathrm{d}s$ is Lévy, for any reasonable function $\varphi : \mathbb{R} \to \mathbb{R}$. If φ is odd, it is furthermore of course symmetric, in the sense that for all $t \geq 0$, Z_t^φ has the same law as $-Z_t^\varphi$. Finally, if one wants Z_t^φ to satisfies the scaling property of α-stable processes, i.e. $Z_t^\varphi \overset{d}{=} c^{-1/\alpha} Z_{ct}^\varphi$ for all $c > 0$ and all $t > 0$, there is no choice for φ: it has to be $\varphi(z) = \mathrm{sgn}(z)|z|^{1/\alpha-2}$, because of the scaling of the Brownian motion and its local time, which tells us that

$$((\tau_t)_{t\geq 0}, (W_t)_{t\geq 0}) \overset{d}{=} (c^{-2}\tau_{ct}, c^{-1}W_{c^2 t}).$$

All this classically implies that $(K_{\tau_t})_{t\geq 0}$ is a symmetric α-stable process. The computation of κ_α is rather tedious and involves special functions.

The above arguments are perfectly rigorous when $\alpha \in (0,1)$, but there are some technical difficulties when $\alpha \in [1,2)$ because the integral $\int_0^t \mathrm{sgn}(W_s)|W_s|^{1/\alpha-2}\mathrm{d}s$ is not absolutely convergent.

2.3. **Bessel processes.** Bessel processes are studied in details in Revuz-Yor [23, Chapter XI]. The unfamiliar reader can start directly from Definition 5 below, since we will use nothing more. Let us however briefly recall that, for $\delta \in \mathbb{N}$, a Bessel process $(R_t^{(\delta)})_{t\geq 0}$ of dimension δ is the Euclidean norm of a δ-dimensional Brownian motion. Its square $T_t^{(\delta)} = (R_t^{(\delta)})^2$ then satisfies the 1-dimensional S.D.E.

$$T_t^{(\delta)} = 2\int_0^t [T_s^{(\delta)}]^{1/2}\mathrm{d}W_s + \delta t, \tag{10}$$

for some (other) one-dimensional Brownian motion $(W_t)_{t\geq 0}$. This S.D.E. makes sense and has a unique nonnegative solution for any $\delta \in \mathbb{R}_+$. Hence for any $\delta \in \mathbb{R}_+$, one can define the Bessel process with dimension δ as the square root of the solution to (10).

Bessel processes are nonnegative, and we need a signed symmetric version. Roughly, we would like to take a Bessel process and to change the sign of each excursion, independently, with probability 1/2. Inspired by Donati-Roynette-Vallois-Yor [8], we will rather use the following (equivalent) definition. It would be sufficient, for our purpose, to study the case $\delta \in (0,1)$.

Definition 5. *Let $\delta \in (0,2)$. Consider a Brownian motion $(W_t)_{t\geq 0}$, introduce the time-change*

$$\bar{A}_t = (2-\delta)^{-2}\int_0^t |W_s|^{-2(1-\delta)/(2-\delta)}\mathrm{d}s$$

and its inverse $(\bar{\tau}_t)_{t\geq 0}$. We set

$$U_t^{(\delta)} = \mathrm{sgn}(W_{\bar{\tau}_t})|W_{\bar{\tau}_t}|^{1/(2-\delta)}$$

and say that $(U_t^{(\delta)})_{t\geq 0}$ is a symmetric Bessel process with dimension δ.

Since $2(1-\delta)/(2-\delta) < 1$, $\mathbb{E}[\bar{A}_t] < \infty$ for all $t \geq 0$. The map $t \mapsto \bar{A}_t$ is a.s. continuous, strictly increasing and $\bar{A}_\infty = \infty$ a.s. by recurrence of $(W_t)_{t\geq 0}$, so that $(\bar{\tau}_t)_{t\geq 0}$ is well-defined and continuous.

Also, $U_t^{(1)} = W_t$ (because then $\bar{A}_t = t$, whence $\bar{\tau}_t = t$, so that $U_t^{(1)} = \mathrm{sgn}(W_t)|W_t| = W_t$): the Brownian motion is the symmetric Bessel process of dimension 1.

To justify the terminology, let us mention that $(|U_t^{(\delta)}|)_{t\geq 0}$ is a Bessel process with dimension δ. Indeed, [8, Corollary 2.2] tells us that, for $(R_t^{(\delta)})_{t\geq 0}$ a Bessel process with dimension $\delta \in (0,2)$, there exists a Brownian motion $(W_t)_{t\geq 0}$ such that

$$R_t^{(\delta)} = |W_{C_t}|^{1/(2-\delta)},$$

where $C_t = (2-\delta)^2 \int_0^t [R_s^{(\delta)}]^{2(1-\delta)}\mathrm{d}s$. But $C_t = \bar{\tau}_t$, because

$$\bar{A}_{C_t} = (2-\delta)^{-2} \int_0^{C_t} |W_s|^{-2(1-\delta)/(2-\delta)}\mathrm{d}s = \int_0^t |W_{C_u}|^{-2(1-\delta)/(2-\delta)} [R_u^{(\delta)}]^{2(1-\delta)}\mathrm{d}u = t.$$

As a conclusion, $R_t^{(\delta)} = |W_{\bar{\tau}_t}|^{1/(2-\delta)} = |U_t^{(\delta)}|$.

3. Proofs

Here is the strategy of the proof. In Subsection 3.1, we write down the *explicit* solution to (9), when $X_0 = V_0 = 0$, using the classical theory of speed measures and scale functions, in terms of a time changed Brownian motion. More precisely, we introduce some explicit (in terms of Θ) functions $\Psi_\epsilon : \mathbb{R} \to \mathbb{R}$ and $\sigma_\epsilon : \mathbb{R} \to (0,\infty)$ such that, for $(W_t)_{t\geq 0}$ a Brownian motion and for $(\tau_t^\epsilon)_{t\geq 0}$ the inverse function of $(A_t^\epsilon)_{t\geq 0}$ defined by

$$A_t^\epsilon = \int_0^t [\sigma_\epsilon(W_s)]^{-2}\mathrm{d}s,$$

the processes $(V_{t/\epsilon})_{t\geq 0}$ (where $(V_t)_{t\geq 0}$ solves (9)) and $(\Psi_\epsilon(W_{\tau_t^\epsilon}))_{t\geq 0}$ have the same law.

In Subsection 3.2, we recall some facts about the convergence of (generalized) inverse functions.

In Subsection 3.3, we prove our main theorem when $\beta \in (0,1)$ and $X_0 = V_0 = 0$. We start from

$$(\epsilon^{3/2} X_{t/\epsilon}, \epsilon^{1/2} V_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} \Big(\int_0^t \epsilon^{1/2}\Psi_\epsilon(W_{\tau_s^\epsilon})\mathrm{d}s, \epsilon^{1/2}\Psi_\epsilon(W_{\tau_t^\epsilon}) \Big)_{t\geq 0}.$$

We show that $\epsilon^{1/2}\Psi_\epsilon(z)$ resembles $\mathrm{sgn}(z)|z|^{1/(1+\beta)}$ and that $\sigma_\epsilon(z)$ resembles $(\beta+1)|z|^{\beta/(\beta+1)}$. Recalling Definition 5 with $\delta = 1-\beta$, we thus have

$$A_t^\epsilon \simeq (\beta+1)^{-2} \int_0^t |W_s|^{-2\beta/(\beta+1)}\mathrm{d}s = \bar{A}_t, \quad \text{whence} \quad \tau_t^\epsilon \simeq \bar{\tau}_t,$$

so that

$$\epsilon^{1/2}\Psi_\epsilon(W_{\tau_t^\epsilon}) \simeq \mathrm{sgn}(W_{\bar{\tau}_t})|W_{\bar{\tau}_t}|^{1/(\beta+1)} = U_t^{(1-\beta)} \quad \text{and} \quad \epsilon^{3/2} X_{t/\epsilon} \simeq \int_0^t U_s^{(1-\beta)}\mathrm{d}s.$$

We study the case $\beta \in [1,5)$ (and $X_0 = V_0 = 0$ again) in Subsection 3.4. Up to a logarithmic correction when $\beta = 1$, we write, with $\alpha = (\beta+1)/3$,

$$(\epsilon^{1/\alpha} X_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} \Big(\epsilon^{1/\alpha-1}\int_0^t \Psi_\epsilon(W_{\tau_s^\epsilon})\mathrm{d}s\Big)_{t\geq 0} = \Big(\epsilon^{1/\alpha-1}\int_0^{\tau_t^\epsilon} \Psi_\epsilon(W_s)[\sigma_\epsilon(W_s)]^{-2}\mathrm{d}s\Big)_{t\geq 0}.$$

We first show, using the occupation times formula, that

$$A_t^\epsilon \simeq L_t^0, \quad \text{whence} \quad \tau_t^\epsilon \simeq \tau_t,$$

where $(\tau_t)_{t\geq 0}$ is the generalized inverse of $(L_t^0)_{t\geq 0}$. We also verify that

$$\epsilon^{1/\alpha-1}\Psi_\epsilon(z)[\sigma_\epsilon(z)]^{-2} \simeq \mathrm{sgn}(z)|z|^{(1-2\beta)/(1+\beta)} = \mathrm{sgn}(z)|z|^{1/\alpha-2}.$$

All in all, we deduce that

$$(\epsilon^{1/\alpha} X_{t/\epsilon})_{t\geq 0} \overset{d}{\simeq} \Big(\int_0^{\tau_t} \mathrm{sgn}(W_s)|W_s|^{1/\alpha-2}\mathrm{d}s \Big)_{t\geq 0},$$

which is a symmetric α-stable process by Theorem 4. This part is rather technical and some constants actually appear everywhere, but we indicate the points of the proof required to understand the case $\beta \in (1,2)$, which is the less involved.

In Subsection 3.5, we quickly check Theorem 2 in the normal diffusive case $\beta > 5$, still when $X_0 = V_0 = 0$. This enters the classical theory of limit theorems for stochastic processes that can be found in Jacod-Shiryaev [12, Chapter VIII, Section 3f]. We also study the case $\beta = 5$, making use of some other material checked in Subsection 3.4.

We extend the above results to all initial conditions in Subsection 3.6 and the last subsection is devoted to the proof of Theorem 3 concerning the kinetic Fokker-Planck equation.

3.1. **Scale function and speed measure.** We introduce some notation, closely linked with the theory of scale function and speed measure. This is a classical way to solve explicitly one-dimensional S.D.E.s. The reason why we introduce h, σ and ϕ below will appear clearly in the proof of Lemma 6. The goal is to rewrite $(X_t)_{t\geq 0}$ in such a way that it resembles the objects appearing in Theorem 4 and Definition 5.

Recall our conditions on Θ, see (6) and that $F = -\Theta'/\Theta$. First, the function from $\mathbb{R}$ to $\mathbb{R}$

$$h(v) = (\beta+1)\int_0^v [\Theta(u)]^{-\beta}\mathrm{d}u$$

is odd, increasing, bijective, solves $h'' = \beta F h'$, and we have the asymptotics

$$h(v) \overset{|v|\to\infty}{\sim} \mathrm{sgn}(v)|v|^{\beta+1} \quad \text{and} \quad h^{-1}(z) \overset{|z|\to\infty}{\sim} \mathrm{sgn}(z)|z|^{1/(\beta+1)}.$$

Next, the function on $\mathbb{R}$

$$\sigma(z) = h'(h^{-1}(z))$$

is even, bounded below by some constant $c > 0$ and

$$\sigma(z) \overset{|z|\to\infty}{\sim} (\beta+1)|z|^{\beta/(\beta+1)}.$$

The function from $\mathbb{R}$ to $\mathbb{R}$

$$\phi(z) = h^{-1}(z)/\sigma^2(z)$$

is odd and

$$\phi(z) \overset{|z|\to\infty}{\sim} (\beta+1)^{-2}\mathrm{sgn}(z)|z|^{(1-2\beta)/(\beta+1)}.$$

When $\beta = 5$, we define for subsequent use the function on $\mathbb{R}$

$$\psi(z) = [g'(h^{-1}(z))]^2/\sigma^2(z),$$

where $g'(v) = 2\Theta^{-5}(v)\int_v^\infty u\Theta^5(u)\mathrm{d}u \overset{|v|\to\infty}{\sim} 2|v|^2/3$. Observe that g' is even, because Θ is even and $\int_{\mathbb{R}} u\Theta^5(u)\mathrm{d}u = 0$. The function ψ is even, bounded and

$$\psi(z) \overset{|z|\to\infty}{\sim} 1/(81|z|).$$

Lemma 6. *Fix $\beta > 0$, $\epsilon > 0$ and $a_\epsilon > 0$. Consider a Brownian motion $(W_t)_{t\geq 0}$. Define*

$$A_t^\epsilon = \epsilon a_\epsilon^{-2}\int_0^t [\sigma(W_s/a_\epsilon)]^{-2}\mathrm{d}s$$

and its inverse $(\tau_t^\epsilon)_{t\geq 0}$, which is a continuous increasing bijection from $[0,\infty)$ into itself. Set

$$V_t^\epsilon = h^{-1}(W_{\tau_t^\epsilon}/a_\epsilon) \quad \textit{and} \quad X_t^\epsilon = H^\epsilon_{\tau_t^\epsilon} \quad \textit{where} \quad H_t^\epsilon = a_\epsilon^{-2}\int_0^t \phi(W_s/a_\epsilon)\mathrm{d}s.$$

For $(X_t, V_t)_{t\geq 0}$ the unique solution of (9) starting from $(0,0)$, we have

$$(X_{t/\epsilon}, V_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} (X_t^\epsilon, V_t^\epsilon)_{t\geq 0}.$$

The result holds for any value of $a_\epsilon > 0$, but in each situation, we will choose it judiciously, in such a way that $(A_t^\epsilon)_{t\geq 0}$ a.s. converges, as $\epsilon \to 0$, to the desired limit time-change.

Remark 7. *For any $\beta > 0$, the solution $(V_t)_{t\geq 0}$ to (9) is global, regular and recurrent. Indeed, choosing $\epsilon = a_\epsilon = 1$ in Lemma 6, we see that the solution $(V_t)_{t\geq 0}$ to (9) with $V_0 = 0$ has the same law as $(h^{-1}(W_{\tau_t}))_{t\geq 0}$, for some Brownian motion $(W_t)_{t\geq 0}$, some (random) continuous bijective time change $\tau_t : [0,\infty) \to [0,\infty)$ and some continuous bijective function $h : \mathbb{R} \to \mathbb{R}$. Hence $(V_t)_{t\geq 0}$ is non-exploding and thus global, and it is regular and recurrent (when starting from any initial condition).*

Proof of Lemma 6. We fix $\beta > 0$, $\epsilon > 0$ and $a_\epsilon > 0$ and set $\sigma_\epsilon(w) = \epsilon^{-1/2} a_\epsilon \sigma(w/a_\epsilon)$, so that

$$A_t^\epsilon = \int_0^t [\sigma_\epsilon(W_s)]^{-2}\mathrm{d}s.$$

Since σ_ϵ is bounded below, $t \mapsto A_t^\epsilon$ is a.s. continuous and strictly increasing. By recurrence of the Brownian motion, we also have $A_\infty^\epsilon = \infty$ a.s. Hence τ_t^ϵ is well-defined, continuous, bijective from $[0,\infty) \to [0,\infty)$ and $Y_t^\epsilon = W_{\tau_t^\epsilon}$ classically solves, see e.g. Revuz-Yor [23, Proposition 1.13 page 373],

$$Y_t^\epsilon = \int_0^t \sigma_\epsilon(Y_s^\epsilon)\mathrm{d}B_s^\epsilon,$$

for some Brownian motion $(B_t^\epsilon)_{t\geq 0}$. We then set $\varphi_\epsilon(y) = h^{-1}(y/a_\epsilon)$ and use the Itô formula to write

$$V_t^\epsilon = \varphi_\epsilon(Y_t^\epsilon) = \int_0^t \varphi_\epsilon'(Y_s^\epsilon)\sigma_\epsilon(Y_s^\epsilon)\mathrm{d}B_s^\epsilon + \frac{1}{2}\int_0^t \varphi_\epsilon''(Y_s^\epsilon)\sigma_\epsilon^2(Y_s^\epsilon)\mathrm{d}s.$$

But the functions σ_ϵ and φ_ϵ, built from σ and h, have been precisely designed in such a way that $\varphi_\epsilon'(y)\sigma_\epsilon(y) = \epsilon^{-1/2}$ and $\varphi_\epsilon''(y)\sigma_\epsilon^2(y) = -\beta\epsilon^{-1}F(\varphi_\epsilon(y))$.

Indeed, using that $(h^{-1})' = 1/\sigma$, we find

$$\varphi_\epsilon'(y)\sigma_\epsilon(y) = \frac{1}{a_\epsilon \sigma(y/a_\epsilon)}\epsilon^{-1/2}a_\epsilon\sigma(y/a_\epsilon) = \epsilon^{-1/2}.$$

And since $\sigma' = [h'(h^{-1})]' = (h^{-1})'h''(h^{-1}) = h''(h^{-1})/h'(h^{-1}) = \beta F(h^{-1})$,

$$\varphi_\epsilon''(y)\sigma_\epsilon^2(y) = \frac{-\sigma'(y/a_\epsilon)}{a_\epsilon^2\sigma^2(y/a_\epsilon)}\epsilon^{-1}a_\epsilon^2\sigma^2(y/a_\epsilon) = -\epsilon^{-1}\sigma'(y/a_\epsilon) = -\beta\epsilon^{-1}F(h^{-1}(y/a_\epsilon)) = -\beta\epsilon^{-1}F(\varphi_\epsilon(y)).$$

We end with

$$V_t^\epsilon = \frac{1}{\sqrt{\epsilon}}B_t^\epsilon - \frac{\beta}{2\epsilon}\int_0^t F(\varphi_\epsilon(Y_s^\epsilon))\mathrm{d}s = \frac{1}{\sqrt{\epsilon}}B_t^\epsilon - \frac{\beta}{2\epsilon}\int_0^t F(V_s^\epsilon)\mathrm{d}s.$$

Next, starting from (9) (with $V_0 = 0$), we find

$$V_{t/\epsilon} = B_{t/\epsilon} - \frac{\beta}{2}\int_0^{t/\epsilon} F(V_s)\mathrm{d}s = \frac{1}{\sqrt{\epsilon}}(\sqrt{\epsilon}B_{t/\epsilon}) - \frac{\beta}{2\epsilon}\int_0^t F(V_{s/\epsilon})\mathrm{d}s.$$

Hence $(V_t^\epsilon)_{t\geq 0}$ and $(V_{t/\epsilon})_{t\geq 0}$ are two solutions of the same well-posed S.D.E., driven by different Brownian motions, namely $(B_t^\epsilon)_{t\geq 0}$ and $(\sqrt{\epsilon}B_{t/\epsilon})_{t\geq 0}$. They thus have the same law.

Since $X_{t/\epsilon} = \int_0^{t/\epsilon} V_s \mathrm{d}s = \epsilon^{-1}\int_0^t V_{s/\epsilon}\mathrm{d}s$, we conclude that

$$(X_{t/\epsilon}, V_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} \Big(\epsilon^{-1}\int_0^t V_s^\epsilon \mathrm{d}s, V_t^\epsilon\Big)_{t\geq 0}.$$

But using the substitution $u = \tau_s^\epsilon$, i.e. $s = A_u^\epsilon$, whence $\mathrm{d}s = [\sigma_\epsilon(W_u)]^{-2}\mathrm{d}u$, we find

$$\epsilon^{-1}\int_0^t V_s^\epsilon \mathrm{d}s = \epsilon^{-1}\int_0^t \varphi_\epsilon(W_{\tau_s^\epsilon})\mathrm{d}s = \epsilon^{-1}\int_0^{\tau_t^\epsilon} \frac{\varphi_\epsilon(W_u)}{[\sigma_\epsilon(W_u)]^2}\mathrm{d}u = a_\epsilon^{-2}\int_0^{\tau_t^\epsilon}\phi(W_u/a_\epsilon)\mathrm{d}u = H^\epsilon_{\tau_t^\epsilon}.$$

as desired. We used that

$$\epsilon^{-1}[\sigma_\epsilon(w)]^{-2}\varphi_\epsilon(w) = a_\epsilon^{-2}[\sigma(w/a_\epsilon)]^{-2}h^{-1}(w/a_\epsilon) = a_\epsilon^{-2}\phi(w/a_\epsilon).$$ □

3.2. **Inverting time-changes.** We recall the following classical and elementary results.

Lemma 8. *Consider, for each $n \geq 1$, a continuous increasing bijective function $(a_t^n)_{t\geq 0}$ from $[0,\infty)$ into itself, as well as its inverse $(r_t^n)_{t\geq 0}$.*

(a) Assume that $(a_t^n)_{t\geq 0}$ converges pointwise to some (nondecreasing) function $(a_t)_{t\geq 0}$ such that $\lim_{t\to\infty} a_t = \infty$, denote by $r_t = \inf\{u \geq 0 : a_u > t\}$ its right-continuous generalized inverse and set

$$J = \{s \in [0,\infty) : r_{t-} < r_t\}.$$

For all $t \in [0,\infty)\setminus J$, we have $\lim_{t\to\infty} r_t^n = r_t$.

(b) If $(a_t^n)_{t\geq 0}$ converges locally uniformly to some strictly increasing function $(a_t)_{t\geq 0}$ such that $\lim_{t\to\infty} a_t = \infty$, then $(r_t^n)_{t\geq 0}$ converges locally uniformly to $(r_t)_{t\geq 0}$, the (classical) inverse of $(a_t)_{t\geq 0}$.

3.3. **The integrated Bessel regime.** We can now give the

Proof of Theorem 2-(e) when $X_0 = V_0 = 0$. Let $\beta \in (0,1)$ be fixed. It suffices to verify that

$$(\epsilon^{1/2}V_{t/\epsilon})_{t\geq 0} \stackrel{d}{\longrightarrow} (U_t^{(1-\beta)})_{t\geq 0}. \tag{11}$$

This will indeed imply that

$$(\epsilon^{3/2}X_{t/\epsilon}, \epsilon^{1/2}V_{t/\epsilon})_{t\geq 0} = \Big(\int_0^t (\epsilon^{1/2}V_{s/\epsilon})\mathrm{d}s, \epsilon^{1/2}V_{t/\epsilon}\Big)_{t\geq 0} \stackrel{d}{\longrightarrow} \Big(\int_0^t U_s^{(1-\beta)}\mathrm{d}s, U_t^{(1-\beta)}\Big)_{t\geq 0},$$

because we deal with the uniform convergence on compact time intervals.

We consider a Brownian motion $(W_t)_{t\geq 0}$ and, as in Definition 5 with $\delta = 1-\beta \in (0,1)$, we introduce the a.s. continuous strictly increasing bijective time-change

$$\bar{A}_t = (\beta+1)^{-2}\int_0^t |W_s|^{-2\beta/(\beta+1)}\mathrm{d}s,$$

its inverse $(\bar{\tau}_t)_{t\geq 0}$ and the symmetric Bessel process with dimension $1-\beta$

$$U_t^{(1-\beta)} = \mathrm{sgn}(W_{\bar{\tau}_t})|W_{\bar{\tau}_t}|^{1/(\beta+1)}.$$

We now apply Lemma 6 with the choice $a_\epsilon = \epsilon^{(\beta+1)/2}$: with the same Brownian motion as above, we consider, for each $\epsilon > 0$, the time-change

$$A_t^\epsilon = \epsilon^{-\beta}\int_0^t [\sigma(W_s/\epsilon^{(\beta+1)/2})]^{-2}\mathrm{d}s,$$

its inverse τ_t^ϵ, and the process

$$V_t^\epsilon = h^{-1}(W_{\tau_t^\epsilon}/\epsilon^{(\beta+1)/2}).$$

Since $V_0 = 0$, $(V_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} (V_t^\epsilon)_{t\geq 0}$ by Lemma 6. Since our goal is (11), it suffices to prove that

$$\lim_{\epsilon\to 0}\sup_{[0,T]}\big|\epsilon^{1/2}V_t^\epsilon - U_t^{(1-\beta)}\big| = 0 \quad \text{a.s., for all } T \geq 0. \tag{12}$$

Since $\sigma(z) \overset{|z|\to\infty}{\sim} (\beta+1)|z|^{\beta/(\beta+1)}$, whence $\sigma^{-2}(z) \le C|z|^{-2\beta/(\beta+1)}$ (because $\sigma(x) \ge c > 0$),

$$\lim_{\epsilon\to 0} \sup_{[0,T]} |A^\epsilon_t - \bar{A}_t| \le \lim_{\epsilon\to 0} \int_0^T \Big|\epsilon^{-\beta}[\sigma(W_s/\epsilon^{(\beta+1)/2})]^{-2} - [(\beta+1)|W_s|^{\beta/(\beta+1)}]^{-2}\Big| \mathrm{d}s = 0 \quad \text{a.s.}$$

by dominated convergence. Indeed, we have $\sup_{\epsilon>0} \epsilon^{-\beta}[\sigma(W_s/\epsilon^{(\beta+1)/2})]^{-2} \le C|W_s|^{-2\beta/(\beta+1)}$, and $\int_0^T |W_s|^{-2\beta/(\beta+1)}\mathrm{d}s < \infty$ a.s. because $2\beta/(\beta+1) < 1$.

By Lemma 8-(b), we deduce that a.s., for all $T \ge 0$,

$$\lim_{\epsilon\to 0} \sup_{[0,T]} |\tau^\epsilon_t - \bar{\tau}_t| = 0,$$

whence, by continuity of $(W_t)_{t\ge 0}$,

$$\lim_{\epsilon\to 0} \sup_{[0,T]} |W_{\tau^\epsilon_t} - W_{\bar{\tau}_t}| = 0 \quad \text{a.s. for all } T > 0. \tag{13}$$

We next claim that for all $M > 0$,

$$\kappa_\epsilon(M) = \sup_{|z|\le M} |\epsilon^{1/2} h^{-1}(z/\epsilon^{(\beta+1)/2}) - \mathrm{sgn}(z)|z|^{1/(\beta+1)}| \to 0.$$

Indeed, since h^{-1} is C^1 with $h^{-1}(0) = 0$, since $\beta > 0$ and since $h^{-1}(z) \overset{|z|\to\infty}{\sim} \mathrm{sgn}(z)|z|^{1/(\beta+1)}$ the function

$$\gamma(z) = \frac{h^{-1}(z)}{\mathrm{sgn}(z)|z|^{1/(\beta+1)}} - 1 \qquad (\text{with } \gamma(0) = -1)$$

is continuous on $\mathbb{R}$ and $\lim_{|z|\to\infty} \gamma(z) = 0$. Hence,

$$\begin{aligned}\kappa_\epsilon(M) = &\sup_{|z|\le M} |z|^{1/(\beta+1)} |\gamma(z/\epsilon^{(\beta+1)/2})| \\ \le &\epsilon^{1/4}||\gamma||_\infty + M^{1/(\beta+1)} \sup_{|z|\ge \epsilon^{(\beta+1)/4}} |\gamma(z/\epsilon^{(\beta+1)/2})| \\ = &\epsilon^{1/4}||\gamma||_\infty + M^{1/(\beta+1)} \sup_{|z|\ge \epsilon^{-(\beta+1)/4}} |\gamma(z)|,\end{aligned}$$

which tends to 0 as $\epsilon \to 0$.

All in all, denoting by $M_T = \sup_{[0,T]} \sup_{\epsilon\in(0,1)} |W_{\tau^\epsilon_t}|$, which is a.s. finite by (13),

$$\begin{aligned}\sup_{[0,T]} |\epsilon^{1/2} V^\epsilon_t - U^{(1-\beta)}_t| = &\sup_{[0,T]} |\epsilon^{1/2} h(W_{\tau^\epsilon_t}/\epsilon^{(\beta+1)/2}) - \mathrm{sgn}(W_{\bar{\tau}_t})|W_{\bar{\tau}_t}|^{1/(\beta+1)}| \\ \le &\kappa_\epsilon(M_T) + \sup_{[0,T]} \Big|\mathrm{sgn}(W_{\tau^\epsilon_t})|W_{\tau^\epsilon_t}|^{1/(\beta+1)} - \mathrm{sgn}(W_{\bar{\tau}_t})|W_{\bar{\tau}_t}|^{1/(\beta+1)}\Big| \to 0\end{aligned}$$

a.s., by (13) and by continuity of $w \to \mathrm{sgn}(w)|w|^{1/(\beta+1)}$. This shows (12) and completes the proof. □

3.4. **The Lévy regime.** Here we treat the case $\beta \in [1, 5)$ and by the way prepare the case $\beta = 5$. We start with the following crucial lemma, that will allow us to pass to the limit in the explicit expression of the solution to (9) written in Lemma 6 and to find as limit the stable process of Theorem 4.

It might be helpful at first reading to look only at the proofs of (a) below when $\beta > 1$, of (b) below when $\beta \in (1, 2)$ and of Theorem 2-(c). This is sufficient to understand the proof when $\beta \in (1, 2)$. The other cases share the same spirit, with possibly important additional technical difficulties.

Lemma 9. *Fix $\beta \in [1, 5]$ and a Brownian motion $(W_t)_{t\ge 0}$, denote by $(L^0_t)_{t\ge 0}$ its local time at 0 and by $(K_t)_{t\ge 0}$ the process defined in Theorem 4 with $\alpha = (\beta+1)/3$. For each $\epsilon > 0$, consider the processes $(A^\epsilon_t)_{t\ge 0}$ and $(H^\epsilon_t)_{t\ge 0}$ built in Lemma 6 with the choice*

$$a_\epsilon = \epsilon/[(\beta+1)c_\beta] \quad \textit{if } \beta \in (1, 5] \qquad \textit{and} \quad a_\epsilon = \epsilon|\log\epsilon|/2 \quad \textit{if } \beta = 1$$

and with the same Brownian motion $(W_t)_{t\geq 0}$ as above.

(a) We always have a.s., for all $T>0$,

$$\lim_{\epsilon\to 0}\sup_{[0,T]}\left|A_t^\epsilon - L_t^0\right| = 0.$$

(b) If $\beta\in(1,5)$, then a.s., for all $T>0$,

$$\lim_{\epsilon\to 0}\sup_{[0,T]}\left|\epsilon^{1/\alpha}H_t^\epsilon - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_t\right| = 0.$$

(c) If $\beta=1$, then a.s., for all $T>0$,

$$\lim_{\epsilon\to 0}\sup_{[0,T]}\left||\epsilon\log\epsilon|^{3/2}H_t^\epsilon - K_t/\sqrt{2}\right| = 0.$$

(d) If $\beta=5$, then a.s., for all $T>0$,

$$\lim_{\epsilon\to 0}\sup_{[0,T]}\left|T_t^\epsilon - \sigma_5^2L_t^0\right| = 0, \quad \textit{where} \quad T_t^\epsilon = \frac{\epsilon}{a_\epsilon^2|\log\epsilon|}\int_0^t \psi(W_s/a_\epsilon)\mathrm{d}s.$$

We recall that $c_\beta = 1/[\int_{\mathbb{R}}[\Theta(v)]^\beta \mathrm{d}v]$ (when $\beta>1$), that $\sigma_5^2 = 4c_5/27$ that the functions h, σ, ϕ and ψ were introduced at the beginning of Subsection 3.1.

Proof. Point (a) when $\beta>1$. We set $\gamma=(\beta+1)c_\beta$ and recall that $a_\epsilon=\epsilon/\gamma$, whence

$$A_t^\epsilon = \gamma^2\epsilon^{-1}\int_0^t [\sigma(\gamma W_s/\epsilon)]^{-2}\mathrm{d}s.$$

Using the occupation times formula, see Revuz-Yor [23, Corollary 1.6 page 224], we may write

$$A_t^\epsilon = \int_{\mathbb{R}}\frac{\gamma^2 L_t^x\mathrm{d}x}{\epsilon\sigma^2(\gamma x/\epsilon)} = \int_{\mathbb{R}}\frac{\gamma L_t^{\epsilon y/\gamma}\mathrm{d}y}{\sigma^2(y)},$$

where $(L_t^x)_{t\geq 0}$ is the local time of $(W_t)_{t\geq 0}$ at x. Observe now that

$$\int_{\mathbb{R}}\frac{\gamma\mathrm{d}y}{\sigma^2(y)} = \int_{\mathbb{R}}\frac{\gamma\mathrm{d}y}{[h'(h^{-1}(y))]^2} = \int_{\mathbb{R}}\frac{\gamma\mathrm{d}v}{h'(v)} = \int_{\mathbb{R}}\frac{\gamma\Theta^\beta(v)\mathrm{d}v}{(\beta+1)} = \frac{\gamma}{(\beta+1)c_\beta} = 1.$$

Consequently,

$$\sup_{[0,T]}|A_t^\epsilon - L_t^0| = \sup_{[0,T]}\left|\gamma\int_{\mathbb{R}}\frac{L_t^{\epsilon y/\gamma}-L_t^0}{\sigma^2(y)}\mathrm{d}y\right| \leq \gamma\int_{\mathbb{R}}\frac{\sup_{[0,T]}|L_t^{\epsilon y/\gamma}-L_t^0|}{\sigma^2(y)}\mathrm{d}y,$$

which a.s. tends to 0 as $\epsilon\to 0$ by dominated convergence, since $\sup_{[0,T]}|L_t^{\epsilon y/\gamma}-L_t^0|$ a.s. tends to 0 for each fixed y by [23, Corollary 1.8 page 226], since $\sup_{[0,T]\times\mathbb{R}}L_t^x$ is a.s. finite and since $\int_{\mathbb{R}}\sigma^{-2}(y)\mathrm{d}y<\infty$.

Point (b) when $\beta\in(1,2)$. This step is useless since included in the more general case $\beta\in(1,5)$ below. Since it is much easier, we present it for the sake of pedagogy. We have $a_\epsilon=\epsilon/\gamma$ with $\gamma=(\beta+1)c_\beta$. Recall that $\alpha=(\beta+1)/3$, whence $1/\alpha-2=(1-2\beta)/(\beta+1)$. Since $\alpha\in(0,1)$, no principal value is needed and it holds true that (recall that K_t and K_t^η were defined in Theorem 4)

$$K_t = \lim_{\eta\to 0}\int_0^t \mathrm{sgn}(W_s)|W_s|^{1/\alpha-2}\mathbf{1}_{\{|W_s|\geq\eta\}}\mathrm{d}s = \int_0^t \mathrm{sgn}(W_s)|W_s|^{1/\alpha-2}\mathrm{d}s.$$

Recalling Lemma 6, we have

$$\epsilon^{1/\alpha}H_t^\epsilon = \gamma^2\epsilon^{1/\alpha-2}\int_0^t\phi(\gamma W_s/\epsilon)\mathrm{d}s.$$

Hence

$$\lim_{\epsilon\to 0}\sup_{[0,T]}|\epsilon^{1/\alpha}H^\epsilon_t-(\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_t|$$
$$\le \lim_{\epsilon\to 0}\int_0^T\Big|\gamma^2\epsilon^{1/\alpha-2}\phi(\gamma W_s/\epsilon)-(\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}\mathrm{sgn}(W_s)|W_s|^{1/\alpha-2}\Big|\mathrm{d}s=0$$

a.s. by dominated convergence. Indeed, it suffices to use that

$$\phi(z)\stackrel{|z|\to\infty}{\sim}(\beta+1)^{-2}\mathrm{sgn}(z)|z|^{(1-2\beta)/(\beta+1)}=(\beta+1)^{-2}\mathrm{sgn}(z)|z|^{1/\alpha-2},$$

whence

$$\gamma^2\epsilon^{1/\alpha-2}\phi(\gamma w/\epsilon)\to\gamma^{1/\alpha}(\beta+1)^{-2}\mathrm{sgn}(w)|w|^{1/\alpha-2}=(\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}\mathrm{sgn}(w)|w|^{1/\alpha-2}$$

by definition of γ, the bound $|\phi(z)|\le C|z|^{1/\alpha-2}$ (recall that ϕ is continuous and that $1/\alpha-2<0$ because $\alpha>2/3>1/2$) which implies that

$$\sup_{\epsilon\in(0,1)}\gamma^2\epsilon^{1/\alpha-2}|\phi(\gamma W_s/\epsilon)|\le C\gamma^{1/\alpha}|W_s|^{1/\alpha-2}$$

and the fact that, since $1/\alpha-2>-1$ (because $\alpha<1$),

$$\int_0^T|W_s|^{1/\alpha-2}\mathrm{d}s<\infty\quad\text{a.s.}$$

Point (a) when $\beta=1$. Here $a_\epsilon=\epsilon|\log\epsilon|/2$. Also, $\sigma(z)$ is bounded below and $\sigma(z)\stackrel{|z|\to\infty}{\sim}2|z|^{1/2}$, from which

$$\int_{-x}^x\frac{\mathrm{d}z}{\sigma^2(z)}\stackrel{x\to\infty}{\sim}\frac{\log x}{2}. \tag{14}$$

We now fix $\delta>0$ and write $A^\epsilon_t=I^{\epsilon,\delta}_t+J^{\epsilon,\delta}_t$, where

$$I^{\epsilon,\delta}_t=\int_0^t\frac{\epsilon\mathrm{d}s}{a_\epsilon^2\sigma^2(W_s/a_\epsilon)}\mathbf{1}_{\{|W_s|\le\delta\}}\quad\text{and}\quad J^{\epsilon,\delta}_t=\int_0^t\frac{\epsilon\mathrm{d}s}{a_\epsilon^2\sigma^2(W_s/a_\epsilon)}\mathbf{1}_{\{|W_s|>\delta\}}.$$

There is $c>0$ such that $\sigma^2(z)\ge c(1+|z|)$, from which one verifies, using only that

$$|W_s|>\delta\quad\Longrightarrow\quad\sigma^2(W_s/a_\epsilon)\ge c(1+\delta/a_\epsilon)\ge c\delta/a_\epsilon,$$

that $\sup_{[0,T]}|J^{\epsilon,\delta}_t|\le T\epsilon/(ca_\epsilon\delta)$, which tends to 0 as $\epsilon\to0$ because $a_\epsilon=\epsilon|\log\epsilon|/2$. We next use the occupation times formula to write

$$I^{\epsilon,\delta}_t=\int_{-\delta}^\delta\frac{\epsilon L^x_t\mathrm{d}x}{a_\epsilon^2\sigma^2(x/a_\epsilon)}=\Big(\int_{-\delta}^\delta\frac{\epsilon\mathrm{d}x}{a_\epsilon^2\sigma^2(x/a_\epsilon)}\Big)L^0_t+\int_{-\delta}^\delta\frac{\epsilon(L^x_t-L^0_t)\mathrm{d}x}{a_\epsilon^2\sigma^2(x/a_\epsilon)}=r_{\epsilon,\delta}L^0_t+R^{\epsilon,\delta}_t,$$

the last identity standing for a definition. But a substitution and (14) allow us to write

$$r_{\epsilon,\delta}=\int_{-\delta/a_\epsilon}^{\delta/a_\epsilon}\frac{\epsilon\mathrm{d}y}{a_\epsilon\sigma^2(y)}\stackrel{\epsilon\to0}{\sim}\frac{\epsilon\log(\delta/a_\epsilon)}{2a_\epsilon}\longrightarrow1$$

as $\epsilon\to0$ since $a_\epsilon=\epsilon|\log\epsilon|/2$. Recalling that $A^\epsilon_t=r_{\epsilon,\delta}L^0_t+R^{\epsilon,\delta}_t+J^{\epsilon,\delta}_t$, we have proved that a.s.,

$$\text{for all }\delta>0,\quad\limsup_{\epsilon\to0}\sup_{[0,T]}|A^\epsilon_t-L^0_t|\le\limsup_{\epsilon\to0}\sup_{[0,T]}|R^{\epsilon,\delta}_t|.$$

But

$$\sup_{[0,T]}|R^{\epsilon,\delta}_t|\le r_{\epsilon,\delta}\times\sup_{[0,T]\times[-\delta,\delta]}|L^x_t-L^0_t|,$$

which implies that

$$\limsup_{\epsilon\to0}\sup_{[0,T]}|A^\epsilon_t-L^0_t|\le\sup_{[0,T]\times[-\delta,\delta]}|L^x_t-L^0_t|$$

a.s. Letting $\delta \to 0$, using [23, Corollary 1.8 page 226], completes the proof.

Point (d). Here $\beta = 5$. This is very similar to point (a) when $\beta = 1$ and we only sketch the proof. We set $\gamma = 6c_5$ and recall that $a_\epsilon = \epsilon/\gamma$. Since ψ is bounded on $\mathbb{R}$ and satisfies $\psi(z) \overset{|z|\to\infty}{\sim} |81z|^{-1}$,

$$\int_{-x}^{x} \psi(z)\mathrm{d}z \overset{x\to\infty}{\sim} \frac{2\log x}{81}. \tag{15}$$

Proceeding as previously, we can show rigorously that, for any $\delta > 0$, uniformly in $t \in [0, T]$,

$$T_t^\epsilon = \int_0^t \frac{\gamma^2\psi(\gamma W_s/\epsilon)\mathrm{d}s}{\epsilon|\log\epsilon|} \simeq \int_0^t \frac{\gamma^2\psi(\gamma W_s/\epsilon)\mathrm{d}s}{\epsilon|\log\epsilon|}\mathbf{1}_{\{|W_s|\le\delta\}} = \int_{-\delta}^{\delta} \frac{\gamma^2\psi(\gamma x/\epsilon)L_t^x\mathrm{d}x}{\epsilon|\log\epsilon|},$$

whence

$$T_t^\epsilon \simeq \Big(\frac{\gamma}{|\log\epsilon|}\int_{-\delta\gamma/\epsilon}^{\delta\gamma/\epsilon}\psi(x)\mathrm{d}x\Big)\Big(L_t^0 \pm \sup_{[0,T]\times[-\delta,\delta]}|L_t^x - L_t^0|\Big) \simeq \frac{2\gamma}{81}\Big(L_t^0 \pm \sup_{[0,T]\times[-\delta,\delta]}|L_t^x - L_t^0|\Big)$$

by (15). We conclude by letting δ tend to 0, since $2\gamma/81 = 4c_5/27 = \sigma_5^2$.

Point (b), general case. Here $\beta \in (1, 5)$ and $a_\epsilon = \epsilon/\gamma$ with $\gamma = (\beta+1)c_\beta$. Recall that $\alpha = (\beta+1)/3$, whence $1/\alpha - 2 = (1-2\beta)/(\beta+1)$. First, recalling that K_t^η was defined in Theorem 4 and using the occupation times formula,

$$K_t^\eta = \int_{\mathbb{R}} \mathrm{sgn}(x)|x|^{(1-2\beta)/(\beta+1)}\mathbf{1}_{\{|x|\ge\eta\}}L_t^x\mathrm{d}x = \int_{|x|\le S_T} \mathrm{sgn}(x)|x|^{(1-2\beta)/(\beta+1)}\mathbf{1}_{\{|x|\ge\eta\}}L_t^x\mathrm{d}x,$$

where $S_T = \sup_{t\in[0,T]}|W_t|$, since $L_t^x = 0$ for all $t \in [0, T]$, all $|x| > S_T$. By symmetry, we may write

$$K_t^\eta = \int_{|x|\le S_T} \mathrm{sgn}(x)|x|^{(1-2\beta)/(\beta+1)}\mathbf{1}_{\{|x|\ge\eta\}}(L_t^x - L_t^0)\mathrm{d}x.$$

But we know from [23, Corollary 1.8 page 226] that for all $\theta \in (0, 1/2)$, all $T > 0$,

$$M_{\theta,T} = \sup_{[0,T]\times\mathbb{R}} |x|^{-\theta}|L_t^x - L_t^0| < \infty \quad \text{a.s.}$$

Since $(1-2\beta)/(\beta+1) > -3/2$ (because $\beta < 5$), we deduce that $(K_t^\eta)_{t\ge0}$ a.s. converges uniformly on $[0, T]$, as $\eta \to 0$, to

$$K_t = \int_{|x|\le S_T} \mathrm{sgn}(x)|x|^{(1-2\beta)/(\beta+1)}(L_t^x - L_t^0)\mathrm{d}x.$$

By oddness of ϕ (and since $\epsilon^{1/\alpha}a_\epsilon^{-2} = \gamma^2\epsilon^{1/\alpha-2} = \gamma^2\epsilon^{(1-2\beta)/(\beta+1)}$), we can also write

$$\begin{aligned}
\epsilon^{1/\alpha}H_t^\epsilon &= \gamma^2\epsilon^{(1-2\beta)/(\beta+1)}\int_0^t \phi(\gamma W_s/\epsilon)\mathrm{d}s \\
&= \int_{\mathbb{R}} \gamma^2\epsilon^{(1-2\beta)/(\beta+1)}\phi(\gamma x/\epsilon)L_t^x\mathrm{d}x \\
&= \int_{|x|\le S_T} \gamma^2\epsilon^{(1-2\beta)/(\beta+1)}\phi(\gamma x/\epsilon)(L_t^x - L_t^0)\mathrm{d}x.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sup_{[0,T]}|\epsilon^{1/\alpha}H_t^\epsilon - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_t| \\
\le &\int_{|x|\le S_T}\Big|\gamma^2\epsilon^{(1-2\beta)/(\beta+1)}\phi(\gamma x/\epsilon) - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}\mathrm{sgn}(x)|x|^{(1-2\beta)/(\beta+1)}\Big|\sup_{[0,T]}|L_t^x - L_t^0|\mathrm{d}x \\
\le &M_{\theta,T}\int_{|x|\le S_T}\Big|\gamma^2\epsilon^{(1-2\beta)/(\beta+1)}\phi(\gamma x/\epsilon) - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}\mathrm{sgn}(x)|x|^{(1-2\beta)/(\beta+1)}\Big||x|^\theta\mathrm{d}x
\end{aligned}$$

for any $\theta \in (0,1/2)$. Using the equivalence $\phi(z) \overset{|z|\to\infty}{\sim} (\beta+1)^{-2}\mathrm{sgn}(z)|z|^{(1-2\beta)/(\beta+1)}$, we see that

$$\begin{aligned}\gamma^2\epsilon^{(1-2\beta)/(\beta+1)}\phi(\gamma x/\epsilon) \to&\gamma^{2+(1-2\beta)/(\beta+1)}(\beta+1)^{-2}\mathrm{sgn}(x)|x|^{(1-2\beta)/(\beta+1)}\\ =&(\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}\mathrm{sgn}(x)|x|^{(1-2\beta)/(\beta+1)}.\end{aligned}$$

Using furthermore the bound $|\phi(z)|\le C|z|^{(1-2\beta)/(\beta+1)}$ and that $(1-2\beta)/(\beta+1) > -3/2$, we may choose $\theta\in(0,1/2)$ such that $\theta+(1-2\beta)/(\beta+1) > -1$ and conclude by dominated convergence that

$$\lim_{\epsilon\to 0}\sup_{[0,T]}|\epsilon^{1/\alpha}H_t^\epsilon - (\beta+1)^{-2}K_t| = 0 \quad \text{a.s.}$$

Point (c). Here $\beta=1$, $\alpha=2/3$ and $a_\epsilon = \epsilon|\log\epsilon|/2$. No principal value is needed here and we have, with the notation of Theorem 4,

$$K_t = \lim_{\eta\to 0}\int_0^t \mathrm{sgn}(W_s)|W_s|^{-1/2}\mathbf{1}_{\{|W_s|\ge\eta\}}\mathrm{d}s = \int_0^t \mathrm{sgn}(W_s)|W_s|^{-1/2}\mathrm{d}s.$$

Also, we have

$$|\epsilon\log\epsilon|^{3/2}H_t^\epsilon = |\epsilon\log\epsilon|^{3/2}a_\epsilon^{-2}\int_0^t\phi(W_s/a_\epsilon)\mathrm{d}s = 4|\epsilon\log\epsilon|^{-1/2}\int_0^t\phi(2W_s/|\epsilon\log\epsilon|)\mathrm{d}s.$$

Using that $\phi(z)\overset{|z|\to\infty}{\sim}\mathrm{sgn}(z)|z|^{-1/2}/4$, that $|\phi(z)|\le C|z|^{-1/2}$ and that $\int_0^T|W_s|^{-1/2}\mathrm{d}s<\infty$ a.s., one verifies, by dominated convergence, that

$$\lim_{\epsilon\to 0}\sup_{[0,T]}\Big||\epsilon\log\epsilon|^{3/2}H_t^\epsilon - K_t/\sqrt{2}\Big| \le \lim_{\epsilon\to 0}\int_0^T\Big|4|\epsilon\log\epsilon|^{-1/2}\phi(2W_s/|\epsilon\log\epsilon|) - \mathrm{sgn}(W_s)|W_s|^{-1/2}/\sqrt{2}\Big|\mathrm{d}s = 0$$

a.s., as desired. □

We now give the

Proof of Theorem 2-(c)-(d) when $X_0=V_0=0$. Fix $\beta\in[1,5)$ and a Brownian motion $(W_t)_{t\ge0}$, denote by $(L_t^0)_{t\ge0}$ its local time and by $\tau_t=\inf\{u\ge0 : L_u^0>t\}$. Consider the process $(K_t)_{t\ge0}$ defined in Theorem 4 with $\alpha=(\beta+1)/3$.

For each $\epsilon>0$, consider the processes $(A_t^\epsilon)_{t\ge0}$, $(\tau_t^\epsilon)_{t\ge0}$, $(V_t^\epsilon)_{t\ge0}$ and $(H_t^\epsilon)_{t\ge0}$ built in Lemma 6 with the choice

$$a_\epsilon = \epsilon/[(\beta+1)c_\beta] \quad \text{if } \beta\in(1,5) \qquad \text{and} \quad a_\epsilon=\epsilon|\log\epsilon|/2 \quad \text{if } \beta=1.$$

Since $X_0=V_0=0$, we know that

$$(X_{t/\epsilon})_{t\ge0} \overset{d}{=} (H^\epsilon_{\tau_t^\epsilon})_{t\ge0}. \tag{16}$$

Point (c): $\beta\in(1,5)$. We want to prove that $(\epsilon^{1/\alpha}X_{t/\epsilon})_{t\ge0}\overset{f.d.}{\to}(\sigma_\beta S_t^{(\alpha)})_{t\ge0}$. By (16), it is sufficient to verify that for each $t\ge0$ fixed,

$$\Delta_t(\epsilon) = |\epsilon^{1/\alpha}H^\epsilon_{\tau_t^\epsilon} - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_{\tau_t}| \to 0 \quad \text{a.s.}$$

Indeed, Theorem 4 tells us that $S_t^{(\alpha)} := \sigma_\beta^{-1}(\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_{\tau_t}$ is a symmetric α-stable process with

$$\mathbb{E}[\exp(i\xi S_t^{(\alpha)})] = \exp(-\kappa_\alpha t|\sigma_\beta^{-1}(\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}\xi|^\alpha) = \exp(-t|\xi|^\alpha)$$

by definition of σ_β and κ_α, see Subsections 1.4 and 2.2.

By Lemma 9-(a), $\sup_{[0,T]} |A_t^\epsilon - L_t^0| \to 0$ a.s. Since $(\tau_t)_{t\geq 0}$, the generalized inverse of $(L_t^0)_{t\geq 0}$, has no deterministic time of jump, we deduce from Lemma 8-(a) that $\tau_t^\epsilon \to \tau_t$ a.s. (for each deterministic $t \geq 0$ fixed). And by Lemma 9-(b),

$$\lim_{\epsilon\to 0} \sup_{[0,T]} |\epsilon^{1/\alpha} H_t^\epsilon - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} K_t| = 0 \quad \text{a.s. for all } T > 0. \tag{17}$$

All in all,

$$\Delta_t(\epsilon) \leq |\epsilon^{1/\alpha} H^\epsilon_{\tau_t^\epsilon} - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} K_{\tau_t^\epsilon}| + (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} |K_{\tau_t^\epsilon} - K_{\tau_t}|,$$

which a.s. tends to 0: for the first term, we use that $T = \sup_{\epsilon\in(0,1)} \tau_t^\epsilon < \infty$ a.s. and (17). For the second one, we use that $(K_t)_{t\geq 0}$ is a.s. continuous and that $\tau_t^\epsilon \to \tau_t$ a.s.

Point (d): $\beta = 1$. We want to verify that $(|\epsilon \log \epsilon|^{3/2} X_{t/\epsilon})_{t\geq 0} \stackrel{f.d.}{\to} (\sigma_1 S_t^{(2/3)})_{t\geq 0}$. By (16), it is sufficient to verify that for each $t \geq 0$ fixed, a.s.,

$$\Delta'_t(\epsilon) = \Big| |\epsilon \log \epsilon|^{3/2} H^\epsilon_{\tau_t^\epsilon} - K_{\tau_t}/\sqrt{2} \Big| \to 0.$$

Indeed, Theorem 4 tells us that $(S_t^{(2/3)})_{t\geq 0} = (\sqrt{2}\sigma_1)^{-1} K_{\tau_t}$ is a symmetric α-stable process with

$$\mathbb{E}[\exp(i\xi S_t^{(2/3)})] = \exp(-\kappa_{2/3} t |(\sqrt{2}\sigma_1)^{-1}\xi|^{2/3}) = \exp(-t|\xi|^{2/3})$$

by definition of σ_1 and $\kappa_{2/3}$, see Subsections 1.4 and 2.2.

As in point (c), Lemmas 9-(a) and 8-(a) imply that $|\tau_t^\epsilon - \tau_t| \to 0$ a.s. for each $t \geq 0$, and

$$\lim_{\epsilon\to 0} \sup_{[0,T]} ||\epsilon \log \epsilon|^{3/2} H_t^\epsilon - K_t/\sqrt{2}| = 0 \quad \text{a.s. for all } T > 0. \tag{18}$$

by Lemma 9-(d). Thus

$$\Delta'_t(\epsilon) \leq \Big| |\epsilon \log \epsilon|^{3/2} H^\epsilon_{\tau_t^\epsilon} - K_{\tau_t^\epsilon}/\sqrt{2} \Big| + |K_{\tau_t^\epsilon} - K_{\tau_t}|/\sqrt{2} \to 0$$

since $T = \sup_{\epsilon\in(0,1)} \tau_t^\epsilon < \infty$ a.s. and by continuity of $(K_t)_{t\geq 0}$. □

3.5. **The diffusive case.** This case is standard, see Jacod-Shiryaev [12, Chapter VIII, Section 3f].

Proof of Theorem 2-(a) when $X_0 = V_0 = 0$. We assume that $\beta > 5$. Since $(V_t)_{t\geq 0}$ is a regular diffusion, see Remark 7, and since μ_β is a probability measure (because $\beta > 1$), we classically deduce, see e.g. Kallenberg [14, Lemma 23.17 page 466 and Thm 23.14 page 464], that

(i) V_t tends in law to μ_β as $t \to \infty$,

(ii) for all $\varphi \in L^1(\mathbb{R}, \mu_\beta)$, $\lim_{t\to\infty} t^{-1} \int_0^t \varphi(V_s)\mathrm{d}s = \int_{\mathbb{R}} \varphi \mathrm{d}\mu_\beta$ a.s.

The function

$$g(v) = 2\int_0^v \Theta^{-\beta}(x) \int_x^\infty u \Theta^\beta(u) \mathrm{d}u \mathrm{d}x$$

is odd (since Θ is even and $\int_{\mathbb{R}} u\Theta^\beta(u)\mathrm{d}u\mathrm{d}x = 0$) and solves the Poisson equation

$$g''(v) - \beta F(v) g'(v) = -2v,$$

whence, by the Itô formula,

$$g(V_t) = \int_0^t g'(V_s)\mathrm{d}B_s - \int_0^t V_s \mathrm{d}s, \quad \textit{i.e.} \quad X_t = \int_0^t g'(V_s)\mathrm{d}B_s - g(V_t).$$

Consequently, we have $\epsilon^{1/2} X_{t/\epsilon} = M_t^\epsilon - \epsilon^{1/2} g(V_{t/\epsilon})$, where $M_t^\epsilon = \epsilon^{1/2} \int_0^{t/\epsilon} g'(V_s)\mathrm{d}B_s$.

For each $t \geq 0$, $\epsilon^{1/2} g(V_{t/\epsilon})$ tends to 0 in probability: this follows from point (i) above. Here is why we deal with finite-dimensional distributions: it is not clear that $\sup_{t\in[0,1]} |\epsilon^{1/2} g(V_{t/\epsilon})|$ tends to 0.

We now show that $(M^\epsilon_t)_{t\geq 0}$ tends in law (in the usual sense of continuous processes) to $(\sigma_\beta W_t)_{t\geq 0}$, and this will complete the proof. It suffices, see e.g. Jacod-Shiryaev [12, Theorem VIII-3.11 page 473], to verify that for each $t \geq 0$, $\lim_{\epsilon\to 0}\langle M^\epsilon\rangle_t = \sigma_\beta^2 t$ in probability. But

$$\langle M^\epsilon\rangle_t = \epsilon\int_0^{t/\epsilon}[g'(V_s)]^2\mathrm{d}s,$$

which a.s. tends to $\sigma_\beta^2 t$ by point (ii). Indeed, using a symmetry argument,

$$\int_{\mathbb{R}}[g'(v)]^2\mu_\beta(\mathrm{d}v) = 8\int_0^\infty\Big[\Theta^{-\beta}(v)\int_v^\infty u\Theta^\beta(u)\mathrm{d}u\Big]^2\mu_\beta(\mathrm{d}v) = 8c_\beta\int_0^\infty\Theta^{-\beta}(v)\Big[\int_v^\infty u\Theta^\beta(u)\mathrm{d}u\Big]^2\mathrm{d}v = \sigma_\beta^2,$$

recall Subsection 1.4. This value is finite, since $\Theta^{-\beta}(v)[\int_v^\infty u\Theta^\beta(u)\mathrm{d}u]^2 \sim (\beta-2)^{-2}v^{4-\beta}$ as $v\to\infty$ by (6) and since $\beta > 5$. ☐

Remark 10. *When $\beta = 5$, our goal is to prove that $(\epsilon^{1/2}|\log\epsilon|^{-1/2}X_{t/\epsilon})_{t\geq 0} \overset{f.d.}{\longrightarrow} (\sigma_5 W_t)_{t\geq 0}$. We can use exactly the same proof, provided we can show that for each $t\geq 0$, in probability, as $\epsilon\to 0$,*

$$\frac{\epsilon}{|\log\epsilon|}\int_0^{t/\epsilon}[g'(V_s)]^2\mathrm{d}s \longrightarrow \sigma_5^2 t. \tag{19}$$

Proof of Theorem 2-(b) when $X_0 = V_0 = 0$. Here $\beta = 5$ and our goal is to verify (19) or equivalently, that for each $t\geq 0$,

$$J_t^\epsilon = |\log\epsilon|^{-1}\int_0^t[g'(V_s^\epsilon)]^2\mathrm{d}s \longrightarrow \sigma_5^2 t,$$

where $V_t^\epsilon = h^{-1}(W_{\tau_t^\epsilon}/a_\epsilon)$ and where $(\tau_t^\epsilon)_{t\geq 0}$, the inverse of $A_t^\epsilon = \epsilon a_\epsilon^{-2}\int_0^t[\sigma(W_s/a_\epsilon)]^{-2}\mathrm{d}s$, were introduced in Lemma 6. We choose $a_\epsilon = \epsilon/[6c_5]$, as prescribed by Lemma 9.

As usual,

$$J_t^\epsilon = \int_0^t\frac{[g'(h^{-1}(W_{\tau_s^\epsilon}/a_\epsilon))]^2}{|\log\epsilon|}\mathrm{d}s = \int_0^{\tau_t^\epsilon}\frac{\epsilon[g'(h^{-1}(W_u/a_\epsilon))]^2}{a_\epsilon^2|\log\epsilon|[\sigma(W_u/a_\epsilon)]^2}\mathrm{d}u = \frac{\epsilon}{a_\epsilon^2|\log\epsilon|}\int_0^{\tau_t^\epsilon}\psi(W_u/a_\epsilon)\mathrm{d}u = T^\epsilon_{\tau_t^\epsilon}$$

with the notation of Lemma 9.

By Lemma 9-(a), $\sup_{[0,T]}|A_t^\epsilon - L_t^0|\to 0$ a.s. Since $(\tau_t)_{t\geq 0}$, the generalized inverse of $(L_t^0)_{t\geq 0}$, has no deterministic time of jump, we deduce from Lemma 8-(a) that $\tau_t^\epsilon\to\tau_t$ a.s. (for each deterministic $t\geq 0$ fixed). And by Lemma 9-(d),

$$\sup_{[0,T]}|T_t^\epsilon - \sigma_5^2 L_t^0|\to 0 \quad \text{a.s. for all } T>0. \tag{20}$$

All in all, for $t\geq 0$ fixed,

$$|J_t^\epsilon - \sigma_5^2 t| \leq |T^\epsilon_{\tau_t^\epsilon} - \sigma_5^2 L^0_{\tau_t^\epsilon}| + |\sigma_5^2 L^0_{\tau_t^\epsilon} - \sigma_5^2 L^0_{\tau_t}| + |\sigma_5^2 L^0_{\tau_t} - \sigma_5^2 t|$$

which a.s. tends to 0: for the first term, we use that $T = \sup_{\epsilon\in(0,1)}\tau_t^\epsilon$ is a.s. finite and (20). For the second one, we use that $(L_t^0)_{t\geq 0}$ is a.s. continuous and that $\tau_t^\epsilon\to\tau_t$ a.s.. For the last one, we use that (for $t\geq 0$ fixed) $L^0_{\tau_t} = t$ a.s. ☐

3.6. **Conclusion.** We have proved Theorem 2 when $X_0 = V_0 = 0$. It remains to check the following.

Lemma 11. *If Theorem 2 holds when $X_0 = V_0 = 0$ a.s., then it holds for any initial condition.*

Proof. We divide the proof into three steps.

Step 1. We first verify that for any $\beta > 0$ (we will use only the case $\beta\geq 1$), there is a constant $C > 0$ such that, for $(V_t)_{t\geq 0}$ the solution to (9) with $V_0 = 0$, for all $t\geq 0$,

$$\mathbb{E}[V_t^2 + |V_t|^{\beta+1}] \leq C(1+t).$$

To this aim, we introduce the even function

$$\ell(v) = 2\int_0^v \Theta^{-\beta}(x)\int_0^x \Theta^\beta(u)\mathrm{d}u\mathrm{d}x,$$

which solves the Poisson equation $\ell''(v) - \beta F(v)\ell'(v) = 2$. Hence

$$\mathbb{E}[\ell(V_t)] = \ell(V_0) + \int_0^t \mathbb{E}\Big[\frac{1}{2}\ell''(V_s) - \frac{1}{2}\beta F(V_s)\ell'(V_s)\Big]\mathrm{d}s = t$$

by the Itô formula and since $\ell(V_0) = \ell(0) = 0$. Using (6), we see that there is a constant $c > 0$ such that, as $|v| \to \infty$,

$$\ell(v) \sim c|v|^{\beta+1} \quad \text{if} \quad \beta > 1, \qquad \ell(v) \sim c|v|^2 \log|v| \quad \text{if} \quad \beta = 1 \qquad \text{and} \quad \ell(v) \sim cv^2 \quad \text{if} \quad \beta \in (0,1).$$

Thus in any case, we can find a constant C such that $v^2 + |v|^{\beta+1} \le C(\ell(v)+1)$ for all $v \in \mathbb{R}$, whence

$$\mathbb{E}[V_t^2 + |V_t|^{\beta+1}] \le C(1 + \mathbb{E}[\ell(V_t)]) = C(1+t).$$

Step 2. We prove the lemma when $\beta \ge 1$, i.e. in the cases (a)-(b)-(c)-(d) of Theorem 1. Assume that for some fixed $\beta \ge 1$, Theorem 2 holds when starting from $(0,0)$ and consider the solution $(X_t, V_t)_{t\ge 0}$ to (9) starting from some (X_0, V_0). We introduce

$$\tau = \inf\{t \ge 0 : V_t = 0\},$$

which is a.s. finite by recurrence, see Remark 7. Then

$$(\hat{X}_t, \hat{V}_t) = (X_{\tau+t} - X_\tau, V_{\tau+t})$$

solves (9), starts from $(0,0)$, and is independent of τ by the strong Markov property. We thus know that $(v_\epsilon^{(\beta)}\hat{X}_{t/\epsilon})_{t\ge0} \stackrel{f.d.}{\to} (X_t^{(\beta)})_{t\ge0}$, where $v_\epsilon^{(\beta)} \to 0$ and $(X_t^{(\beta)})_{t\ge0}$ are the rate and limit process appearing in Theorem 2. We now prove that for each $t \ge 0$, $v_\epsilon^{(\beta)}|X_{t/\epsilon} - \hat{X}_{t/\epsilon}| \to 0$ in probability, and this will complete the proof.

We introduce $D^1 = |X_0| + \int_0^{2\tau}|V_s|\mathrm{d}s$ and $D_t^{2,\epsilon} = \mathbf{1}_{\{t/\epsilon\ge\tau\}}\int_{t/\epsilon-\tau}^{t/\epsilon}|\hat{V}_s|\mathrm{d}s$ and observe that

$$|X_{t/\epsilon} - \hat{X}_{t/\epsilon}| \le D^1 + D_t^{2,\epsilon},$$

because if $t/\epsilon \le \tau$, then

$$|X_{t/\epsilon} - \hat{X}_{t/\epsilon}| \le |X_{t/\epsilon}| + |X_{\tau+t/\epsilon} - X_\tau| \le |X_0| + \int_0^{t/\epsilon}|V_s|\mathrm{d}s + \int_\tau^{\tau+t/\epsilon}|V_s|\mathrm{d}s \le D^1,$$

while if $t/\epsilon > \tau$, then

$$|X_{t/\epsilon} - \hat{X}_{t/\epsilon}| = |X_\tau + \hat{X}_{t/\epsilon-\tau} - \hat{X}_{t/\epsilon}| \le |X_0| + \int_0^\tau |V_s|\mathrm{d}s + \int_{t/\epsilon-\tau}^{t/\epsilon}|\hat{V}_s|\mathrm{d}s \le D^1 + D_t^{2,\epsilon}.$$

Of course, $v_\epsilon^{(\beta)}D^1$ a.s. tends to 0, because D^1 is finite. And using Step 1,

$$\mathbb{E}[v_\epsilon^{(\beta)}D_t^{2,\epsilon}|\mathcal{F}_\tau] \le \mathbf{1}_{\{t/\epsilon\ge\tau\}}Cv_\epsilon^{(\beta)}\int_{t/\epsilon-\tau}^{t/\epsilon}(1+s)^{1/(\beta+1)}\mathrm{d}s \le C\tau v_\epsilon^{(\beta)}(1+t/\epsilon)^{1/(\beta+1)},$$

which a.s. tends to 0 for all values of $\beta \ge 1$. Hence $v_\epsilon^{(\beta)}D_t^{2,\epsilon}$ tends to 0 in probability.

Step 3. We finally prove the lemma when $\beta \in (0,1)$. First,

$$(\epsilon^{1/2}V_{t/\epsilon})_{t\ge0} \stackrel{d}{\longrightarrow} (U_t^{(1-\beta)})_{t\ge0} \quad \Longrightarrow \quad (\epsilon^{3/2}X_{t/\epsilon}, \epsilon^{1/2}V_{t/\epsilon})_{t\ge0} \stackrel{d}{\longrightarrow} \Big(\int_0^t U_s^{(1-\beta)}\mathrm{d}s, U_t^{(1-\beta)}\Big)_{t\ge0},$$

simply because $\epsilon^{3/2}X_{t/\epsilon} = \epsilon^{3/2}X_0 + \int_0^t(\epsilon^{1/2}V_{s/\epsilon})\mathrm{d}s$ and because we deal with uniform convergence on compact time intervals.

We assume that $(\epsilon^{1/2}V_{t/\epsilon})_{t\geq 0} \xrightarrow{d} (U_t^{(1-\beta)})_{t\geq 0}$ holds true when $V_0=0$, consider any other solution $(V_t)_{\geq 0}$, introduce $\tau=\inf\{t\geq 0 : V_t=0\}$ and $\hat V_t = V_{\tau+t}$ as previously. We will check that, in probability, for all $T>0$,

$$\Delta_T^\epsilon = \epsilon^{1/2}\sup_{[0,T]}|V_{t/\epsilon}-\hat V_{t/\epsilon}| \to 0$$

and this will complete the proof, since we know that $(\epsilon^{1/2}\hat V_{t/\epsilon})_{t\geq 0} \xrightarrow{d} (U_t^{(1-\beta)})_{t\geq 0}$.

It holds that $\Delta_T^\epsilon \leq \Delta^{1,\epsilon}+\Delta_T^{2,\epsilon}$, where

$$\Delta^{1,\epsilon}=2\epsilon^{1/2}\sup_{[0,2\tau]}|V_s| \quad\text{and}\quad \Delta_T^{2,\epsilon}=\sup_{[0,T]}\epsilon^{1/2}|\hat V_{(t+\epsilon\tau)/\epsilon}-\hat V_{t/\epsilon}|,$$

because if $t\in[0,T]$ and $t/\epsilon\leq\tau$, then

$$\epsilon^{1/2}|V_{t/\epsilon}-\hat V_{t/\epsilon}| \leq \epsilon^{1/2}|V_{t/\epsilon}|+\epsilon^{1/2}|V_{\tau+t/\epsilon}| \leq \Delta^{1,\epsilon},$$

while if $t\in[0,T]$ and $t/\epsilon>\tau$, then

$$\epsilon^{1/2}|V_{t/\epsilon}-\hat V_{t/\epsilon}| = \epsilon^{1/2}|\hat V_{t/\epsilon-\tau}-\hat V_{t/\epsilon}| \leq \Delta_T^{2,\epsilon}.$$

First, $\Delta^{1,\epsilon}$ a.s. tends to 0. Next, it is not hard to check that $\Delta_T^{2,\epsilon}$ tends in probability to 0, using that $(\epsilon^{1/2}\hat V_{t/\epsilon})_{t\geq 0}$ tends in law, in $C([0,\infty),\mathbb{R})$, to the continuous process $(U_t^{(1-\beta)})_{t\geq 0}$. □

3.7. **Decoupling.** We now give the

Proof of Theorem 3. Let $\beta>1$ be fixed, as well as the solution $(X_t,V_t)_{t\geq 0}$ to (9), starting from some given initial condition (X_0,V_0) and driven by some Brownian motion $(B_t)_{t\geq 0}$. We introduce $\mathcal{F}_t=\sigma(X_0,V_0,B_s,s\leq t)$. We know that

$$(v_\epsilon^{(\beta)}X_{t/\epsilon})_{t\geq 0} \overset{f.d.}{\to} (X_t^{(\beta)})_{t\geq 0},$$

where $v_\epsilon^{(\beta)}\to 0$ and $(X_t^{(\beta)})_{t\geq 0}$ are the rate and limit process appearing in Theorem 2 (case (a), (b) or (c)). We fix $t>0$, $\varphi\in C_b^1(\mathbb{R})$ and $\psi\in B_b(\mathbb{R})$, and our goal is to verify that, setting $\mu_\beta(\psi)=\int_\mathbb{R}\psi d\mu_\beta$,

$$\Delta_\epsilon = \Big|\mathbb{E}[\varphi(v_\epsilon^{(\beta)}X_{t/\epsilon})\psi(V_{t/\epsilon})]-\mathbb{E}[\varphi(X_t^{(\beta)})]\mu_\beta(\psi)\Big| \to 0 \quad\text{as } \epsilon\to 0.$$

Step 1. We check here that for any fixed $h\in(0,t)$,

$$\delta_{\epsilon,h}=\mathbb{E}\Big[\Big|\mathbb{E}[\psi(V_{t/\epsilon})|\mathcal{F}_{(t-h)/\epsilon}]-\mu_\beta(\psi)\Big|\Big] \to 0$$

as $\epsilon\to 0$. We use the common notation notation $P_t\psi(v)=\mathbb{E}_v[\psi(V_t)]$ (although we still use $\mathbb{E}$ and $\mathbb{P}$ without subscript when working with the initial condition V_0). We introduce the total variation norm $||\cdot||_{TV}$. By the Markov property,

$$\delta_{\epsilon,h}=\mathbb{E}[|P_{h/\epsilon}\psi(V_{(t-h)/\epsilon})-\mu_\beta(\psi)|] \leq \delta_{\epsilon,h}^1+\delta_{\epsilon,h}^2,$$

where, introducing some μ_β-distributed $\bar V_{\epsilon,h}$ such that $\mathbb{P}(\bar V_{\epsilon,h}\neq V_{(t-h)/\epsilon})=||\text{Law}(V_{(t-h)/\epsilon})-\mu_\beta||_{TV}$,

$$\begin{aligned}\delta_{\epsilon,h}^1 =&\mathbb{E}[|P_{h/\epsilon}\psi(V_{(t-h)/\epsilon})-P_{h/\epsilon}\psi(\bar V_{\epsilon,h})|]\\ \leq&2||P_{h/\epsilon}\psi||_\infty\mathbb{P}(\bar V_{\epsilon,h}\neq V_{(t-h)/\epsilon})\\ \leq&2||\psi||_\infty||\text{Law}(V_{(t-h)/\epsilon})-\mu_\beta||_{TV},\end{aligned}$$

and

$$\delta_{\epsilon,h}^2=\mathbb{E}[|P_{h/\epsilon}\psi(\bar V_{\epsilon,h})-\mu_\beta(\psi)|]=\int_\mathbb{R}|P_{h/\epsilon}\psi(v)-\mu_\beta(\psi)|\mu_\beta(dv).$$

But, by Remark (7) and [14, Lemma 23.17 page 466], $||\text{Law}(V_s)-\mu_\beta||_{TV}\to 0$ as $s\to\infty$. Hence $\delta_{\epsilon,h}^1$ tends to 0 as $\epsilon\to 0$. We also have $\lim_{s\to\infty}P_s\psi(v)=\mu_\beta(\psi)$ for all $v\in\mathbb{R}$, so that $\delta_{\epsilon,h}^2$ tends to 0 by dominated convergence.

Step 2. For any $h \in (0,t)$, we write $\Delta_\epsilon \le \Delta^1_{\epsilon,h} + \Delta^2_{\epsilon,h} + \Delta^3_{\epsilon,h} + \Delta^4_{\epsilon,h}$, where

$$\begin{aligned}\Delta^1_{\epsilon,h} &= \Big|\mathbb{E}[\varphi(v^{(\beta)}_\epsilon X_{t/\epsilon})\psi(V_{t/\epsilon})] - \mathbb{E}[\varphi(v^{(\beta)}_\epsilon X_{(t-h)/\epsilon})\psi(V_{t/\epsilon})]\Big|,\\ \Delta^2_{\epsilon,h} &= \Big|\mathbb{E}[\varphi(v^{(\beta)}_\epsilon X_{(t-h)/\epsilon})\psi(V_{t/\epsilon})] - \mathbb{E}[\varphi(v^{(\beta)}_\epsilon X_{(t-h)/\epsilon})]\mu_\beta(\psi)\Big|,\\ \Delta^3_{\epsilon,h} &= \Big|\mathbb{E}[\varphi(v^{(\beta)}_\epsilon X_{(t-h)/\epsilon})]\mu_\beta(\psi) - \mathbb{E}[\varphi(X^{(\beta)}_{t-h})]\mu_\beta(\psi)\Big|,\\ \Delta^4_{\epsilon,h} &= \Big|\mathbb{E}[\varphi(X^{(\beta)}_{t-h})]\mu_\beta(\psi) - \mathbb{E}[\varphi(X^{(\beta)}_t)]\mu_\beta(\psi)\Big|.\end{aligned}$$

By Theorem 2, $\lim_{\epsilon\to 0}\Delta^3_{\epsilon,h} = 0$.

By Theorem 2 again, with $C = ||\psi||_\infty(2||\varphi||_\infty + ||\varphi'||_\infty)$

$$\limsup_{\epsilon\to 0}\Delta^1_{\epsilon,h} \le C\limsup_{\epsilon\to 0}\mathbb{E}[|v^{(\beta)}_\epsilon X_{t/\epsilon} - v^{(\beta)}_\epsilon X_{(t-h)/\epsilon}|\wedge 1] = C\mathbb{E}[|X^{(\beta)}_t - X^{(\beta)}_{t-h}|\wedge 1].$$

We also have $\Delta^4_{\epsilon,h} \le C\mathbb{E}[|X^{(\beta)}_t - X^{(\beta)}_{t-h}|\wedge 1]$.

By Step 1, $\Delta^2_{\epsilon,h} \le ||\varphi||_\infty\mathbb{E}[|\mathbb{E}[\psi(V_{t/\epsilon})|\mathcal{F}_{(t-h)/\epsilon}] - \mu_\beta(\psi)|] \to 0$ as $\epsilon\to 0$.

All in all,

$$\limsup_{\epsilon\to 0}\Delta_\epsilon \le 2C\mathbb{E}[|X^{(\beta)}_t - X^{(\beta)}_{t-h}|\wedge 1]$$

for any $h \in (0,t)$. Letting $h \downarrow 0$ ends the proof, by dominated convergence and since the process $(X^{(\beta)}_t)_{t\ge 0}$ is a.s. continuous at $t \ge 0$: a Lévy process is generally discontinuous but never has any deterministic jump time. □

3.8. **The kinetic Fokker-Planck equation.** We end the paper with the proof of the P.D.E. statement, which follows from Theorems 2 and 3.

Proof of Theorem 1. Fix $\beta > 0$ and $f_0 \in \mathcal{P}(\mathbb{R}\times\mathbb{R})$, and consider the solution $(X_t, V_t)_{t\ge 0}$ to (9) starting from $(X_0, V_0) \sim f_0$. Denote, for each $t \ge 0$, by f_t the law of (X_t, V_t). By a.s. continuity of $t \mapsto (X_t, V_t)$, it is clear that $(f_t)_{t\ge 0} \in C([0,\infty), \mathcal{P}(\mathbb{R}\times\mathbb{R}))$. A simple application of the Itô formula shows that for all $\varphi \in C^2_c(\mathbb{R}\times\mathbb{R})$,

$$\mathbb{E}[\varphi(X_t, V_t)] = \mathbb{E}[\varphi(X_0, V_0)] + \int_0^t \mathbb{E}\Big[V_s\partial_x\varphi(X_s, V_s) + \frac{1}{2}\partial_{vv}\varphi(X_s, V_s) - \frac{\beta}{2}F(V_s)\partial_v\varphi(X_s, V_s)\Big]\mathrm{d}s,$$

which may be written as

$$\begin{aligned}(21)\qquad \int_{\mathbb{R}\times\mathbb{R}}\varphi(x,v)f_t(\mathrm{d}x,\mathrm{d}v) &= \int_{\mathbb{R}\times\mathbb{R}}\varphi(x,v)f_0(\mathrm{d}x,\mathrm{d}v)\\ &+ \int_0^t\int_{\mathbb{R}\times\mathbb{R}}\Big[v\partial_x\varphi(x,v) + \frac{1}{2}\partial_{vv}\varphi(x,v) - \frac{\beta}{2}F(v)\partial_v\varphi(x,v)\Big]f_s(\mathrm{d}x,\mathrm{d}v)\mathrm{d}s.\end{aligned}$$

In other words, $(f_t)_{t\ge 0}$ solves (7) in the sense of distributions.

Points (a), (b) and (c) then immediately follow from Theorem 3. For example concerning (c), where $\beta \in (1,5)$, we know that for each $t > 0$, $(\epsilon^{1/\alpha}X_{t/\epsilon}, V_{t/\epsilon})$ converges in law, as $\epsilon \to 0$, to $(\sigma_\beta S^{(\alpha)}_t, \bar{V})$, where $\alpha = (\beta+1)/3$, where $\mathbb{E}[\exp(i\xi S^{(\alpha)}_t)] = \exp(-t|\xi|^\alpha)$ and where $\bar{V}$ is μ_β-distributed and independent of $S^{(\alpha)}_t$. In other words, the law of $(\sigma_\beta S^{(\alpha)}_t, \bar{V})$ is $g_t \otimes \mu_\beta$, where g_t is characterized by its Fourier transform $\int_{\mathbb{R}} g_t(x)e^{i\xi x}\mathrm{d}x = \exp(-t|\sigma_\beta\xi|^\alpha)$. Since finally the law of $(\epsilon^{1/\alpha}X_{t/\epsilon}, V_{t/\epsilon})$ is $\epsilon^{-1/\alpha}f_{\epsilon^{-1}t}(\epsilon^{-1/\alpha}x, v)$, with the abuse of notation introduced before the statement of Theorem 1, we conclude that indeed, for each $t > 0$,

$$\lim_{\epsilon\to 0}\epsilon^{-1/\alpha}f_{\epsilon^{-1}t}(\epsilon^{-1/\alpha}x, v) = g_t \otimes \mu_\beta \quad \text{in } \mathcal{P}(\mathbb{R}\times\mathbb{R}).$$

Similarly, point (d), where $\beta = 1$, follows from Theorem 2-(d).

We finally check (e), where $\beta \in (0,1)$. By Theorem 2-(e), we know that for each $t > 0$ fixed, $(\epsilon^{3/2} X_{t/\epsilon}, \epsilon^{1/2} V_{t/\epsilon})$ converges in law, as $\epsilon \to 0$, to $(\int_0^t U_s^{(1-\beta)} \mathrm{d}s, U_t^{(1-\beta)})$. Denoting by $h_t \in \mathcal{P}(\mathbb{R}\times\mathbb{R})$ the law of this last random variable, we conclude that for each $t > 0$,

$$\lim_{\epsilon\to 0} \epsilon^{-2} f_{\epsilon^{-1}t}(\epsilon^{-3/2}x, \epsilon^{-1/2}v) = h_t \quad \text{in } \mathcal{P}(\mathbb{R}\times\mathbb{R}).$$

We of course have $h_t(-x,-v) = h_t(x,v)$ for all $t \geq 0$ and

$$h_t(\mathbb{R}\times\{0\}) = \mathbb{P}(U_t^{(1-\beta)} = 0) = 0 \quad \text{for all } t > 0$$

by Definition 5 of $U_t^{(1-\beta)}$. It remains to check (8), i.e. that

$$\int_{\mathbb{R}\times\mathbb{R}} \varphi(x,v) h_t(\mathrm{d}x,\mathrm{d}v) = \varphi(0,0) + \int_0^t \int_{\mathbb{R}\times\mathbb{R}} \Big[v\partial_x\varphi(x,v) + \frac{1}{2}\partial_{vv}\varphi(x,v) - \frac{\beta}{2v}\partial_v\varphi(x,v)\Big] h_s(\mathrm{d}x,\mathrm{d}v)\mathrm{d}s \tag{22}$$

for all $t \geq 0$ and all $\varphi \in C_c^2(\mathbb{R}\times\mathbb{R}_*)$. To this end, we use that h_t is the limit, in $\mathcal{P}(\mathbb{R}\times\mathbb{R})$, of $f_t^\epsilon(x,v) = \epsilon^{-2} f_{\epsilon^{-1}t}(\epsilon^{-3/2}x, \epsilon^{-1/2}v)$, where $(f_t)_{t\geq 0}$ is the weak solution to (7) with $f_0 = \delta_{(0,0)}$ and $F(v) = v/(1+v^2)$. Starting from (21) and using a change of variables, we find that

$$\begin{aligned} \int_{\mathbb{R}\times\mathbb{R}} \varphi(x,v) f_t^\epsilon(\mathrm{d}x,\mathrm{d}v) = {}& \varphi(0,0) \\ & + \int_0^t \int_{\mathbb{R}\times\mathbb{R}} \Big[v\partial_x\varphi(x,v) + \frac{1}{2}\partial_{vv}\varphi(x,v) - \frac{\beta}{2} F_\epsilon(v)\partial_v\varphi(x,v)\Big] f_s^\epsilon(\mathrm{d}x,\mathrm{d}v)\mathrm{d}s \end{aligned} \tag{23}$$

for all $t \geq 0$, all $\varphi \in C_c^2(\mathbb{R}\times\mathbb{R})$, where

$$F_\epsilon(v) = \epsilon^{-1/2} F(\epsilon^{-1/2}v) = \frac{v}{\epsilon + v^2}.$$

Letting $\epsilon \to 0$ in (23), we conclude that (22) holds true if $\varphi \in C_c^2(\mathbb{R}\times\mathbb{R}_*)$. There is no issue to check that

$$\lim_{\epsilon\to 0} \int_0^t \int_{\mathbb{R}\times\mathbb{R}} F_\epsilon(v)\partial_v\varphi(x,v) f_s^\epsilon(\mathrm{d}x,\mathrm{d}v)\mathrm{d}s = \int_0^t \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{v}\partial_v\varphi(x,v) h_s(\mathrm{d}x,\mathrm{d}v)\mathrm{d}s,$$

because since φ is supported in a compact subset of $\mathbb{R}\times\mathbb{R}_*$, we have

$$\lim_{\epsilon\to 0} \sup_{(x,v)\in\mathbb{R}\times\mathbb{R}} |v^{-1} - F_\epsilon(v)||\partial_v\varphi(x,v)| = 0,$$

and the function $(x,v) \mapsto v^{-1}\partial_v\varphi(x,v)$ is continuous and bounded on $\mathbb{R}\times\mathbb{R}$. □

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