A RECURSIVE ALGORITHM AND A SERIES EXPANSION RELATED TO THE HOMOGENEOUS BOLTZMANN EQUATION FOR HARD POTENTIALS WITH ANGULAR CUTOFF

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Abstract. We consider the spatially homogeneous Boltzmann equation for hard potentials with angular cutoff. This equation has a unique conservative weak solution $(f_t)_{t\geq 0}$, once the initial condition f_0 with finite mass and energy is fixed. Taking advantage of the energy conservation, we propose a recursive algorithm that produces a $(0,\infty)\times\mathbb{R}^3$ random variable (M_t, V_t) such that $\mathbb{E}[M_t \mathbf{1}_{\{V_t\in\cdot\}}] = f_t$. We also write down a series expansion of f_t. Although both the algorithm and the series expansion might be theoretically interesting in that they explicitly express f_t in terms of f_0, we believe that the algorithm is not very efficient in practice and that the series expansion is rather intractable. This is a tedious extension to non-Maxwellian molecules of Wild's sum [18] and of its interpretation by McKean [10, 11].

1. Introduction

We consider a spatially homogeneous gas modeled by the Boltzmann equation: the density $f_t(v) \geq 0$ of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$ solves

$$\partial_t f_t(v) = \int_{\mathbb{R}^3} \mathrm{d}v^* \int_{\mathbb{S}^2} \mathrm{d}\sigma B(v-v^*,\sigma)\big[f_t(v')f_t(v'^*) - f_t(v)f_t(v^*)\big], \tag{1}$$

where

$$v' = v'(v,v^*,\sigma) = \frac{v+v^*}{2} + \frac{|v-v^*|}{2}\sigma \quad \text{and} \quad v'^* = v'^*(v,v^*,\sigma) = \frac{v+v^*}{2} - \frac{|v-v^*|}{2}\sigma. \tag{2}$$

The cross section B is a nonnegative function given by physics. We refer to Cercignani [4] and Villani [16] for very complete books on the subject. We are concerned here with hard potentials with angular cutoff: the cross section satisfies

$$\begin{cases} B(v-v^*,\sigma) = |v-v^*|^\gamma b(\langle \frac{v-v^*}{|v-v^*|}, \sigma\rangle) \quad \text{for some } \gamma \in [0,1] \\ \\ \text{and some bounded measurable } b : [-1,1] \mapsto [0,\infty). \end{cases} \tag{3}$$

The important case where $\gamma = 1$ and b is constant corresponds to a gas of hard spheres. If $\gamma = 0$, the cross section is velocity independent and one talks about Maxwellian molecules with cutoff.

We classically assume without loss of generality that the initial mass $\int_{\mathbb{R}^3} f_0(v)\mathrm{d}v = 1$ and we denote by $e_0 = \int_{\mathbb{R}^3} |v|^2 f_0(v)\mathrm{d}v > 0$ the initial kinetic energy. It is then well-known, see Mischler-Wennberg [12], that (1) has a unique weak solution such that for all $t \geq 0$, f_t is a probability density on $\mathbb{R}^3$ with energy $\int_{\mathbb{R}^3} |v|^2 f_t(v)\mathrm{d}v = e_0$. Some precise statements are recalled in the next section.

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In the whole paper, we denote, for E a topological space, by $\mathcal{P}(E)$ (resp. $\mathcal{M}(E)$) the set of probability measures (resp. nonnegative measures) on E endowed with its Borel σ-field $\mathcal{B}(E)$. For $v, v^* \in \mathbb{R}^3$ and $\sigma \in \mathbb{S}^2$, we put

$$\beta_{v,v^*}(\sigma) = b(\langle \tfrac{v-v^*}{|v-v^*|}, \sigma \rangle) \quad \text{and we observe that} \quad \kappa = \int_{\mathbb{S}^2} \beta_{v,v^*}(\sigma) \mathrm{d}\sigma \tag{4}$$

does not depend on v, v^* and is given by $\kappa = 2\pi \int_0^\pi b(\theta) \sin\theta \mathrm{d}\theta$. If $v = v^*$, then we have $v' = v'^* = v$, see (2), so that the definition of $\beta_{v,v}(\sigma)$ is not important, we can e.g. set $\beta_{v,v}(\sigma) = |\mathbb{S}^2|^{-1}\kappa$.

In the rest of this introduction, we informally recall how (1) can be solved, in the case of Maxwellian molecules, by using the Wild sum, we quickly explain its interpretation by McKean, and we write down a closely related recursive simulation algorithm. We also recall that Wild's sum can be used for theoretical and numerical analysis of Maxwellian molecules. Then we briefly recall how the Wild sum and the algorithm can be easily extended to the case of any *bounded* cross section, by introducing fictitious jumps. Finally, we quickly explain our strategy to deal with hard potentials with angular cutoff.

1.1. **Wild's sum.** Let us first mention that some introductions to Wild's sum and its probabilistic interpretation by McKean can be found in the book of Villani [16, Section 4.1] and in Carlen-Carvalho-Gabetta [1, 2]. Wild [18] noted that for Maxwellian molecules, i.e. when $\gamma = 0$, so that the cross section $B(v - v^*, \sigma) = \beta_{v,v^*}(\sigma)$ does not depend on the relative velocity, (1) rewrites

$$\partial_t f_t(v) = \kappa Q(f_t, f_t) - \kappa f_t(v)$$

where, for f, g two probability densities on $\mathbb{R}^3$, $Q(f,g)(v) = \kappa^{-1} \int_{\mathbb{S}^2} f(v')g(v'^*)\beta_{v,v^*}(\sigma)\mathrm{d}\sigma$.

It holds that $Q(f,g)$ is also a probability density on $\mathbb{R}^3$, that can be interpreted as the law of $V' = (V + V^* + |V - V^*|\sigma)/2$, where V and V^* are two independent $\mathbb{R}^3$-valued random variables with densities f and g and where σ is, conditionally on (V, V^*), a $\kappa^{-1}\beta_{V,V^*}(\sigma)d\sigma$-distributed $\mathbb{S}^2$-valued random variable. Wild [18] proved that given f_0, the solution f_t to (1) is given by

$$f_t = e^{-\kappa t} \sum_{n \geq 1} (1 - e^{-\kappa t})^{n-1} Q_n(f_0), \tag{5}$$

where $Q_n(f_0)$ is defined recursively by $Q_1(f_0) = f_0$ and, for $n \geq 1$, by

$$Q_{n+1}(f_0) = \frac{1}{n} \sum_{k=1}^n Q(Q_k(f_0), Q_{n-k}(f_0)).$$

McKean [10, 11] provided an interpretation of the Wild sum in terms of binary trees, see also Villani [16] and Carlen-Carvalho-Gabetta [1]. Let $\mathcal{T}$ be the set of all discrete finite rooted ordered binary trees. By *ordered*, we mean that each node of $\Upsilon \in \mathcal{T}$ with two children has a *left* child and a *right* child. We denote by $\ell(\Upsilon)$ the number of leaves of $\Upsilon \in \mathcal{T}$. If $\circ$ is the trivial tree (the one with only one node: the root), we set $Q_\circ(f_0) = f_0$. If now $\Upsilon \in \mathcal{T} \setminus \{\circ\}$, we put $Q_\Upsilon(f_0) = Q(Q_{\Upsilon_\ell}(f_0), Q_{\Upsilon_r}(f_0))$, where Υ_ℓ (resp. Υ_r) is the subtree of Υ consisting of the left (resp. right) child of the root with its whole progeny. Then (5) can be rewritten as

$$f_t = e^{-\kappa t} \sum_{\Upsilon \in \mathcal{T}} (1 - e^{-\kappa t})^{|\Upsilon|-1} Q_\Upsilon(f_0). \tag{6}$$

In words, (6) can be interpreted as follows. For each $\Upsilon \in \mathcal{T}$, the term $e^{-\kappa t}(1 - e^{-\kappa t})^{|\Upsilon|-1}$ is the probability that a typical particle has Υ as (ordered) collision tree, while $Q_\Upsilon(f_0)$ is the density of its velocity knowing that it has Υ as (ordered) collision tree.

Finally, let us mention a natural algorithmic interpretation of (1) closely related to (6). The *dynamical* probabilistic interpretation of Maxwellian molecules, initiated by Tanaka [15], can be roughly summarized as follows. Consider a typical particle in the gas. Initially, its velocity V_0 is f_0-distributed. Then, at *rate* κ, that is, after an $\mathrm{Exp}(\kappa)$-distributed random time τ, it collides with another particle: its velocity V_0 is replaced by $(V_0 + V_\tau^* + |V_0 - V_\tau^*|\sigma)/2$, where V_τ^* is the velocity of an independent particle *undergoing the same process* (stopped at time τ) and σ is a $\kappa^{-1}\beta_{V_0,V_\tau^*}(\sigma)d\sigma$-distributed $\mathbb{S}^2$-valued random variable. Then, at *rate* κ, it collides again, etc. This produces a stochastic process $(V_t)_{t\geq 0}$ such that for all $t \geq 0$, V_t is f_t-distributed.

Consider now the following recursive algorithm.

```
Function velocity(t):
..  Simulate a f_0-distributed random variable v, set s = 0.
..  While s < t do
..  ..  simulate an exponential random variable τ with parameter κ,
..  ..  set s = s + τ,
..  ..  if s < t, do
..  ..  ..  set v* =velocity(s),
..  ..  ..  simulate a κ^{-1} β_{v,v*}(σ)dσ-distributed S^2-valued random variable σ,
..  ..  ..  set v = (v + v* + |v − v*|σ)/2,
..  ..  end if,
..  end while.
..  Return velocity(t) = v.
```

Of course, each new random variable is simulated independently of the previous ones. In particular, line 7 of the algorithm, all the random variables required to produce $v^* =$`velocity`(s) are independent of all that has already been simulated.

Comparing the above paragraph and the algorithm, it appears clearly that `velocity`(t) produces a f_t-distributed random variable. We have never seen this fact written precisely as it is here, but it is more or less well-known folklore. In the present paper, we will prove such a fact, in a slightly more complicated situation.

In spirit, the algorithm produces a binary ordered tree: each time the recursive function calls itself, we add a branch (on the right). So it is closely related to (6) and, actually, one can get convinced that `velocity`(t) is precisely an algorithmic interpretation of (6). But entering into the details would lead us to tedious and technical explanations.

1.2. **Utility of Wild's sum.** The Wild sum has often been used for numerical computations: one simply cutoffs (5) at some well-chosen level and, possibly, adds a Gaussian distribution with adequate mean and covariance matrix to make it have the desired mass and energy. See Carlen-Salvarini [3] for a very precise study in this direction. And actually, Pareschi-Russo [13] also managed to use the Wild sum, among many other things, to solve numerically the *inhomogeneous* Boltzmann equation for non Maxwellian molecules.

A completely different approach is to use a large number N of times the perfect simulation algorithm previously described to produce some i.i.d. f_t-distributed random variables $V_t^1, \ldots, V_t^N$, and to approximate f_t by $N^{-1}\sum_1^N \delta_{V_t^i}$. We believe that this is not very efficient in practice, especially when compared to the use of a classical interacting particle system in the spirit of Kac [8], see e.g. [7]. The main reason is that the computational cost of the above *perfect* simulation algorithm increases exponentially with time, while the one of Kac's particle system increases linearly. So the

cost to remove the bias is disproportionate. See [6] for such a study concerning the Smoluchowski equation, which has the same structure (at the rough level) as the Boltzmann equation.

The Wild sum has also been intensively used to study the rate of approach to equilibrium of Maxwellian molecules. This was initiated by McKean [10], with more recent studies by Carlen-Carvalho-Gabetta [1, 2], themselves followed by Dolera-Gabetta-Regazzini [5] and many other authors.

1.3. **Bounded cross sections.** If $B(v-v^*,\sigma)=\Phi(|v-v^*|)b(\langle\frac{v-v^*}{|v-v^*|},\sigma\rangle)$ with Φ bounded, e.g. by 1, we can introduce fictitious jumps to write (1) as $\partial_t f_t=\kappa Q(f_t,f_t)-\kappa f_t$, with $Q(f,g)(v)=\kappa^{-1}\int_{\mathbb{S}^2}\int_0^1 f(v'')g(v''^*)\mathrm{d}a\beta_{v,v^*}(\sigma)\mathrm{d}\sigma$, where $v''=v+[v'-v]\mathbf{1}_{\{a\le\Phi(|v-v^*|)\}}$ and where $v''^*=v^*+[v'^*-v]\mathbf{1}_{\{a\le\Phi(|v-v^*|)\}}$. Indeed, it suffices to note that

$$\begin{aligned}\partial_t f_t(v)=&\Phi(|v-v^*|)\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\big[f_t(v')f_t(v'^*)-f_t(v)f_t(v^*)\big]\beta_{v,v^*}(\sigma)\mathrm{d}\sigma\mathrm{d}v^*\\=&\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_0^1\big[f_t(v'')f_t(v''^*)-f_t(v)f_t(v^*)\big]\mathrm{d}a\beta_{v,v^*}(\sigma)\mathrm{d}\sigma\mathrm{d}v^*\\=&\kappa Q(f_t,f_t)(v)-\kappa f_t(v).\end{aligned}$$

All the previous study directly applies: we have the formulas (5) and (6), with exactly the same construction of Q_n and Q_Υ as in Subsection 1.1, the only modification being the definition of $Q(f,g)$. However, the resulting Wild sum does not seem to allow for a precise study of the large time behavior of f_t, because it leads to intractable computations. We can also produce a f_t-distributed random variable using the algorithm `velocity(`t`)` with its lines 8 and 9 replaced by

.. .. .. simulate a $\kappa^{-1}\beta_{v,v^*}(\sigma)d\sigma$-distributed $\mathbb{S}^2$-valued random variable σ,
.. .. .. simulate a uniformly distributed random variable a in $[0,1]$,
.. .. .. set $v=v+[v'-v]\mathbf{1}_{\{a\le\Phi(|v-v^*|)\}}$.

1.4. **Hard potentials with angular cutoff.** Of course, the angular cutoff (that is, we assume that $\kappa<\infty$) is crucial to hope for a perfect simulation algorithm and for a series expansion in the spirit of Wild's sum. Indeed, $\kappa=\infty$ implies that a particle is subjected to infinitely many collisions on each finite time interval. So our goal is to extend, at the price of many complications, the algorithm and series expansion to hard potentials with cutoff. Since the cross section is unbounded in the relative velocity variable, some work is needed.

We work with weak forms of PDEs for simplicity. It is classical, see e.g. [16, Section 2.3] that a family $(f_t)_{t\ge0}\subset\mathcal{P}(\mathbb{R}^3)$ is a weak solution to (1) if it satisfies, for all reasonable $\phi\in C_b(\mathbb{R}^3)$,

$$\frac{d}{dt}\int_{\mathbb{R}^3}\phi(v)f_t(\mathrm{d}v)=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v^*|^\gamma[\phi(v')-\phi(v)]\beta_{v,v^*}(\sigma)\mathrm{d}\sigma f_t(\mathrm{d}v^*)f_t(\mathrm{d}v).\tag{7}$$

As already mentioned, one also has $\int_{\mathbb{R}^3}|v|^2f_t(\mathrm{d}v)=e_0$ for all $t\ge0$. For $v\in\mathbb{R}^3$, we introduce

$$\Lambda(v)=(1+e_0)(1+|v|^\gamma)\in[1,\infty)\quad\text{which satisfies}\quad\Lambda(v)\ge\sup_{v^*\in\mathbb{R}^3}\frac{(1+e_0)|v-v^*|^\gamma}{1+|v^*|^2}.$$

For $v,v^*\in\mathbb{R}^3$ and $\sigma\in\mathbb{S}^2$ and $a\in[0,1]$, we put

$$v''(v,v^*,\sigma,a)=v+\Big[v'-v\Big]\mathbf{1}_{\{a\le q(v,v^*)\}}\in\mathbb{R}^3\quad\text{where}\quad q(v,v^*)=\frac{(1+e_0)|v-v^*|^\gamma}{(1+|v^*|^2)\Lambda(v)}\in[0,1].$$

We now rewrite (7), using that $\Lambda(v)q(v,v^*)(1+|v^*|^2)/(1+e_0) = |v-v^*|^\gamma$, as

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3}\phi(v)f_t(\mathrm{d}v) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_0^1 \Lambda(v)\frac{1+|v^*|^2}{1+e_0}[\phi(v'')-\phi(v)]\beta_{v,v^*}(\sigma)\mathrm{d}\sigma f_t(\mathrm{d}v^*)f_t(\mathrm{d}v)\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_0^1 \Lambda(v)\Big[\frac{1+|v^*|^2}{1+e_0}\phi(v'')-\phi(v)\Big]\beta_{v,v^*}(\sigma)\mathrm{d}a\mathrm{d}\sigma f_t(\mathrm{d}v^*)f_t(\mathrm{d}v),\end{aligned} \tag{8}$$

since $\int_{\mathbb{S}^2}\int_0^1 \beta_{v,v^*}(\sigma)\mathrm{d}a\mathrm{d}\sigma$ does not depend on v, v^* and since $\int_{\mathbb{R}^3}(1+|v^*|^2)f_t(\mathrm{d}v^*) = 1+e_0$.

Equation (8) enjoys the pleasant property that the *rate of collision*, given by $\Lambda(v)$, does not depend on v^*. Hence, one is able to predict when a particle will collide (possibly fictitiously) from its sole velocity, knowing nothing of the *environment* represented by $f_t(\mathrm{d}v^*)$. The function Λ is not bounded as a function of v, but, as we will see, this it does not matter too much. On the contrary, the presence of $(1+|v^*|^2)/(1+e_0)$ in front of the gain term is problematic. It means that, in some sense, particles are not all taken into account equally. To overcome this problem, we consider an equation with an additional *weight* variable $m \in (0,\infty)$, with unknown $(F_t)_{t\geq 0} \subset \mathcal{P}((0,\infty)\times\mathbb{R}^3)$: for all reasonnable $\Phi : (0,\infty)\times\mathbb{R}^3 \mapsto \mathbb{R}$,

$$\begin{aligned}\frac{d}{dt}\int_{(0,\infty)\times\mathbb{R}^3}\Phi(m,v)F_t(\mathrm{d}m,\mathrm{d}v) &= \int_{(0,\infty)\times\mathbb{R}^3}\int_{(0,\infty)\times\mathbb{R}^3}\int_{\mathbb{S}^2}\int_0^1 \Lambda(v)\Big[\Phi(m'',v'')-\Phi(m,v)\Big]\\ &\qquad \beta_{v,v^*}(\sigma)\mathrm{d}a\mathrm{d}\sigma F_t(\mathrm{d}m^*,\mathrm{d}v^*)F_t(\mathrm{d}m,\mathrm{d}v),\end{aligned} \tag{9}$$

where $m'' = mm^*(1+|v^*|^2)/(1+e_0)$. A straightforward informal computation using that $F_t \in \mathcal{P}(E)$ for all $t \geq 0$ and test functions of the form $\Phi(m,v) = m\phi(v)$ shows that if $(F_t)_{t\geq 0}$ solves (9), then $(f_t)_{t\geq 0}$ defined by $f_t(v) = \int_0^\infty mF_t(m,v)\mathrm{d}m$ solves (8) and thus (7). And (9) is a nice equation, in that

- it is the Kolmogorov equation of a nonlinear Markov process, which was not the case of (8);
- the rate of jump $\Lambda(v)$ depends only on the characteristics (m,v) of the particle under consideration (as opposed to (7), for which the rate $|v-v^*|^\gamma$ also depends on v^*);
- the strong form of (9) can be written, setting $y = (m,v) \in E = (0,\infty)\times\mathbb{R}^3$,

$$\partial_t F_t(y) = Q(F_t,F_t)(y) - \kappa\Lambda(y)F_t(y) \tag{10}$$

where $\Lambda(y) = \Lambda(v)$, where, for $F, G \in \mathcal{P}(E)$ and $y' \in E$, setting $H = \mathbb{S}^2\times[0,1]$, $\nu_{y,y^*}(\mathrm{d}z) = \beta_{v,v^*}(\sigma)\mathrm{d}a\mathrm{d}\sigma$ and $h(y,y^*,z) = (m'',v'')$,

$$Q(F,G)(y') = \int_E\int_E\int_H \kappa\Lambda(y)\delta_{\{h(y,y^*,z)=y'\}}\nu_{y,y^*}(\mathrm{d}z).$$

We will see in the next section that, at the informal level, any equation of the form (10) (with, importantly, $\kappa = \nu_{y,y^*}(H)$ for all $y, y^* \in E$) allows for a perfect simulation algorithm and for a series expansion in the spirit of Wild. The perfect simulation algorithm is almost as simple as the one previously described, except that the rate of collision $\Lambda(v)$ now depends on the state of the particle. On the contrary, this state-dependent rate complicates subsequently the series expansion because the time and phase variables do not separate anymore.

The additional variable m has unfortunately no interesting physical interpretation, in particular because its value is modified during fictitious collisions. The main idea is that F_t consists of a *biased* velocity-distribution, with some more information that allows us to remove this bias.

1.5. **Plan of the paper.** In the next section, we expose our main results: we properly define weak solutions to (1) and (10), state that (10) has a unique solution F_t that, once integrated in m, produces the solution f_t to (1). We then propose an algorithm that perfectly simulates an F_t-distributed random variable, we write down a series expansion for F_t in the spirit of (6) and discuss briefly the relevance of our results. The proofs are then handled: the algorithm is studied in Section 3, the series expansion established in Section 4, the well-posedness of (10) checked in Section 5, and the link between (10) and (1) shown in Section 6.

2. Main results

2.1. **Weak solutions.** We use a classical definition of weak solutions, see e.g. [16, Section 2.3].

Definition 1. *Assume* (3) *and recall* (4). *A measurable family* $f=(f_t)_{t\geq 0}\subset\mathcal{P}(\mathbb{R}^3)$ *is a weak solution to (1) if for all* $t\geq 0$, $\int_{\mathbb{R}^3}|v|^2 f_t(\mathrm{d}v)=\int_{\mathbb{R}^3}|v|^2 f_0(\mathrm{d}v)<\infty$ *and for all* $\phi\in C_b(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3}\phi(v)f_t(\mathrm{d}v)=\int_{\mathbb{R}^3}\phi(v)f_0(\mathrm{d}v)+\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathcal{A}\phi(v,v^*)f_s(\mathrm{d}v^*)f_s(\mathrm{d}v)\mathrm{d}s, \tag{11}$$

where $\mathcal{A}\phi(v,v^*)=|v-v_*|^\gamma\int_{\mathbb{S}^2}[\phi(v'(v,v^*,\sigma))-\phi(v)]\beta_{v,v^*}(\sigma)\mathrm{d}\sigma$.

Everything is well-defined in (11) by boundedness of mass and energy and since $|\mathcal{A}\phi(v,v^*)|\leq 2\kappa||\phi||_\infty|v-v^*|^\gamma\leq 2\kappa||\phi||_\infty(1+|v|^2)(1+|v^*|^2)$.

For any given $f_0\in\mathcal{P}(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3}|v|^2f_0(\mathrm{d}v)<\infty$, the existence of unique weak solution starting from f_0 is now well-known. See Mischler-Wennberg [12] when f_0 has a density and Lu-Mouhot [9] for the general case. Let us also mention that the conservation assumption is important in Definition 1, since Wennberg [17] proved that there also solutions with increasing energy.

2.2. **An equation with an additional variable.** We briefly summarize the notation of the introduction. We fix $e_0>0$ and define, for $v\in\mathbb{R}^3$,

$$\Lambda(v)=(1+e_0)(1+|v|^\gamma)\in[1,\infty)\quad\text{which satisfies}\quad\Lambda(v)\geq\sup_{v^*\in\mathbb{R}^3}\frac{(1+e_0)|v-v^*|^\gamma}{1+|v^*|^2}.$$

For $v,v^*\in\mathbb{R}^3$ and $z=(\sigma,a)\in H=\mathbb{S}^2\times[0,1]$, we put

$$q(v,v^*)=\frac{(1+e_0)|v-v^*|^\gamma}{(1+|v^*|^2)\Lambda(v)}\in[0,1]\quad\text{and}\quad v''(v,v^*,z)=v+\Big[v'(v,v^*,\sigma)-v\Big]\mathbf{1}_{\{a\leq q(v,v^*)\}}\in\mathbb{R}^3.$$

We also introduce $E=(0,\infty)\times\mathbb{R}^3$ and, for $y=(m,v)$ and $y^*=(m^*,v^*)$ in E and $z\in H$,

$$h(y,y^*,z)=\Big(\frac{mm^*(1+|v^*|^2)}{1+e_0},v''(v,v^*,z)\Big)\in E\quad\text{and}\quad\Lambda(y)=\Lambda(v)\in[1,\infty)$$

with a small abuse of notation. We finally consider, for $y=(m,v)$ and $y^*=(m^*,v^*)$ in E,

$$\nu_{y,y^*}(\mathrm{d}z)=\beta_{v,v^*}(\sigma)\mathrm{d}\sigma\mathrm{d}a$$

which is a measure on H with total mass $\nu_{y,y^*}(H)=\kappa$, see (3). We now define properly weak solutions to (10).

Definition 2. *Assume* (3). *A measurable family* $F=(F_t)_{t\geq 0}\subset\mathcal{P}(E)$ *is said to solve (A) if for all* $T>0$, $\sup_{[0,T]}\int_E|v|^\gamma F_t(\mathrm{d}m,\mathrm{d}v)<\infty$, *and for all* $\Phi\in C_b(E)$, *all* $t\geq 0$,

$$\int_E\Phi(y)F_t(\mathrm{d}y)=\int_E\Phi(y)F_0(\mathrm{d}y)+\int_0^t\int_E\int_E\mathcal{B}\Phi(y,y^*)F_s(\mathrm{d}y^*)F_s(\mathrm{d}y)\mathrm{d}s, \tag{12}$$

where $\mathcal{B}\Phi(y,y^*)=\Lambda(y)\int_H[\Phi(h(y,y^*,z))-\Phi(y)]\nu_{y,y^*}(\mathrm{d}z)$.

All is well-defined in (12) thanks to the conditions on F and since $|\mathcal{B}\Phi(y,y^*)| \le 2\kappa||\Phi||_\infty\Lambda(y) \le C_\Phi(1+|v|^\gamma)$ (with the notation $y=(m,v)$). As already mentioned in the introduction, the important point is that the function Λ does not depend on y^*. Hence a particle, when in the state y, jumps at rate $\kappa\Lambda(y)$, independently of everything else.

Proposition 3. *Assume* (3). *For any $F_0 \in \mathcal{P}(E)$ such that $\int_E |v|^\gamma F_0(\mathrm{d}m,\mathrm{d}v) < \infty$, (A) has exactly one solution F starting from F_0.*

We will also verify the following estimate.

Remark 4. *Assume* (3). *A solution to (A) satisfies $\sup_{t\ge0}\int_E |v|^2 F_t(\mathrm{d}m,\mathrm{d}v) = \int_E |v|^2 F_0(\mathrm{d}m,\mathrm{d}v)$.*

The link with the Boltzmann equation is as follows, see Subsection 1.4 for an informal proof.

Proposition 5. *Assume* (3). *Let $F_0 \in \mathcal{P}(E)$ such that $\int_E[|v|^\gamma + m(1+|v|^{2+2\gamma})]F_0(\mathrm{d}m,\mathrm{d}v) < \infty$ and let F be the solution to (A). Introduce, for each $t \ge 0$, the nonnegative measure f_t on $\mathbb{R}^3$ defined by $f_t(A) = \int_E m\mathbf{1}_{\{v\in A\}}F_t(\mathrm{d}m,\mathrm{d}v)$ for all $A \in \mathcal{B}(\mathbb{R}^3)$. If $f_0 \in \mathcal{P}(\mathbb{R}^3)$ and if the quantity e_0 used to define the coefficients of (A) is precisely $e_0 = \int_{\mathbb{R}^3}|v|^2 f_0(\mathrm{d}v)$, then $(f_t)_{t\ge0}$ is the unique weak solution to* (1) *starting from f_0.*

2.3. **A perfect simulation algorithm.** We consider the following procedure.

Algorithm 6. *Fix $e_0 > 0$ and $F_0 \in \mathcal{P}(E)$. For any $t \ge 0$ we define the following recursive function, of which the result is some $E\times\mathbb{N}$-valued random variable.*

```
function (value(t),counter(t)):
..  Simulate a F_0-distributed random variable y, set s = 0 and n = 0.
..  While s < t do
..  ..  simulate an exponential random variable τ with parameter κΛ(y),
..  ..  set s = s + τ,
..  ..  if s < t, do
..  ..  ..  set (y*, n*) =(value(s),counter(s)),
..  ..  ..  simulate z ∈ H with law κ^{-1} ν_{y,y*},
..  ..  ..  set y = h(y, y*, z),
..  ..  ..  set n = n + n* + 1,
..  ..  end if,
..  end while.
..  Return value(t) = y and counter(t) = n.
```

Of course, each time a new random variable is simulated, we implicitly assume that it is independent of everything that has already been simulated. In particular, line 7 of the procedure, all the random variables used to produce $(y^*,n^*) =$`(value(`s`),counter(`s`))` are independent of all the random variables already simulated. By construction, `counter(`t`)` is precisely the number of times the recursive function calls itself.

Proposition 7. *Assume* (3). *Fix $F_0 \in \mathcal{P}(E)$ such that $\int_E |v|^\gamma F_0(\mathrm{d}m,\mathrm{d}v) < \infty$ and fix $t \ge 0$. Algorithm 6 a.s. stops and thus produces a couple (Y_t,N_t) of random variables. The E-valued random variable Y_t is F_t-distributed, where F is the solution to (A) starting from F_0. The $\mathbb{N}$-valued random variable N_t satisfies $\mathbb{E}[N_t] \le \exp(\kappa\int_0^t\int_E \Lambda(y)F_s(\mathrm{d}y)ds) - 1$.*

Let us call F_t the law of Y_t and explain informally why $(F_t)_{t\ge0}$ should solve (A). This does not depend on the specific parameters of equation (A) but only on its structure. In other words, this should work generally, i.e. with another phase space E, other functions $\Lambda : E \mapsto \mathbb{R}_+$ and

$h : E \times E \times H \mapsto E$, etc, provided one is able to justify the computations and that the algorithm stops. In some obvious sense, Algorithm 6 actually produces an E-valued process $(Y_t)_{t\geq 0}$. Initially, we have $Y_0 \sim F_0$. When in state $y \in E$ at time $s-$, at rate $\kappa\Lambda(y)$, we choose y^* (independently of $(Y_u)_{u<s}$) according to F_s (because the algorithm calls itself at time s), then we choose $z \in H$ with law $\kappa^{-1}\nu_{y,y^*}$ and we set $Y_s = h(y, y^*, z)$. In other words, $(Y_t)_{t\geq 0}$ is a time-inhomogeneous Markov process with initial law F_0 and generator L_s defined, for any reasonable $\Phi : E \mapsto \mathbb{R}$ and any $y \in E$, by

$$L_s\Phi(y) = \kappa\Lambda(y) \int_E \int_H [\Phi(h(y, y^*, z) - \Phi(y)]\kappa^{-1}\nu_{y,y^*}(\mathrm{d}z)F_s(\mathrm{d}y).$$

We thus should have $\int_E \Phi(y)F_t(\mathrm{d}y) = \int_E \Phi(y)F_0(\mathrm{d}y) + \int_0^t \int_E L_s\Phi(y)F_s(\mathrm{d}y)\mathrm{d}s$, which is nothing but (12).

2.4. **A series expansion.** We next write down a series expansion of F_t, the solution to (A), in the spirit of Wild's sum (6). Unfortunately, the expressions are more complicated, because the time ($t \geq 0$) and phase ($y \in E$) variables do not separate. This is due to the fact that the jump rate Λ depends on the state of the particle.

For F, G in $\mathcal{M}(E)$, we define $Q(F, G) \in \mathcal{M}(E)$ by, for all Borel subset $A \subset E$,

$$Q(F, G)(A) = \int_E \int_E \int_H \Lambda(y)\mathbf{1}_{\{h(y,y^*,z)\in A\}}\nu_{y,y^*}(\mathrm{d}z)G(\mathrm{d}y^*)F(\mathrm{d}y).$$

Observe that (12) may be written, at least formally, $\partial_t F_t = -\kappa\Lambda F_t + Q(F_t, F_t)$, provided F_t is a probability measure on E for all $t \geq 0$. Also, note that $Q(F, G) \neq Q(G, F)$ in general.

For $J \in \mathcal{M}(\mathbb{R}_+ \times E)$, consider the measurable family $(\Gamma_t(J))_{t\geq 0} \subset \mathcal{M}(E)$ defined by

$$\Gamma_t(J)(A) = \int_0^t \int_E \mathbf{1}_{\{y\in A\}} e^{-\kappa\Lambda(y)(t-s)} J(\mathrm{d}s, \mathrm{d}y)$$

for all Borel subset $A \subset E$.

We finally consider the set $\mathcal{T}$ of all finite binary (discrete) ordered trees: such a tree is constituted of a finite number of nodes, including the root, each of these nodes having either 0 or two children (ordered, in the sense that a node having two children has a *left* child and a *right* child). We denote by $\circ \in \mathcal{T}$ the trivial tree, composed of the root as only node.

Proposition 8. *Assume* (3). *Let $F_0 \in \mathcal{P}(E)$ such that $\int_E |v|^\gamma F_0(\mathrm{d}m, \mathrm{d}v) < \infty$. The unique solution $(F_t)_{t\geq 0}$ to (A) starting from F_0 is given by*

$$F_t = \sum_{\Upsilon\in\mathcal{T}} \Gamma_t(J_\Upsilon(F_0)),$$

with $J_\Upsilon(F_0) \in \mathcal{M}(\mathbb{R}_+ \times E)$ defined by induction: $J_\circ(F_0)(\mathrm{d}t, \mathrm{d}x) = \delta_0(\mathrm{d}t)F_0(\mathrm{d}x)$ and, if $\Upsilon \in \mathcal{T}\backslash\{\circ\}$,

$$J_\Upsilon(F_0)(\mathrm{d}t, \mathrm{d}x) = \mathrm{d}tQ(\Gamma_t(J_{\Upsilon_\ell}(F_0)), \Gamma_t(J_{\Upsilon_r}(F_0)))(\mathrm{d}x),$$

where Υ_ℓ (resp. Υ_r) is the subtree of Υ consisting of the left (resp. right) child of the root with its whole progeny.

We will prove this formula by a purely analytic method. We do not want to discuss precisely its connection with Algorithm 6, but let us mention that in spirit, the algorithm produces a (random) ordered tree Υ_t of interactions together with the value of Y_t, and that $\Gamma_t(J_\Upsilon(F_0))$ can be interpreted as the probability distribution of Y_t restricted to the event that $\Upsilon_t = \Upsilon$.

Let us informally explain why $F_t := \sum_{\Upsilon\in\mathcal{T}} \Gamma_t(J_\Upsilon(F_0))$ should solve (A), i.e., in strong form, $\partial_t F_t = -\kappa\Lambda F_t + Q(F_t, F_t)$. Exactly as for the perfect simulation algorithm, this informal proof does not depend on the specific parameters of the equation, but only on its structure. We assume that all the involved probability measures have densities (except $J_\circ(F_0)$) and we observe that

- for any function $J : \mathbb{R}_+ \times E \mapsto \mathbb{R}_+$, $\partial_t \Gamma_t(J)(y) = J(t,y) - \kappa\Lambda(y)\Gamma_t(J)(y)$;
- $\Gamma_t(J_\circ(F_0))(y) = e^{-\kappa\Lambda(y)t} F_0(y)$ for all $t \geq 0$, whence $\partial_t \Gamma_t(J_\circ(F_0)) = -\kappa\Lambda(y)\Gamma_t(J_\circ(F_0))$.

We start from the correct initial condition since $\sum_{\Upsilon\in\mathcal{T}} \Gamma_0(J_\Upsilon(F_0)) = F_0$, and we have

$$\begin{aligned}\partial_t F_t(y) =& -\kappa\Lambda(y)\Gamma_t(J_\circ(F_0)) + \sum_{\Upsilon\in\mathcal{T}\setminus\{\circ\}} \big(J_\Upsilon(F_0)(t,y) - \kappa\Lambda(y)\Gamma_t(J_\Upsilon(F_0))\big)\\ =& -\kappa\Lambda(y)F_t(y) + \sum_{\Upsilon\in\mathcal{T}\setminus\{\circ\}} J_\Upsilon(F_0)(t,y).\end{aligned}$$

But by definition,

$$\begin{aligned}\sum_{\Upsilon\in\mathcal{T}\setminus\{\circ\}} J_\Upsilon(F_0)(t,y) =& \sum_{\Upsilon\in\mathcal{T}\setminus\{\circ\}} Q(\Gamma_t(J_{\Upsilon_\ell}(F_0)), \Gamma_t(J_{\Upsilon_r}(F_0)))(y)\\ =& \sum_{\Upsilon\in\mathcal{T},\Upsilon'\in\mathcal{T}} Q(\Gamma_t(J_\Upsilon(F_0)), \Gamma_t(J_{\Upsilon'}(F_0)))(y)\\ =& Q\Big(\sum_{\Upsilon\in\mathcal{T}} \Gamma_t(J_\Upsilon(F_0)), \sum_{\Upsilon'\in\mathcal{T}} \Gamma_t(J_{\Upsilon'}(F_0))\Big)(y),\end{aligned}$$

which equals $Q(F_t, F_t)(y)$ as desired.

2.5. **Conclusion.** Fix $f_0 \in \mathcal{P}(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} |v|^{2+2\gamma} f_0(\mathrm{d}v) < \infty$ and set $F_0 = \delta_1 \otimes f_0 \in \mathcal{P}(E)$, which satisfies $\int_E [|v|^\gamma + m(1+|v|^{2+2\gamma})] F_0(\mathrm{d}m, \mathrm{d}v) = \int_{\mathbb{R}^3} (1 + |v|^\gamma + |v|^{2+2\gamma}) f_0(\mathrm{d}v) < \infty$.

(a) Gathering Propositions 7 and 5, we find that Algorithm 6 used with $e_0 = \int_{\mathbb{R}^3} |v|^2 f_0(\mathrm{d}v)$ and with F_0 produces a random variable (Y_t, N_t), with $Y_t = (M_t, V_t)$ such that $\mathbb{E}[M_t \mathbf{1}_{\{V_t \in A\}}] = f_t(A)$ for all $A \in \mathcal{B}(\mathbb{R}^3)$, where f is the unique weak solution to (1) starting from f_0. Also, the mean number of iterations $\mathbb{E}[N_t]$ is bounded by $\exp[\kappa(1+e_0)(1+e_0^{\gamma/2})t] - 1$.

Indeed, we know from Proposition 7 that $\mathbb{E}[N_t] \leq \exp(\kappa \int_0^t \int_E \Lambda(y) F_s(\mathrm{d}y)ds) - 1$. But we have $\int_E \Lambda(y) F_t(\mathrm{d}y) = (1+e_0)\int_E (1+|v|^\gamma) F_t(\mathrm{d}m, \mathrm{d}v) \leq (1+e_0)(1 + (\int_E |v|^2 F_t(\mathrm{d}m, \mathrm{d}v))^{\gamma/2})$, which is smaller than $(1+e_0)(1 + (\int_E |v|^2 F_0(\mathrm{d}m, \mathrm{d}v))^{\gamma/2}) = (1+e_0)(1+e_0^{\gamma/2})$ by Remark 4.

(b) Gathering Propositions 8 and 5, we conclude that for all $t \geq 0$, all Borel subset $A \subset \mathbb{R}^3$, we have $f_t(A) = \sum_{\Upsilon\in\mathcal{T}} \int_E m \mathbf{1}_{\{v\in A\}} \Gamma_t(J_\Upsilon(F_0))(\mathrm{d}m, \mathrm{d}v)$.

2.6. **Discussion.** It might be possible to prove Proposition 5 assuming only that $F_0 \in \mathcal{P}(E)$ satisfies $\int_E m(1+|v|^2) F_0(\mathrm{d}m, \mathrm{d}v) < \infty$ instead of $\int_E m(1+|v|^{2+2\gamma}) F_0(\mathrm{d}m, \mathrm{d}v) < \infty$, since the Boltzmann equation (1) is known to be well-posed as soon as the initial energy is finite, see [12, 9]. However, it would clearly be more difficult and our condition is rather harmless.

Observe that (A) is well-posed under the condition that $F_0 \in \mathcal{P}(E)$ satisfies $\int_E |v|^\gamma F_0(\mathrm{d}m, \mathrm{d}v)$, which does not at all imply that $e_0 = \int_E m|v|^2 F_0(\mathrm{d}m, \mathrm{d}v) < \infty$. But, recalling that the e_0 has to be finite for the coefficients of (A) to be well-defined, this is not very interesting.

The series expansion of Proposition 8 is of course much more complicated than the original Wild sum, since (a) we had to add the variable m, (b) we had to introduce fictitious jumps, (c)

time and space do not separate. So it is not clear whether the formula can be used theoretically or numerically. However, it provides an explicit formula expressing f_t as a (tedious) function of f_0.

Algorithm 6 is extremely simple. Using it a large number of times, which produces some i.i.d. sample $(M_t^i, V_t^i)_{i=1,\dots,N}$, we may approximate f_t by $N^{-1}\sum_1^N M_t^i \delta_{V_t^i}$. For a central limit theorem to hold true, one needs $\mathbb{E}[M_t^2] = \int_E m^2 F_t(\mathrm{d}m, \mathrm{d}v)$ to be finite. We do not know if this holds true, although we have some serious doubts. Hence the convergence of this Monte-Carlo approximation may be much slower that $N^{-1/2}$. The main interest of Algorithm 6 is thus theoretical.

3. The algorithm

Here we prove Proposition 7. We fix $F_0 \in \mathcal{P}(E)$ such that $\int_E |v|^\gamma F_0(\mathrm{d}m, \mathrm{d}v) < \infty$, which implies that $\int_E \Lambda(y) F_0(\mathrm{d}y) < \infty$. When Algorithm 6 never stops, we take the convention that it returns $(\texttt{value}(t), \texttt{counter}(t)) = (\triangle, \infty)$, where $\triangle$ is a cemetery point. For each $t \geq 0$, we denote by $G_t \in \mathcal{P}((E \times \mathbb{N}) \cup \{(\triangle, \infty)\})$ the law of the random variable produced by Algorithm 6. Also, for $y \in E$, $n \in \mathbb{N}$ and $z \in H$, we take the conventions that $h(y, \triangle, z) = \triangle$ and $n + \infty + 1 = \infty$. We arbitrarily define, for $y \in E$, $\nu_{y,\Delta}(\mathrm{d}z) = |\mathbb{S}^2|^{-1} \kappa \mathrm{d}\sigma \mathrm{d}a$.

Step 1. We now consider the following procedure. It is an *abstract* procedure, because it assumes that for each $t \geq 0$, one can simulate a random variable with law G_t and because the instructions are repeated *ad infinitum* if the cemetery point is not attained.

Simulate a F_0-distributed random variable y, set $s = 0$ and $n = 0$.
While $y \neq \triangle$ do ad infinitum
.. simulate an exponential random variable τ with parameter $\kappa\Lambda(y)$,
.. set $Y_t = y$ and $N_t = n$ for all $t \in [s, s+\tau)$,
.. set $s = s + \tau$,
.. set $(y^*, n^*) = (\texttt{value}(s), \texttt{counter}(s))$, with $(y^*, n^*) = (\triangle, \infty)$ if it never stops,
.. simulate $z \in H$ with law $\kappa^{-1}\nu_{y,y^*}$,
.. set $y = h(y, y^*, z)$,
.. set $n = n + n^* + 1$,
end while.
If $s < \infty$, set $Y_t = \Delta$ and $N_t = \infty$ for all $t \geq s$.

Observe that in the last line, we may have $s < \infty$ either because after a finite number of steps, the simulation of (y^*, n^*) with law G_s has produced $(\triangle, \infty)$, or because we did repeat the loop *ad infinitum*, but the increasing process N became infinite in finite time.

At the end, this produces a process $(Y_t, N_t)_{t \geq 0}$ and one easily gets convinced that for each $t \geq 0$, (Y_t, N_t) is G_t-distributed. Indeed, if one extracts from the above procedure only what is required to produce (Y_t, N_t) (for some fixed t), one precisely re-obtains Algorithm 6 if $(Y_t, N_t) \neq (\triangle, \infty)$ (and in this case Algorithm 6 stops), while $(Y_t, N_t) = (\triangle, \infty)$ implies that Algorithm 6 never stops.

By construction, the process $(Y_t, N_t)_{t \geq 0}$ is a time-inhomogeneous (possibly exploding) Markov process with values in $(E \times \mathbb{N}) \cup \{(\triangle, \infty)\}$ with generator $\mathcal{L}_t$, absorbed at $(\triangle, \infty)$ if this point is reached and set to $(\triangle, \infty)$ after explosion if it explodes, where

$$\mathcal{L}_t \Psi(y,n) = \kappa\Lambda(y) \int_H \int_{(E\times\mathbb{N})\cup\{(\triangle,\infty)\}} \Big[\Psi(h(y,y^*,z), n+n^*+1) - \Psi(y,n)\Big] G_t(\mathrm{d}y^*, \mathrm{d}n^*) \frac{\nu_{y,y^*}(\mathrm{d}z)}{\kappa}$$

for all $t \geq 0$, all $\Psi \in B_b((E \times \mathbb{N}) \cup \{(\triangle, \infty)\})$, all $y \in E$ and all $n \in \mathbb{N}$.

Step 2. Here we handle a preliminary computation: for all $y, y^* \in E$, we have

$$A(y,y^*) = \int_H \Big[\Lambda(h(y,y^*,z^*)) - \Lambda(y)\Big]\nu_{y,y^*}(\mathrm{d}z) \le \kappa(1+e_0). \tag{13}$$

Writing $y=(m,v)$, $y^*=(m^*,v^*)$ and recalling Subsection 2.2, $A(y,y^*)$ equals

$$(1+e_0)\int_H \Big[|v''(v,v^*,z)|^\gamma - |v|^\gamma\Big]\nu_{y,y^*}(\mathrm{d}z) = (1+e_0)q(v,v^*)\int_{\mathbb{S}^2}[|v'(v,v^*,\sigma)|^\gamma - |v|^\gamma]\beta_{v,v^*}(\sigma)\mathrm{d}\sigma.$$

But $|v'(v,v^*,\sigma)| \le |v|+|v^*|$, see (2), so that $|v'(v,v^*,\sigma)|^\gamma - |v|^\gamma \le |v^*|^\gamma$, whence

$$A(y,y^*) \le \kappa(1+e_0)q(v,v^*)|v^*|^\gamma = \kappa(1+e_0)\frac{|v-v^*|^\gamma|v^*|^\gamma}{(1+|v|^\gamma)(1+|v^*|^2)} \le \kappa(1+e_0),$$

because $|v-v^*|^\gamma|v^*|^\gamma \le (|v|^\gamma+|v^*|^\gamma)|v^*|^\gamma \le |v|^\gamma(1+|v^*|^2)+(1+|v^*|^2) = (1+|v|^\gamma)(1+|v^*|^2)$.

Step 3. We now prove that $(Y_t,N_t)_{t\ge0}$ actually does not explode nor reach the cemetery point, that $\mathbb{E}[N_t] \le \exp(\kappa\int_0^t \mathbb{E}[\Lambda(Y_s)]\mathrm{d}s) - 1$ and that $\mathbb{E}[\Lambda(Y_t)] \le \mathbb{E}[\Lambda(Y_0)]\exp(\kappa(1+e_0)t)$.

For $A \in \mathbb{N}_*$, we introduce $\zeta_A = \inf\{t\ge0 : N_t \ge A\}$. The process $(Y_t,N_t)_{t\ge0}$ does not explode nor reach the cemetery point during $[0,\zeta_A)$, so that we can write, with $\Psi(y,n) = n\wedge A$, (recall that $N_0=0$ and that $t\mapsto N_t$ is a.s. non-decreasing),

$$\mathbb{E}[N_t\wedge A] = \mathbb{E}[N_{t\wedge\zeta_A}\wedge A] = \mathbb{E}\Big[\int_0^{t\wedge\zeta_A}\mathcal{L}_s\Psi(Y_s,N_s)\mathrm{d}s\Big] = \int_0^t \mathbb{E}\Big[\mathbf{1}_{\{N_s<A\}}\mathcal{L}_s\Psi(Y_s,N_s)\Big]\mathrm{d}s.$$

Since $0 \le (n+n^*+1)\wedge A - n\wedge A \le 1+n^*\wedge A$, we deduce that

$$0 \le \mathcal{L}_s\Psi(y,n) \le \kappa\Lambda(y)\int_{(E\times\mathbb{N})\cup\{(\triangle,\infty)\}}[1+n^*\wedge A]G_s(\mathrm{d}y^*,\mathrm{d}n^*) = \kappa\Lambda(y)(1+\mathbb{E}[N_s\wedge A]),$$

because (Y_s,N_s) is G_s-distributed. We thus find

$$\mathbb{E}[N_t\wedge A] \le \kappa\int_0^t \mathbb{E}[\Lambda(Y_s)\mathbf{1}_{\{N_s<A\}}]\mathbb{E}[1+N_s\wedge A]\mathrm{d}s,$$

whence, by the Gronwall lemma,

$$\mathbb{E}[N_t\wedge A] \le \exp\Big(\kappa\int_0^t \mathbb{E}[\Lambda(Y_s)\mathbf{1}_{\{N_s<A\}}]\mathrm{d}s\Big) - 1. \tag{14}$$

We next choose $\Psi(y,n) = \Lambda(y)\mathbf{1}_{\{n<A\}}$ and write, as previously,

$$\mathbb{E}[\Lambda(Y_t)\mathbf{1}_{\{N_t<A\}}] = \mathbb{E}[\Lambda(Y_{t\wedge\zeta_A})\mathbf{1}_{\{N_{t\wedge\zeta_A}<A\}}] = \mathbb{E}[\Lambda(Y_0)] + \mathbb{E}\Big[\int_0^{t\wedge\zeta_A}\mathcal{L}_s\Psi(Y_s,N_s)\mathrm{d}s\Big],$$

whence

$$\mathbb{E}[\Lambda(Y_t)\mathbf{1}_{\{N_t<A\}}] = \mathbb{E}[\Lambda(Y_0)] + \int_0^t \mathbb{E}\Big[\mathbf{1}_{\{N_s<A\}}\mathcal{L}_s\Psi(Y_s,N_s)\Big]\mathrm{d}s.$$

But $\mathcal{L}_s\Psi(y,n)$ equals

$$\begin{aligned}&\Lambda(y)\int_H\int_{(E\times\mathbb{N})\cup\{(\triangle,\infty)\}}\Big[\Lambda(h(y,y^*,z^*))\mathbf{1}_{\{n+n^*+1<A\}} - \Lambda(y)\mathbf{1}_{\{n<A\}}\Big]G_s(\mathrm{d}y^*,\mathrm{d}n^*)\nu_{y,y^*}(\mathrm{d}z)\\ \le&\Lambda(y)\mathbf{1}_{\{n<A\}}\int_H\int_{E\times\mathbb{N}}\Big[\Lambda(h(y,y^*,z^*)) - \Lambda(y)\Big]G_s(\mathrm{d}y^*,\mathrm{d}n^*)\nu_{y,y^*}(\mathrm{d}z),\end{aligned}$$

whence $\mathcal{L}_s\Psi(y,n) \le \kappa(1+e_0)\mathbf{1}_{\{n<A\}}\Lambda(y)$ by (13) and since $G_s(E\times\mathbb{N}) \le 1$. Finally, we have checked that $\mathbb{E}[\Lambda(Y_t)\mathbf{1}_{\{N_t<A\}}] \le \mathbb{E}[\Lambda(Y_0)] + \kappa(1+e_0)\int_0^t \mathbb{E}[\mathbf{1}_{\{N_s<A\}}\Lambda(Y_s)]\mathrm{d}s$, whence

$$\mathbb{E}[\Lambda(Y_t)\mathbf{1}_{\{N_t<A\}}] \le \mathbb{E}[\Lambda(Y_0)]\exp(\kappa(1+e_0)t). \tag{15}$$

Gathering (14) and (15) and letting A increase to infinity, we first conclude that $\mathbb{E}[N_t] < \infty$ for all $t \ge 0$. In particular, $N_t < \infty$ a.s. for all $t \ge 0$, and the process $(Y_t, N_t)_{t\ge 0}$ does a.s. not explode and never reach $(\triangle, \infty)$. Consequently, $\mathbb{E}[\Lambda(Y_t)] = \lim_{A\to\infty}\mathbb{E}[\Lambda(Y_t)\mathbf{1}_{\{N_t<A\}}] \le \mathbb{E}[\Lambda(Y_0)]\exp(\kappa(1+e_0)t)$ by (15). Finally, we easily conclude from (14) that $\mathbb{E}[N_t] \le \exp(\kappa\int_0^t \mathbb{E}[\Lambda(Y_s)]\mathrm{d}s) - 1$.

Step 4. By Step 3, we know that G_t (which is the law of (Y_t, N_t)) is actually supported by $E\times\mathbb{N}$ for all $t \ge 0$. Hence Algorithm 6 a.s. stops. The process $(Y_t, N_t)_{t\ge 0}$ is thus an inhomogeneous Markov with generator $\tilde{\mathcal{L}}_t$ defined, for $\Psi \in C_b(E\times\mathbb{N})$, by

$$\tilde{\mathcal{L}}_t\Psi(y,n) = \Lambda(y)\int_H\int_{E\times\mathbb{N}}\Big[\Psi(h(y,y^*,z),n+n^*+1) - \Psi(y,n)\Big]G_t(\mathrm{d}y^*,\mathrm{d}n^*)\nu_{y,y^*}(\mathrm{d}z)$$

and we thus have

$$\int_{E\times\mathbb{N}}\Psi(y,n)G_t(\mathrm{d}y,\mathrm{d}n) = \int_{E\times\mathbb{N}}\Psi(y,n)G_0(\mathrm{d}y,\mathrm{d}n) + \int_0^t\int_{E\times\mathbb{N}}\tilde{\mathcal{L}}_s\Psi(y,n)G_s(\mathrm{d}y,\mathrm{d}n)\mathrm{d}s.$$

Let now $F_t \in \mathcal{P}(E)$ be the law of Y_t (so F_t is the first marginal of G_t). It starts from F_0 and solves (A). Indeed, $\int_E |v|^\gamma F_t(\mathrm{d}m,\mathrm{d}v) \le \int_E \Lambda(y)F_t(\mathrm{d}y) = \mathbb{E}[\Lambda(Y_t)]$ is locally bounded by Step 3 and for all $\Phi \in C_b(E)$, applying the above equation with $\Psi(y,n) = \Phi(y)$, we find $\tilde{\mathcal{L}}_t\Psi(y,n) = \Lambda(y)\int_H\int_E[\Phi(h(y,y^*,z)) - \Phi(y)]F_t(\mathrm{d}y^*)\nu_{y,y^*}(\mathrm{d}z)$, so that

$$\int_E\Phi(y)F_t(\mathrm{d}y) = \int_E\Phi(y)F_0(\mathrm{d}y) + \int_0^t\int_E\int_E\int_H\Lambda(y)\Big[\Phi(h(y,y^*,z)) - \Phi(y)\Big]\nu_{y,y^*}(\mathrm{d}z)F_s(\mathrm{d}y^*)F_s(\mathrm{d}y)\mathrm{d}s$$

as desired. Finally, we have already seen in Step 3 that $\mathbb{E}[N_t] \le \exp(\kappa\int_0^t\mathbb{E}[\Lambda(Y_s)]\mathrm{d}s) - 1 = \exp(\kappa\int_0^t\int_E\Lambda(y)F_s(\mathrm{d}y)\mathrm{d}s) - 1$. We have proved Proposition 7, as well as the existence part of Proposition 3. □

4. Series expansion

The goal of this section is to prove Proposition 8. We thus consider $F_0 \in \mathcal{P}(E)$ such that $\int_E\Lambda(y)F_0(\mathrm{d}y) = (1+e_0)\int_E(1+|v|^\gamma)F_0(\mathrm{d}m,\mathrm{d}v) < \infty$. To shorten notation, we set $J_\Upsilon = J_\Upsilon(F_0)$.

Step 1. Here we check that for all $\Upsilon \in \mathcal{T}$, all $t \ge 0$, $C_\Upsilon(t) = \int_0^t\int_E\Lambda(y)J_\Upsilon(\mathrm{d}s,\mathrm{d}y) < \infty$. We work by induction. First, since $J_\circ(\mathrm{d}t,\mathrm{d}y) = \delta_0(\mathrm{d}t)F_0(\mathrm{d}y)$, we find that $C_\circ(t) = \int_E\Lambda(y)F_0(\mathrm{d}y)$ for all $t \ge 0$, which is finite by assumption. Next, we fix $t \ge 0$, $\Upsilon \in \mathcal{T}\setminus\{\circ\}$, we consider Υ_ℓ and Υ_r as in the statement, we assume by induction that $C_{\Upsilon_\ell}(t) < \infty$ and $C_{\Upsilon_r}(t) < \infty$ and prove that $C_\Upsilon(t) < \infty$. We start from

$$\begin{aligned}
C_\Upsilon(t) &= \int_0^t\int_E\Lambda(x)Q(\Gamma_s(J_{\Upsilon_\ell}),\Gamma_s(J_{\Upsilon_r}))(\mathrm{d}x)\mathrm{d}s\\
&= \int_0^t\int_E\int_E\int_H\Lambda(h(y,y^*,z))\Lambda(y)\nu_{y,y^*}(\mathrm{d}z)\Gamma_s(J_{\Upsilon_r})(\mathrm{d}y^*)\Gamma_s(J_{\Upsilon_\ell})(\mathrm{d}y)\mathrm{d}s\\
&= \int_0^t\int_E\int_E\int_H\int_0^s\int_0^s\Lambda(h(y,y^*,z))\Lambda(y)\nu_{y,y^*}(\mathrm{d}z)\\
&\qquad\qquad J_{\Upsilon_r}(\mathrm{d}u^*,\mathrm{d}y^*)e^{-\kappa\Lambda(y^*)(s-u^*)}J_{\Upsilon_\ell}(\mathrm{d}u,\mathrm{d}y)e^{-\kappa\Lambda(y)(s-u)}\mathrm{d}s.
\end{aligned}$$

But it follows from (13) that $\int_H \Lambda(h(y,y^*,z))\nu_{y,y^*}(\mathrm{d}z) \le \kappa(1+e_0+\Lambda(y)) \le 2\kappa\Lambda(y)$, whence

$$\begin{aligned} C_\Upsilon(t) \le& 2\kappa \int_0^t \int_E \int_E \int_0^s \int_0^s \Lambda^2(y) J_{\Upsilon_r}(\mathrm{d}u^*,\mathrm{d}y^*) J_{\Upsilon_\ell}(\mathrm{d}u,\mathrm{d}y) e^{-\kappa\Lambda(y)(s-u)}\mathrm{d}s \\ \le& 2\kappa C_{\Upsilon_r}(t) \int_0^t \int_E \int_0^s \Lambda^2(y) J_{\Upsilon_\ell}(\mathrm{d}u,\mathrm{d}y) e^{-\kappa\Lambda(y)(s-u)}\mathrm{d}s. \end{aligned}$$

We finally used that for $s\in[0,t]$, $\int_E \int_0^s J_{\Upsilon_r}(\mathrm{d}u^*,\mathrm{d}y^*) \le \int_E \int_0^t \Lambda(y^*) J_{\Upsilon_r}(\mathrm{d}u^*,\mathrm{d}y^*) = C_{\Upsilon_r}(t)$. Next, by the Fubini theorem,

$$C_\Upsilon(t) \le 2\kappa C_{\Upsilon_r}(t) \int_0^t \int_E J_{\Upsilon_\ell}(\mathrm{d}u,\mathrm{d}y) \int_u^t \Lambda^2(y) e^{-\kappa\Lambda(y)(s-u)}\mathrm{d}s \le 2C_{\Upsilon_r}(t) \int_0^t \int_E \Lambda(y) J_{\Upsilon_\ell}(\mathrm{d}u,\mathrm{d}y),$$

so that $C_\Upsilon(t) \le 2C_{\Upsilon_r}(t)C_{\Upsilon_\ell}(t) < \infty$ as desired.

Step 2. We deduce from Step 1 that for all $\Upsilon\in\mathcal{T}$,

$$t \mapsto \int_E \Lambda(y)\Gamma_t(J_\Upsilon)(\mathrm{d}y) = \int_0^t \int_E \Lambda(y) e^{-\Lambda(y)(t-s)} J_\Upsilon(\mathrm{d}s,\mathrm{d}y) \le C_\Upsilon(t)$$

is locally bounded.

Step 3. We fix $k\in\mathbb{N}_*$ and denote by $\mathcal{T}_k \subset \mathcal{T}$ the finite set of all ordered binary trees with at most k nodes. We introduce $F_t^k = \sum_{\Upsilon\in\mathcal{T}_k} \Gamma_t(J_\Upsilon)$. By Step 2, we know that $t\mapsto \int_E \Lambda(y) F_t^k(\mathrm{d}y)$ is locally bounded. We claim that for all $\Phi\in C_b(E)$, all $t\ge 0$,

$$\begin{aligned} \int_E \Phi(y) F_t^k(\mathrm{d}y) =& \int_E \Phi(y) e^{-\kappa\Lambda(y)t} F_0(\mathrm{d}y) \\ &+ \sum_{\Upsilon\in\mathcal{T}_k\setminus\{\circ\}} \int_0^t \int_E \Phi(y) e^{-\kappa\Lambda(y)(t-s)} Q(\Gamma_s(J_{\Upsilon_\ell}),\Gamma_s(J_{\Upsilon_r}))(\mathrm{d}y)\mathrm{d}s, \end{aligned} \tag{16}$$

whence in particular $F_0^k = F_0$. Indeed, we first observe that

$$\int_E \Phi(y)\Gamma_t(J_\circ)(\mathrm{d}y) = \int_0^t \int_E \Phi(y) e^{-\kappa\Lambda(y)(t-s)} J_\circ(\mathrm{d}s,\mathrm{d}y) = \int_E \Phi(y) e^{-\kappa\Lambda(y)t} F_0(\mathrm{d}y)$$

and then that, for $\Upsilon\in\mathcal{T}_k\setminus\{\circ\}$,

$$\int_E \Phi(y)\Gamma_t(J_\Upsilon)(\mathrm{d}y) = \int_0^t \int_E \Phi(y) e^{-\kappa\Lambda(y)(t-s)} J_\Upsilon(\mathrm{d}s,\mathrm{d}y).$$

Since $J_\Upsilon(\mathrm{d}s,\mathrm{d}y) = Q(\Gamma_s(J_{\Upsilon_\ell}),\Gamma_s(J_{\Upsilon_r}))(\mathrm{d}y)\mathrm{d}s$ by definition, the result follows.

Step 4. Differentiating (16), we find that for all $\Phi\in C_b(E)$, all $t\ge 0$,

$$\frac{d}{dt}\int_E \Phi(y) F_t^k(\mathrm{d}y) = -\kappa \int_E \Lambda(y)\Phi(y) F_t^k(\mathrm{d}y) + \sum_{\Upsilon\in\mathcal{T}_k\setminus\{\circ\}} \int_E \Phi(y) Q(\Gamma_t(J_{\Upsilon_\ell}),\Gamma_t(J_{\Upsilon_r}))(\mathrm{d}y). \tag{17}$$

The differentiation is easily justified, using that Φ is bounded, that $t\mapsto \int_E \Lambda(y)F_t^k(\mathrm{d}y)$ is locally bounded, as well as $t\mapsto \int_E \Lambda(y)\Gamma_t(J_\Upsilon)(\mathrm{d}y)$ for all $\Upsilon\in\mathcal{T}$, and that for $F,G\in\mathcal{M}(E)$,

$$Q(F,G)(E) = \kappa G(E) \int_E \Lambda(y) F(\mathrm{d}y).$$

Step 5. Here we verify that $\sup_{[0,\infty)} F_t^k(E) \le 1$ and that $\sup_{k\ge 1}\sup_{[0,T]} \int_E \Lambda(y) F_t^k(\mathrm{d}y) < \infty$ for all $T>0$. First observe that if $\Phi \in C_b(E)$ is nonnegative, then

$$\sum_{\Upsilon\in\mathcal{T}_k\setminus\{\circ\}} \int_E \Phi(y) Q(\Gamma_t(J_{\Upsilon_\ell}),\Gamma_t(J_{\Upsilon_r}))(\mathrm{d}y) \le \sum_{\Upsilon_1\in\mathcal{T}_k,\Upsilon_2\in\mathcal{T}_k} \int_E \Phi(y) Q(\Gamma_t(J_{\Upsilon_1}),\Gamma_t(J_{\Upsilon_2}))(\mathrm{d}y)$$
$$= \int_E \Phi(y) Q(F_t^k,F_t^k)(\mathrm{d}y).$$

We used that the map $\Upsilon \mapsto (\Upsilon_\ell,\Upsilon_r)$ is injective from $\mathcal{T}_k\setminus\{\circ\}$ into $\mathcal{T}_k\times\mathcal{T}_k$, as well as the bilinearity of Q. Consequently, by (17),

$$\frac{d}{dt}\int_E \Phi(y) F_t^k(\mathrm{d}y) \le -\kappa \int_E \Lambda(y)\Phi(y) F_t^k(\mathrm{d}y) + \int_E \Phi(y) Q(F_t^k,F_t^k)(\mathrm{d}y) \tag{18}$$
$$= \int_E\int_E \mathcal{B}\Phi(y,y^*) F_t^k(\mathrm{d}y^*) F_t^k(\mathrm{d}y) + \kappa(F_t^k(E)-1)\int_E \Lambda(y)\Phi(y)F_t^k(\mathrm{d}y).$$

For the last equality, we used that for any $F,G\in\mathcal{M}(E)$, we have

$$\int_E\int_E \mathcal{B}\Phi(y,y^*) G(\mathrm{d}y^*) F(\mathrm{d}y) = \int_E \Phi(y) Q(F,G)(\mathrm{d}y) - \kappa G(E)\int_E \Lambda(y)\Phi(y)F(\mathrm{d}y). \tag{19}$$

Applying (18) with $\Phi=1$, we see that

$$\frac{d}{dt}F_t^k(E) \le \kappa(F_t^k(E)-1)\int_E \Lambda(y) F_t^k(\mathrm{d}y).$$

Since $F_0^k(E)=F_0(E)=1$ and since $t\mapsto \int_E \Lambda(y)F_t^k(\mathrm{d}y)$ is locally bounded, we conclude that $F_t^k(E)\le 1$ for all $t\ge 0$ (because $(d/dt)[(F_t^k(E)-1)\exp(-\kappa\int_0^t\int_E \Lambda(y)F_s^k(\mathrm{d}y)\mathrm{d}s)]\le 0$).

Applying next (18) with $\Phi=\Lambda$ and using that $F_t^k(E)\le 1$, we find that

$$\frac{d}{dt}\int_E \Lambda(y)F_t^k(\mathrm{d}y) \le \int_E\int_E \mathcal{B}\Lambda(y,y^*)F_t^k(\mathrm{d}y^*)F_t^k(\mathrm{d}y) \le \kappa(1+e_0)F_t^k(E)\int_E \Lambda(y)F_t^k(\mathrm{d}y),$$

because $\mathcal{B}\Lambda(y,y^*)=\Lambda(y)\int_H[\Phi(h(y,y^*,z))-\Phi(y)]\nu_{y,y^*}(\mathrm{d}z)\le \kappa(1+e_0)\Lambda(y)$, see (13). Since $F_0^k=F_0$ and since, again, $F_t^k(E)\le 1$, we conclude that $\int_E \Lambda(y)F_t^k(\mathrm{d}y)\le [\int_E \Lambda(y)F_0(\mathrm{d}y)]\exp(\kappa(1+e_0)t)$.

Step 6. By Step 5, the series of nonnegative measures $F_t=\sum_{\Upsilon\in\mathcal{T}}\Gamma_t(J_\Upsilon)$ converges, satisfies $F_t(E)\le 1$, and we know that $t\mapsto\int_E\Lambda(y)F_t(\mathrm{d}y)$ is locally bounded. Passing to the limit in the time-integrated version of (17), we find that for all $\Phi\in C_b(E)$, all $t\ge 0$,

$$\int_E \Phi(y)F_t(\mathrm{d}y) = \int_E \Phi(y)F_0(\mathrm{d}y) - \kappa\int_0^t\int_E \Lambda(y)\Phi(y)F_s(\mathrm{d}y)\mathrm{d}s \tag{20}$$
$$+ \sum_{\Upsilon\in\mathcal{T}\setminus\{\circ\}} \int_0^t\int_E \Phi(y) Q(\Gamma_s(J_{\Upsilon_\ell}),\Gamma_s(J_{\Upsilon_r}))(\mathrm{d}y)\mathrm{d}s.$$

To justify the limiting procedure, it suffices to use that $t\mapsto\int_E\Lambda(y)F_t(\mathrm{d}y)$ is locally bounded, as well as $t\mapsto \sum_{\Upsilon\in\mathcal{T}\setminus\{\circ\}} Q(\Gamma_t(J_{\Upsilon_\ell}),\Gamma_t(J_{\Upsilon_r}))(E) = \sum_{\Upsilon_1\in\mathcal{T},\Upsilon_2\in\mathcal{T}} Q(\Gamma_t(J_{\Upsilon_1}),\Gamma_t(J_{\Upsilon_2}))(E)$, which equals $Q(\sum_{\Upsilon_1\in\mathcal{T}}\Gamma_t(J_{\Upsilon_1}),\sum_{\Upsilon_2\in\mathcal{T}}\Gamma_t(J_{\Upsilon_2}))(E) = Q(F_t,F_t)(E)=\kappa F_t(E)\int_E\Lambda(y)F_t(\mathrm{d}y)$. We used that the map $\Upsilon\mapsto(\Upsilon_\ell,\Upsilon_r)$ is bijective from $\mathcal{T}\setminus\{\circ\}$ into $\mathcal{T}\times\mathcal{T}$, as well as the bilinearity of Q.

By the same way, (20) rewrites as

$$\begin{aligned}\int_E \Phi(y)F_t(\mathrm{d}y) &= \int_E \Phi(y)F_0(\mathrm{d}y) - \kappa\int_0^t\int_E \Lambda(y)\Phi(y)F_s(\mathrm{d}y)\mathrm{d}s + \int_0^t\int_E \Phi(y)Q(F_s,F_s)(\mathrm{d}y)\mathrm{d}s\\ &= \int_E \Phi(y)F_0(\mathrm{d}y) + \int_0^t\int_E\int_E \mathcal{B}\Phi(y,y^*)F_s(\mathrm{d}y^*)F_s(\mathrm{d}y)\mathrm{d}s\\ &\quad + \kappa\int_0^t (F_s(E)-1)\Big(\int_E \Lambda(y)\Phi(y)F_s(\mathrm{d}y)\Big)\mathrm{d}s,\end{aligned}$$

see (19). To conclude that $(F_t)_{t\geq 0}$ solves (A), it only remains to verify that $t\mapsto \int_E |v|^\gamma F_t(\mathrm{d}m,\mathrm{d}v)$ is locally bounded, which follows from the fact that $\int_E |v|^\gamma F_t(\mathrm{d}m,\mathrm{d}v)\leq \int_E \Lambda(y)F_t(\mathrm{d}y)$, and that $F_t(E)=1$ for all $t\geq 0$. Applying the previous equation with $\Phi=1$ (for which $\mathcal{B}\Phi=0$), we find that $F_t(E)=1+\int_0^t (F_s(E)-1)\alpha_s\mathrm{d}s$, where $\alpha_s=\kappa\int_E\Lambda(y)F_s(\mathrm{d}y)$ is locally bounded. Hence $F_t(E)=1$ for all $t\geq 0$ by the Gronwall lemma. The proof of Proposition 8 is complete. □

5. Well-posedness of (A)

We have already checked (twice) the existence part of Proposition 3. We now turn to uniqueness. Let us consider two solutions F and G to (A) with $F_0=G_0$. By assumption, we know that $\alpha_t=\int_E\Lambda(y)(F_t+G_t)(\mathrm{d}y)=(1+e_0)\int_E(1+|v|^\gamma)(F_t+G_t)(\mathrm{d}m,\mathrm{d}v)$ is locally bounded. Hence, setting $\epsilon_t^M=\int_E\Lambda(y)\mathbf{1}_{\{\Lambda(y)\geq M\}}(F_t+G_t)(\mathrm{d}y)$, we have $\lim_{M\to\infty}\int_0^t\epsilon_s^M\mathrm{d}s=0$ for all $t\geq 0$.

We use the total variation distance $u_t=||F_t-G_t||_{TV}=\sup\{u_t^\Phi : \Phi\in C_b(E), ||\Phi||_\infty\leq 1\}$ where $u_t^\Phi=\int_E\Phi(y)(F_t-G_t)(\mathrm{d}y)$. We also have $u_t=\int_E|F_t-G_t|(\mathrm{d}y)$, where for μ a finite signed measure on E, $|\mu|=\mu_+ +\mu_-$ with the usual definitions of μ_+ and μ_-.

We fix $\Phi\in C_b(E)$ such that $||\Phi||_\infty\leq 1$ and we use Definition 2 to write

$$\frac{d}{dt}u_t^\Phi=\int_E\int_E \mathcal{B}\Phi(y,y^*)(F_t(\mathrm{d}y^*)F_t(\mathrm{d}y)-G_t(\mathrm{d}y^*)G_t(\mathrm{d}y))=A_t^\Phi+B_t^\Phi,$$

where $A_t^\Phi=\int_E\int_E\mathcal{B}\Phi(y,y^*)(F_t-G_t)(\mathrm{d}y^*)F_t(\mathrm{d}y)$ and $B_t^\Phi=\int_E\int_E\mathcal{B}\Phi(y,y^*)G_t(\mathrm{d}y^*)(F_t-G_t)(\mathrm{d}y)$.

Using only that $|\mathcal{B}\Phi(y,y^*)|\leq 2\kappa||\Phi||_\infty\Lambda(y)\leq 2\kappa\Lambda(y)$, we get

$$A_t^\Phi\leq 2\kappa\int_E\Lambda(y)F_t(\mathrm{d}y)\int_E|F_t-G_t|(\mathrm{d}y^*)\leq 2\kappa\alpha_t||F_t-G_t||_{TV}=2\kappa\alpha_t u_t.$$

We next recall that $||\Phi||_\infty\leq 1$ and that $\int_E G_t(\mathrm{d}y^*)=1$ and we write

$$\begin{aligned}B_t^\Phi &= \int_E\int_E\int_H \Lambda(y)\Phi(h(y,y^*,z))\nu_{y,y^*}(dz)G_t(\mathrm{d}y^*)(F_t-G_t)(\mathrm{d}y) - \kappa\int_E\Lambda(y)\Phi(y)(F_t-G_t)(\mathrm{d}y)\\ &\leq \kappa\int_E\Lambda(y)|F_t-G_t|(\mathrm{d}y)-\kappa\int_E\Lambda(y)\Phi(y)(F_t-G_t)(\mathrm{d}y).\end{aligned}$$

Since now $|F_t-G_t|(\mathrm{d}y)-\Phi(y)(F_t-G_t)(\mathrm{d}y)$ is a nonnegative measure bounded by $2(F_t+G_t)(\mathrm{d}y)$, we may write, for any $M\geq 1$,

$$\begin{aligned}B_t^\Phi &\leq \kappa M\int_E[|F_t-G_t|(\mathrm{d}y)-\Phi(y)(F_t-G_t)(\mathrm{d}y)]+2\kappa\int_E\Lambda(y)\mathbf{1}_{\{\Lambda(y)\geq M\}}(F_t+G_t)(\mathrm{d}y)\\ &= \kappa M u_t-\kappa M u_t^\Phi+2\kappa\epsilon_t^M.\end{aligned}$$

All this proves that $(d/dt)u_t^\Phi\leq 2\kappa\alpha_t u_t+\kappa M u_t-\kappa M u_t^\Phi+2\kappa\epsilon_t^M$, whence

$$\frac{d}{dt}(u_t^\Phi e^{\kappa M t})\leq[2\kappa\alpha_t u_t+\kappa M u_t+2\kappa\epsilon_t^M]e^{\kappa M t}.$$

Integrating in time (recall that $u_0^\Phi = 0$) and taking the supremum over $\Phi \in C_b(E)$ such that $||\Phi||_\infty \le 1$, we find $u_t e^{\kappa M t} \le \int_0^t [(2\kappa\alpha_s + \kappa M)u_s + 2\kappa\epsilon_s^M] e^{\kappa M s} \mathrm{d}s$.

Recall the following generalized Gronwall lemma: if we have three locally bounded nonnegative functions v, g, h such that $v_t \le \int_0^t (h_s v_s + g_s)\mathrm{d}s$ for all $t \ge 0$, then $v_t \le \int_0^t g_s \exp(\int_s^t h_u \mathrm{d}u)\mathrm{d}s$. Applying this result with $v_t = u_t e^{\kappa M t}$, $g_t = 2\kappa\epsilon_t^M e^{\kappa M t}$ and $h_t = 2\kappa\alpha_t + \kappa M$, we get

$$u_t e^{\kappa M t} \le 2\kappa \int_0^t \epsilon_s^M \exp\Big(\kappa M s + 2\kappa \int_s^t \alpha_u \mathrm{d}u + \kappa M(t-s)\Big)\mathrm{d}s,$$

so that $u_t \le 2\kappa \int_0^t \epsilon_s^M \exp(2\kappa \int_s^t \alpha_u \mathrm{d}u)\mathrm{d}s$. Recalling that α is locally bounded and that $\int_0^t \epsilon_s^M \mathrm{d}s$ tends to 0 as $M \to \infty$, we conclude that $u_t = 0$, which was our goal. The proof of Proposition 3 is now complete. □

We end this section with the

Proof of Remark 4. We fix $A \ge 1$ and apply (12) with $\Phi_A(m, v) = |v|^2 \wedge A$, which belongs to $C_b(E)$. With the notation $y = (m, v)$ and $y^* = (m^*, v^*)$, we find

$$\begin{aligned}
\mathcal{B}\Phi_A(y, y^*) =& \Lambda(v) \int_H [|v''(v, v^*, z)|^2 \wedge A - |v|^2 \wedge A] \nu_{y,y^*}(\mathrm{d}z) \\
=& \Lambda(v) q(v, v^*) \int_{\mathbb{S}^2} [|v'(v, v^*, \sigma)|^2 \wedge A - |v|^2 \wedge A] \beta_{v,v^*}(\sigma)\mathrm{d}\sigma \\
=& \kappa(1 + e_0) \frac{|v - v^*|^\gamma}{1 + |v^*|^2} \Big[\kappa^{-1} \int_{\mathbb{S}^2} (|v'(v, v^*, \sigma)|^2 \wedge A)\beta_{v,v^*}(\sigma)\mathrm{d}\sigma - |v|^2 \wedge A\Big] \\
\le& \kappa(1 + e_0) \frac{|v - v^*|^\gamma}{1 + |v^*|^2} \Big[\Big(\kappa^{-1} \int_{\mathbb{S}^2} |v'(v, v^*, \sigma)|^2 \beta_{v,v^*}(\sigma)\mathrm{d}\sigma\Big) \wedge A - |v|^2 \wedge A\Big].
\end{aligned}$$

But a simple computation, recalling (2) and using that

$$\frac{|v - v^*|}{\kappa} \int_{\mathbb{S}^2} \sigma \beta_{v,v^*}(\sigma)\mathrm{d}\sigma = \frac{|v - v^*|}{\kappa} \int_{\mathbb{S}^2} \sigma b(\langle \tfrac{v - v^*}{|v - v^*|}, \sigma\rangle)\mathrm{d}\sigma = c(v - v^*)$$

where $c = 2\pi\kappa^{-1} \int_0^\pi \sin\theta \cos\theta b(\cos\theta)\mathrm{d}\theta \in [-1, 1]$ (recall (4)) shows that

$$\kappa^{-1} \int_{\mathbb{S}^2} |v'(v, v^*, \sigma)|^2 \beta_{v,v^*}(\sigma)\mathrm{d}\sigma = \frac{1 + c}{2}|v|^2 + \frac{1 - c}{2}|v^*|^2 = (1 - \alpha)|v|^2 + \alpha|v^*|^2, \tag{21}$$

where $\alpha = (1 - c)/2 \in [0, 1]$. Hence,

$$\mathcal{B}\Phi_A(y, y^*) + \mathcal{B}\Phi_A(y^*, y) \le \kappa(1 + e_0)|v - v^*|^\gamma G_A(|v|^2, |v^*|^2),$$

where $G_A(x, x^*) = (1+x^*)^{-1}[((1-\alpha)x + \alpha x^*) \wedge A - x \wedge A] + (1+x)^{-1}[((1-\alpha)x^* + \alpha x) \wedge A - x^* \wedge A]$. One can check that $G_A(x, x^*) \le 0$ if $x \vee x^* \le A$ and it always holds true that $G_A(x, x^*) \le (1 + x^*)^{-1}\alpha x^* + (1 + x)^{-1}\alpha x \le 2$. At the end, $G_A(x, x^*) \le 2(\mathbf{1}_{\{x > A\}} + \mathbf{1}_{\{x^* > A\}})$. Consequently, applying (12) and using a symmetry argument,

$$\begin{aligned}
\int_E (|v|^2 \wedge A) F_t(\mathrm{d}y) =& \int_E (|v|^2 \wedge A) F_0(\mathrm{d}y) + \int_0^t \int_E \int_E \mathcal{B}\Phi_A(y, y^*) F_s(\mathrm{d}y^*) F_s(\mathrm{d}y)\mathrm{d}s \\
=& \int_E (|v|^2 \wedge A) F_0(\mathrm{d}y) + \frac{1}{2} \int_0^t \int_E \int_E [\mathcal{B}\Phi_A(y, y^*) + \mathcal{B}\Phi_A(y^*, y)] F_s(\mathrm{d}y^*) F_s(\mathrm{d}y)\mathrm{d}s \\
\le& \int_E |v|^2 F_0(\mathrm{d}y) + \kappa(1 + e_0) \int_0^t \int_E \int_E |v - v^*|^\gamma [\mathbf{1}_{\{|v|^2 > A\}} + \mathbf{1}_{\{|v^*|^2 > A\}}] F_s(\mathrm{d}y^*) F_s(\mathrm{d}y)\mathrm{d}s.
\end{aligned}$$

Letting $A \to \infty$ and using that $\int_0^t \int_E \int_E |v - v^*|^\gamma F_s(\mathrm{d}y^*) F_s(\mathrm{d}y) \mathrm{d}s < \infty$ (which follows from the fact that $\sup_{[0,t]} \int |v|^\gamma F_s(\mathrm{d}y) < \infty$), we conclude that $\int_E |v|^2 F_t(\mathrm{d}m, \mathrm{d}v) \le \int_E |v|^2 F_0(\mathrm{d}m, \mathrm{d}v)$. □

6. Relation between (A) and the Boltzmann equation

It remains to prove Proposition 5. In the whole section, we consider the solution F to (A) starting from some $F_0 \in \mathcal{P}(E)$ such that $\int_E [|v|^\gamma + m(1+|v|^{2+2\gamma})] F_0(\mathrm{d}y) < \infty$. We define the nonnegative measure f_t on $\mathbb{R}^3$ by $f_t(A) = \int_E m \mathbf{1}_{\{v \in A\}} F_t(\mathrm{d}y)$ for all $A \in \mathcal{B}(\mathbb{R}^3)$ and we assume that $f_0 \in \mathcal{P}(\mathbb{R}^3)$ and that $\int_{\mathbb{R}^3} |v|^2 f_0(\mathrm{d}v) = e_0$, where e_0 was used in Subsection 2.2 to build the coefficients of (A). We want to prove that $f = (f_t)_{t \ge 0}$ is a weak solution to (1).

The main difficulty is to establish properly the following estimate, of which the proof is postponed at the end of the section.

Lemma 9. *For any $T > 0$, $\sup_{[0,T]} \int_E m(1+|v|^{2+\gamma}) F_t(\mathrm{d}m, \mathrm{d}v) < \infty$.*

Next, we handle a few preliminary computations.

Remark 10. *(i) For all $\Phi \in C(E)$ of the form $\Phi(m,v) = m\phi(v)$ with $\phi \in C(\mathbb{R}^3)$, using the notation $y = (m,v)$ and $y^* = (m^*, v^*)$, it holds that*

$$\mathcal{B}\Phi(y, y^*) = m m^* \mathcal{A}\phi(v, v^*) + \kappa m \Lambda(v) \phi(v) \Big(\frac{m^*(1+|v^*|^2)}{1+e_0} - 1 \Big).$$

(ii) Assume furthermore that there is $\alpha \ge 0$ such that for all $v \in \mathbb{R}^3$, $|\phi(v)| \le C(1+|v|^{2+\alpha})$. Then $|\mathcal{B}\Phi(y,y^*)| \le C[\Lambda(y) + \Lambda(y^*)][1 + m(1+|v|^{2+\alpha})][1 + m^*(1+|v^*|^{2+\alpha})]$.

Proof. For (i), it suffices to write

$$\mathcal{B}\Phi(y,y^*) = \Lambda(v) \int_H \Big[\frac{m m^*(1+|v^*|^2)}{1+e_0} \phi(v''(v,v^*,z)) - m\phi(v) \Big] \nu_{y,y^*}(\mathrm{d}z) = \mathcal{B}^1\Phi(y,y^*) + \mathcal{B}^2\Phi(y,y^*),$$

where

$$\begin{aligned}
\mathcal{B}^1\Phi(y,y^*) =& \Lambda(v) \frac{m m^*(1+|v^*|^2)}{1+e_0} \int_H \Big[\phi(v''(v,v^*,z)) - \phi(v) \Big] \nu_{y,y^*}(\mathrm{d}z) \\
=& \Lambda(v) q(v, v_*) \frac{m m^*(1+|v^*|^2)}{1+e_0} \int_{\mathbb{S}^2} \Big[\phi(v'(v,v^*,\sigma)) - \phi(v) \Big] \beta_{v,v^*}(\sigma) \mathrm{d}\sigma,
\end{aligned}$$

which equals $m m^* \mathcal{A}\phi(v,v^*)$, and where

$$\mathcal{B}^2\Phi_A(y,y^*) = \kappa \Lambda(v) \phi(v) \Big[\frac{m m^*(1+|v^*|^2)}{1+e_0} - m \Big].$$

For point (ii), we first observe that $|\mathcal{A}\phi(v,v^*)| \le C|v-v^*|^\gamma (1 + |v|^{2+\alpha} + |v^*|^{2+\alpha})$, because $|v'(v,v^*,\sigma)| \le |v| + |v^*|$, see (2). Using next that $|v-v^*|^\gamma \le |v|^\gamma + |v^*|^\gamma$ and that $\Lambda(y) = \Lambda(v) = (1+e_0)(1+|v|^\gamma)$, we thus find

$$|\mathcal{B}\Phi(y,y^*)| \le C m m^* (|v|^\gamma + |v^*|^\gamma)(1 + |v|^{2+\alpha} + |v^*|^{2+\alpha}) + C\Lambda(v) m (1+|v|^{2+\alpha})(1 + m^*(1+|v^*|^2)),$$

from which the conclusion easily follows. □

We now can give the

Proof of Proposition 5. For any $\phi \in C(\mathbb{R}^3)$ such that $|\phi(v)| \le C(1+|v|^2)$, we can apply (12) with $\Phi(m,v) = m\phi(v)$. To check it it properly, first apply (12) with $\Phi(m,v) = [m \wedge A]\phi_A(v)$ with $\phi_A(v) = \phi(v) \wedge A \vee (-A)$ and let $A \to \infty$. This essentially relies on the facts that

• $|\mathcal{B}\Phi(y,y^*)| \le C[\Lambda(y)+\Lambda(y^*)][1+m(1+|v|^2)][1+m^*(1+|v^*|^2)]$ by Remark 10-(ii) (with $\alpha=0$), whence $|\mathcal{B}\Phi(y,y^*)| \le C[1+|v|^\gamma+m(1+|v|^{2+\gamma})][1+|v^*|^\gamma+m^*(1+|v^*|^{2+\gamma})]$ and

• $t \mapsto \int_E [1+|v|^\gamma+m(1+|v|^{2+\gamma})]F_t(\mathrm{d}m,\mathrm{d}v) < \infty$ is locally bounded by Lemma 9 and Definition 2.

So, applying (12) and using the formula of Remark 10-(i), we find

$$\int_E m\phi(v)F_t(\mathrm{d}y) = \int_E m\phi(v)F_0(\mathrm{d}y) + \int_0^t \int_E \int_E mm^* \mathcal{A}\phi(v,v^*)F_s(\mathrm{d}y^*)F_s(\mathrm{d}y)\mathrm{d}s \\ + \kappa \int_0^t \int_E \int_E m\Lambda(v)\phi(v)\Big(\frac{m^*(1+|v^*|^2)}{1+e_0}-1\Big)F_s(\mathrm{d}y^*)F_s(\mathrm{d}y)\mathrm{d}s.$$

This precisely rewrites, by definition of f_t,

$$\int_{\mathbb{R}^3} \phi(v)f_t(\mathrm{d}v) = \int_{\mathbb{R}^3} \phi(v)f_0(\mathrm{d}v) + \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathcal{A}\phi(v,v^*)f_s(\mathrm{d}v^*)f_s(\mathrm{d}v)\mathrm{d}s \\ + \kappa \int_0^t (\Theta_s-1)\Big(\int_{\mathbb{R}^3} \Lambda(v)\phi(v)f_s(\mathrm{d}v)\Big)\mathrm{d}s, \tag{22}$$

where $\Theta_t = (1+e_0)^{-1}\int_{\mathbb{R}^3}(1+|v|^2)f_t(\mathrm{d}v)$.

But, with $\phi(v) = (1+e_0)^{-1}(1+|v|^2)$, recalling (21), it holds that

$$\mathcal{A}\phi(v,v^*)+\mathcal{A}\phi(v^*,v) = \frac{\kappa}{1+e_0}\big[(1-\alpha)|v|^2+\alpha|v^*|^2-|v|^2+(1-\alpha)|v^*|^2+\alpha|v|^2-|v^*|^2\big] = 0. \tag{23}$$

Hence applying (22) and using a symmetry argument, we find

$$\Theta_t = 1+\kappa\int_0^t (\Theta_s-1)\Big(\int_{\mathbb{R}^3}\Lambda(v)\phi(v)f_s(\mathrm{d}v)\Big)\mathrm{d}s.$$

Hence $\Theta_t = 1$ for all $t\ge 0$ by the Gronwall Lemma, because $\Theta_t = (1+e_0)^{-1}\int_E m(1+|v|^2)F_t(\mathrm{d}m,\mathrm{d}v)$ and $\int_{\mathbb{R}^3}\Lambda(v)\phi(v)f_t(\mathrm{d}v) = \int_E m(1+|v|^2)(1+|v|^\gamma)F_t(\mathrm{d}m,\mathrm{d}v)$ are locally bounded by Lemma 9.

Coming back to (22), we thus see that for all $\phi \in C_b(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \phi(v)f_t(\mathrm{d}v) = \int_{\mathbb{R}^3} \phi(v)f_0(\mathrm{d}v) + \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathcal{A}\phi(v,v^*)f_s(\mathrm{d}v^*)f_s(\mathrm{d}v)\mathrm{d}s.$$

To complete the proof, it only remains to prove that $f_t(\mathbb{R}^3)=1$ for all $t\ge 0$, which follows from the choice $\phi(v)=1$ (for which $\mathcal{A}\phi(v,v_*)=0$), and to check that $\int_{\mathbb{R}^3}|v|^2 f_t(\mathrm{d}v) = e_0$ for all $t\ge 0$, which holds true because $\int_{\mathbb{R}^3}|v|^2 f_t(\mathrm{d}v) = (1+e_0)\Theta_t - f_t(\mathbb{R}^3)$. □

It only remains to prove Lemma 9.

Proof of Lemma 9. The proof relies on the series expansion $F_t = \sum_{\Upsilon\in\mathcal{T}}\Gamma_t(J_\Upsilon(F_0))$, see Proposition 8. We write $J_\Upsilon = J_\Upsilon(F_0)$ for simplicity. We will make use of the functions $\Phi_0(m,v) = m(1+|v|^2)/(1+e_0)$, $\Phi_1(m,v) = m(1+|v|^{2+\gamma})$ and $\Phi_2(m,v) = m(1+|v|^{2+2\gamma})$.

Step 1. Here we verify that for all $\Upsilon\in\mathcal{T}$, all $t\ge 0$, $D_\Upsilon(t) = \int_0^t\int_E \Phi_2(y)J_\Upsilon(\mathrm{d}s,\mathrm{d}y) < \infty$. We proceed by induction as in the proof of Proposition 8, Step 1. First, $D_\circ(t) = \int_E \Phi_2(y)F_0(\mathrm{d}y) < \infty$ by assumption. Next, we fix $t\ge 0$, $\Upsilon\in\mathcal{T}\setminus\{\circ\}$, we assume by induction that $D_{\Upsilon_\ell}(t)<\infty$ and $D_{\Upsilon_r}(t)<\infty$ and prove that $D_\Upsilon(t)<\infty$. We start from

$$D_\Upsilon(t) = \int_0^t\int_E\int_E\int_H\int_0^s\int_0^s \Phi_2(h(y,y^*,z))\Lambda(y)\nu_{y,y^*}(\mathrm{d}z) \\ J_{\Upsilon_r}(\mathrm{d}u^*,\mathrm{d}y^*)e^{-\kappa\Lambda(y^*)(s-u^*)}J_{\Upsilon_\ell}(\mathrm{d}u,\mathrm{d}y)e^{-\kappa\Lambda(y)(s-u)}\mathrm{d}s.$$

But we see from Remark 10-(ii) (with $\alpha = 2\gamma$) that

$$\begin{aligned}\Lambda(y)\int_H \Phi_2(h(y,y^*,z))\nu_{y,y^*}(\mathrm{d}z) =& \mathcal{B}\Phi_2(y,y^*) + \kappa\Lambda(y)\Phi_2(y)\\ \leq& C[\Lambda(y)+\Lambda(y^*)](1+\Phi_2(y))(1+\Phi_2(y^*)).\end{aligned}$$

Together with the Fubini theorem, this gives us

$$\begin{aligned}D_\Upsilon(t) \leq& C\int_E\int_0^t\int_E\int_0^t (1+\Phi_2(y))(1+\Phi_2(y^*))J_{\Upsilon_r}(\mathrm{d}u^*,\mathrm{d}y^*)J_{\Upsilon_\ell}(\mathrm{d}u,\mathrm{d}y)\\ &\qquad\qquad \int_{u\vee u^*}^t [\Lambda(y)+\Lambda(y^*)]e^{-\kappa\Lambda(y^*)(s-u^*)}e^{-\kappa\Lambda(y)(s-u)}\mathrm{d}s\\ \leq& C\int_E\int_0^t\int_E\int_0^t (1+\Phi_2(y))(1+\Phi_2(y^*))J_{\Upsilon_r}(\mathrm{d}u^*,\mathrm{d}y^*)J_{\Upsilon_\ell}(\mathrm{d}u,\mathrm{d}y)\\ =& C[J_{\Upsilon_\ell}([0,t]\times E)+D_{\Upsilon_\ell}(t)][J_{\Upsilon_r}([0,t]\times E)+D_{\Upsilon_r}(t)].\end{aligned}$$

We conclude by induction and since we already know from Step 1 of the proof of Proposition 8 that $J_\Upsilon([0,t]\times E) \leq \int_0^t\int_E \Lambda(y)J_\Upsilon(\mathrm{d}s,\mathrm{d}y) < \infty$ for all $\Upsilon \in \mathcal{T}$.

Step 2. For $k \in \mathbb{N}^*$, we define $F_t^k = \sum_{\Upsilon\in\mathcal{T}_k}\Gamma_t(J_\Upsilon(F_0))$ as in the proof of Proposition 8, Step 3. We know that $F_0^k = F_0$ and that for all nonnegative $\Phi \in C_b(E)$, see (18) and recall that $F_t^k(E) \leq 1$,

$$\int_E \Phi(y)F_t^k(\mathrm{d}y) \leq \int_E \Phi(y)F_0(\mathrm{d}y) + \int_0^t\int_E\int_E \mathcal{B}\Phi(y,y^*)F_s^k(\mathrm{d}y^*)F_s^k(\mathrm{d}y)\mathrm{d}s. \tag{24}$$

Also, we immediately deduce from Step 1 that $t \mapsto \int_E \Phi_2(y)F_t^k(\mathrm{d}y)$ is locally bounded, as well as $t \mapsto \int_E \Lambda(y)F_t^k(\mathrm{d}y)$, see Step 2 of the proof of Proposition 8. It is then easy to extend (24) to any function $\Phi \in C(E)$ of the form $\Phi(m,v) = m\phi(v)$, with $0 \leq \phi(v) \leq C(1+|v|^{2+\gamma})$. This follows from the fact that, by Remark 10-(ii) (with $\alpha = \gamma$),

$$|\mathcal{B}\Phi(y,y^*)| \leq C[\Lambda(y)+\Lambda(y^*)](1+\Phi_1(y))(1+\Phi_1(y^*)) \leq C(1+\Lambda(y)+\Phi_2(y))(1+\Lambda(y^*)+\Phi_2(y^*)).$$

Step 3. We now verify that $\int_E \Phi_0(y)F_t^k(\mathrm{d}y) \leq 1$ for all $t \geq 0$. To this end, we apply (24) with $\Phi = \Phi_0$ for which, by Remark 10-(i),

$$\mathcal{B}\Phi_0(y,y^*) = mm^*\mathcal{A}\phi(v,v^*) + \kappa\Lambda(y)\Phi_0(y)[\Phi_0(y^*)-1],$$

where $\phi(v) = (1+|v|^2)/(1+e_0)$. Using that $\mathcal{A}\phi(v,v^*)+\mathcal{A}\phi(v^*,v) = 0$ (recall (23)) and a symmetry argument, we find that

$$\begin{aligned}\int_E \Phi_0(y)F_t^k(\mathrm{d}y) \leq& \int_E \Phi_0(y)F_0(\mathrm{d}y) + \int_0^t\int_E\int_E \kappa\Lambda(y)\Phi_0(y)[\Phi_0(y^*)-1]F_s^k(\mathrm{d}y^*)F_s^k(\mathrm{d}y)\mathrm{d}s\\ =& 1 + \int_0^t \Big(\int_E \kappa\Lambda(y)\Phi_0(y)F_s^k(\mathrm{d}y)\Big)\Big(\int_E \Phi_0(y)F_s^k(\mathrm{d}y) - 1\Big)\mathrm{d}s.\end{aligned}$$

Setting $u_t = \int_E \Phi_0(y)F_t^k(\mathrm{d}y) - 1$ and $\alpha_t = \int_E \kappa\Lambda(y)\Phi_0(y)F_t^k(\mathrm{d}y) \geq 0$, we know that u and α are locally bounded by Step 2 (because $\Phi_0(y)+\Lambda(y)\Phi_0(y) \leq C\Phi_2(y)$) and that $u_t \leq \int_0^t \alpha_s u_s\mathrm{d}s$. This implies that $u_t \leq 0$ for all $t \geq 0$, which was our goal.

Step 4. We finally apply (24) with $\Phi = \Phi_1$. By Remark 10, we see that, with $\phi(v) = 1+|v|^{2+\gamma}$,

$$\mathcal{B}\Phi_1(y,y^*) = mm^*\mathcal{A}\phi(v,v^*) + \kappa\Lambda(y)\Phi_1(y)(\Phi_0(y^*)-1).$$

Hence

$$\begin{aligned}\int_E \Phi_1(y)F_t^k(\mathrm{d}y) &\le \int_E \Phi_1(y)F_0(\mathrm{d}y) + \int_0^t \int_E\int_E mm^*\mathcal{A}\phi(v,v^*)F_s^k(\mathrm{d}y^*)F_s^k(\mathrm{d}y)\mathrm{d}s\\ &\quad + \int_0^t \int_E\int_E \kappa\Lambda(y)\Phi_1(y)[\Phi_0(y^*)-1]F_s^k(\mathrm{d}y^*)F_s^k(\mathrm{d}y)\mathrm{d}s\\ &\le \int_E \Phi_1(y)F_0(\mathrm{d}y) + \int_0^t \int_E\int_E mm^*\mathcal{A}\phi(v,v^*)F_s^k(\mathrm{d}y^*)F_s^k(\mathrm{d}y)\mathrm{d}s\end{aligned}$$

by Step 3. We next recall a Povzner lemma [14] in the version found in [12, Lemma 2.2-(i)]: for $\alpha > 0$, setting $\phi_\alpha(v) = |v|^{2+\alpha}$, there is a $C_\alpha > 0$ such that for all $v, v^* \in \mathbb{R}^3$, $\mathcal{A}\phi_\alpha(v,v^*) + \mathcal{A}\phi_\alpha(v^*,v) \le C_\alpha|v-v^*|^\gamma(|v||v^*|)^{1+\alpha/2}$. Actually, the result of [12] is much stronger. Since $\phi = 1+\phi_\gamma$ and since $\mathcal{A}1 = 0$, we conclude that

$$\begin{aligned}\mathcal{A}\phi(v,v^*) + \mathcal{A}\phi(v^*,v) &\le C|v-v^*|^\gamma(|v||v^*|)^{1+\gamma/2}\\ &\le C|v|^{1+3\gamma/2}|v^*|^{1+\gamma/2} + C|v|^{1+\gamma/2}|v^*|^{1+3\gamma/2}\\ &\le C(1+|v|^{2+\gamma})(1+|v^*|^2) + C(1+|v|^2)(1+|v^*|^{2+\gamma}),\end{aligned}$$

so that

$$mm^*[\mathcal{A}\phi(v,v^*) + \mathcal{A}\phi(v^*,v)] \le C[\Phi_1(y)\Phi_0(y^*) + \Phi_1(y^*)\Phi_0(y)]$$

Finally, using twice a symmetry argument,

$$\begin{aligned}\int_E \Phi_1(y)F_t^k(\mathrm{d}y) &\le \int_E \Phi_1(y)F_0(\mathrm{d}y) + C\int_0^t \int_E\int_E \Phi_1(y)\Phi_0(y^*)F_s^k(\mathrm{d}y^*)F_s^k(\mathrm{d}y)\mathrm{d}s\\ &\le \int_E \Phi_1(y)F_0(\mathrm{d}y) + C\int_0^t \int_E \Phi_1(y)F_s^k(\mathrm{d}y)\mathrm{d}s\end{aligned}$$

by Step 3 again. Hence $\int_E \Phi_1(y)F_t^k(\mathrm{d}y) \le e^{Ct}\int_E \Phi_1(y)F_0(\mathrm{d}y)$ by the Gronwall lemma. It then suffices to let k increase to infinity, by monotone convergence, to complete the proof. □

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