# STATISTICAL INFERENCE VERSUS MEAN FIELD LIMIT FOR HAWKES PROCESSES

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ABSTRACT. We consider a population of $N$ individuals, of which we observe the number of *actions* until time $t$. For each couple of individuals $(i, j)$, $j$ may or not *influence* $i$, which we model by i.i.d. Bernoulli($p$)-random variables, for some unknown parameter $p \in (0, 1]$. Each individual acts *autonomously* at some unknown rate $\mu > 0$ and acts *by mimetism* at some rate proportional to the sum of some function $\varphi$ of the ages of the actions of the individuals which influence him. The function $\varphi$ is unknown but assumed, roughly, to be decreasing and with fast decay. The goal of this paper is to estimate $p$, which is the main characteristic of the *graph of interactions*, in the asymptotic $N \to \infty$, $t \to \infty$. The main issue is that the mean field limit (as $N \to \infty$) of this model is unidentifiable, in that it only depends on the parameters $\mu$ and $p\varphi$. Fortunately, this mean field limit is not valid for large times. We distinguish the subcritical case, where, roughly, the mean number $m_t$ of actions per individual increases linearly and the supercritical case, where $m_t$ increases exponentially. Although the nuisance parameter $\varphi$ is non-parametric, we are able, in both cases, to estimate $p$ without estimating $\varphi$ in a nonparametric way, with a precision of order $N^{-1/2} + N^{1/2}m_t^{-1}$, up to some arbitrarily small loss. We explain, using a Gaussian toy model, the reason why this rate of convergence might be (almost) optimal.

## 1. Introduction and main results

1.1. **Setting.** We consider some unknown parameters $p \in (0, 1]$, $\mu > 0$ and $\varphi : [0, \infty) \mapsto [0, \infty)$. For $N \geq 1$, we consider an i.i.d. family $(\pi^i(dt, dz))_{i=1,\dots,N}$ of Poisson measures on $[0, \infty) \times [0, \infty)$ with intensity measure $dtdz$, independent of an i.i.d. family $(\theta_{ij})_{i,j=1,\dots,N}$ of Bernoulli($p$)-distributed random variables. We also consider the system of equations, for $i = 1, \dots, N$,

$$(1) \qquad Z_t^{i,N} = \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \lambda_s^{i,N}\}} \pi^i(ds, dz) \quad \text{where} \quad \lambda_t^{i,N} = \mu + \frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^{t-} \varphi(t-s) dZ_s^{j,N}.$$

Here and in the whole paper, $\int_0^t$ means $\int_{[0,t]}$ and $\int_0^{t-}$ means $\int_{[0,t)}$. The solution $((Z_t^{i,N})_{t\geq 0})_{i=1,\dots,N}$ is a family of $N$ counting processes (that is, a.s. integer-valued, càdlàg and non-decreasing). The following well-posedness result is more or less well-known, see e.g. Brémaud-Massoulié [9] and [13] (we will apply directly the latter reference).

**Proposition 1.** *Assume that $\varphi$ is locally integrable and fix $N \geq 1$. The system* (1) *has a unique càdlàg $(\mathcal{F}_t)_{t\geq 0}$-adapted solution $((Z_t^{i,N})_{t\geq 0})_{i=1,\dots,N}$ such that $\sum_{i=1}^N \mathbb{E}[Z_t^{i,N}] < \infty$ for all $t \geq 0$, where $\mathcal{F}_t = \sigma(\pi^i(A) : A \in \mathcal{B}([0, t] \times [0, \infty)), i = 1, \dots, N) \vee \sigma(\theta_{ij} : i, j = 1, \dots, N)$.*

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Let us provide a brief heuristic description of this process. We have $N$ individuals and $Z_t^{i,N}$ stands for the number of actions of the $i$-th individual until $t$. We say that $j$ *influences* $i$ if and only if $\theta_{ij} = 1$ (with possibly $i = j$). Each individual $i$ acts, at time $t$, with rate $\lambda_t^{i,N}$. In other words, each individual has an *autonomous rate of action* $\mu$ as well as a *subordinate rate of action* $N^{-1}\sum_{j=1}^N \theta_{ij}\int_0^t \varphi(t-s)dZ_s^{j,N}$, which depends on the number of actions of the individuals that influence him, with a weight $N^{-1}$ and taking into account the age of these actions through $\varphi$. If for example $\varphi = a\mathbf{1}_{[0,K]}$, then the subordinate rate of action of $i$ is simply $a/N$ times the total number of actions, during $[t-K,t]$, of all the individuals that influence him.

As is well-known, a phase-transition occurs for such a model, see Hawkes-Oakes [18] (or [13] for such considerations on large networks): setting $\Lambda = \int_0^\infty \varphi(t)dt$,

- in the subcritical case where $\Lambda p < 1$, we will see that $Z_t^{1,N}$ increases linearly with time, at least on the event where the family $(\theta_{ij})_{i,j=1,\dots,N}$ behaves reasonably;
- in the supercritical case where $\Lambda p > 1$, we will see that $Z_t^{1,N}$ increases exponentially fast with time, at least on the event where the family $(\theta_{ij})_{i,j=1,\dots,N}$ behaves reasonably.

The limit theorems, and thus the statistical inference, completely differ in both cases, so that the present paper contains essentially two independent parts.

We will not study the critical case where $\Lambda p = 1$ because it is a very particular case. However, it would be very interesting to understand what happens *near* the critical case. Our results say nothing about this problem.

1.2. **Assumptions.** Recalling that $\Lambda = \int_0^\infty \varphi(s)ds$, we will work under one of the two following conditions: either for some $q \geq 1$,

$$(H(q)) \qquad \mu \in (0,\infty), \quad \Lambda p \in (0,1) \quad \text{and} \quad \int_0^\infty s^q \varphi(s)ds < \infty$$

or

$$(A) \qquad \mu \in (0,\infty), \quad \Lambda p \in (1,\infty) \quad \text{and} \quad \int_0^t |d\varphi(s)| \text{ increases at most polynomially.}$$

In many applications, $\varphi$ is smooth and has a fast decay, so that, except in the critical case, either $H(q)$ is satisfied for all $q \geq 1$ or $A$ is satisfied.

1.3. **References and fields of application.** Hawkes processes have been introduced by Hawkes [17] and Oakes-Hawkes [18] have found a noticeable representation of such processes in terms of Galton-Watson trees. Since then, there has been a huge literature on Hawkes processes, see e.g. Daley and Vere-Jones [12] for an introduction, Massoulié [24], Brémaud-Massoulié [9] and [13] for stability results, Brémaud-Nappo-Torrisi [10], Zhu [35, 36] and [3] for limit theorems, etc. Hawkes processes are used in various fields of applications:

- earthquake replicas in seismology, see Helmstetter-Sornette [19], Kagan [23], Ogata [26], Bacry-Muzy [5],
- spike trains for brain activity in neuroscience, see Grün et al. [15], Okatan et al. [27], Pillow et al. [28], Reynaud et al. [31, 32],
- genome analysis, see Reynaud-Schbath [30],
- various fields of mathematical finance, see Ait-Sahalia et al. [1], Bauwens-Hautsch [6], Hewlett [20], Bacry et al. [2], Bacry-Muzy [4, 5],
- social networks interactions, see Blundell et al. [8] and Zhou et al. [34].

Concerning the statistical inference for Hawkes processes, only the case of fixed finite dimension $N$ has been studied, to our knowledge, in the asymptotic $t \to \infty$ (for possibly more general shapes of interaction). Some parametric and nonparametric estimation procedures for $\mu$ and $\varphi$ have been proposed, with or without rigorous proofs. Let us mention Ogata [25], Bacry-Muzzy [5], [2], the various recent results of Hansen et al. [16] and Reynaud et al. [30, 31, 32], as well as the Bayesian study of Rasmussen [29].

1.4. **Goals and motivation.** In many applications, the number of individuals is very large (think of neurons, financial agents or of social networks). Then we need some estimators in the asymptotic where $N$ and $t$ tend simultaneously to infinity. This problem seems to be completely open.

We assume that we observe $(Z^{i,N}_s)_{i=1,\dots,N,s\in[0,t]}$ (or, for convenience, $(Z^{i,N}_s)_{i=1,\dots,N,s\in[0,2t]}$), that is all the actions of the individuals on some (large) time interval.

In our point of view, we only observe the activity of the individuals, we do not know the *graph of interactions*. A very similar problem was studied in [32], although in fixed finite dimension $N$. Our goal is to estimate $p$, which can be seen as the main characteristic of the graph of interactions, since it represents the proportion of open edges. We consider $\mu$ and $\varphi$ as nuisance parameters, although this is debatable. In the supercritical case, we will be able to estimate $p$ without estimating $\mu$ nor $\varphi$. In the subcritical case, we will be able to recover $p$ estimating only $\mu$ and the integral $\Lambda$ of $\varphi$. In any case, we will *not* need to provide a nonparametric estimation of $\varphi$, and we believe it is a very good point: it would require regularity assumptions and would complicate a lot the study.

The main goal of this paper is to provide the basic tools for the statistical estimation of Hawkes processes when both the graph size and the observation time increase. Of course, this is only a toy model and we have no precise idea of real world applications, although we can think e.g. of neurons spiking: they are clearly numerous (so $N$ is large), we can only observe their activities (each time they spike), and we would like to have an idea of the graph of interactions. See again [32] for a more convincing biological background. Think also of financial agents: they are also numerous, we can observe their actions (each time they buy or sell a product), and we would like to recover the interaction graph.

1.5. **Mean field limit.** We quickly describe the expected *chaotic* behavior of $((Z^{i,N}_t)_{t\geq 0})_{i=1,\dots,N}$ as $N \to \infty$. We refer to Sznitman [33] for an introduction to *propagation of chaos*. Extending the method of [13, Theorem 8], it is not hard to check, assuming that $\int_0^\infty \varphi^2(s)ds < \infty$, that for each given $k \geq 1$ and $T > 0$, the sample $((Z^{i,N}_t)_{t\in[0,T]})_{i=1,\dots,k}$ goes in law, as $N \to \infty$, to a family $((Y^i_t)_{t\in[0,T]})_{i=1,\dots,k}$ of i.i.d. inhomogeneous Poisson processes with intensity $(\lambda_t)_{t\geq 0}$, unique locally bounded nonnegative solution to $\lambda_t = \mu + \int_0^t p\varphi(t-s)\lambda_s ds$.

On the one hand, approximate independence is of course a good point for statistical inference. On the other hand, the mean-field limit (i.e. the $(Y^i_t)_{t\geq 0}$'s) depends on $p$ and $\varphi$ only through $(\lambda_t)_{t\geq 0}$ and thus through $p\varphi$, which is a negative point: the mean-field limit is *unidentifiable*. The situation is however not hopeless because roughly, the mean-field limit *does not* hold true for the whole sample $(Z^{i,N})_{i=1,\dots,N}$ and is less and less true as time becomes larger and larger.

1.6. **Main result in the subcritical case.** For $N \geq 1$ and for $((Z^{i,N}_t)_{t\geq 0})_{i=1,\dots,N}$ the solution to (1), we introduce $\bar Z^N_t = N^{-1}\sum_{i=1}^N Z^{i,N}_t$. We mention in the following remark, that we will prove later, that the number of actions per individual increases linearly in the subcritical case.

**Remark 2.** *Assume $H(1)$. Then for all $\varepsilon > 0$,*

$$\lim_{(N,t)\to(\infty,\infty)} \Pr\Big(\Big|\frac{\bar Z^N_t}{t} - \frac{\mu}{1-\Lambda p}\Big| \geq \varepsilon\Big) = 0.$$

We next introduce

$$\mathcal{E}^N_t = \frac{\bar Z^N_{2t} - \bar Z^N_t}{t}, \qquad \mathcal{V}^N_t = \sum_{i=1}^N \Big(\frac{Z^{i,N}_{2t} - Z^{i,N}_t}{t} - \mathcal{E}^N_t\Big)^2 - \frac{N}{t}\mathcal{E}^N_t,$$

$$\mathcal{W}^N_{\Delta,t} = 2\mathcal{Z}^N_{2\Delta,t} - \mathcal{Z}^N_{\Delta,t}, \quad \text{where} \quad \mathcal{Z}^N_{\Delta,t} = \frac{N}{t}\sum_{k=t/\Delta+1}^{2t/\Delta}\Big(\bar Z^N_{k\Delta} - \bar Z^N_{(k-1)\Delta} - \Delta\mathcal{E}^N_t\Big)^2.$$

In the last expression, $\Delta \in (0,t)$ is required to be such that $t/(2\Delta) \in \mathbb{N}^*$.

**Theorem 3.** *Assume $H(q)$ for some $q > 3$. For $t \geq 1$, put $\Delta_t = t/(2\lfloor t^{1-4/(q+1)}\rfloor)$: it holds that $t/(2\Delta_t) \in \mathbb{N}^*$ and that $\Delta_t \sim t^{4/(q+1)}/2$ as $t \to \infty$. There is a constant $C$ depending only on $p$, $\mu$, $\varphi$ and $q$ such that for all $\varepsilon \in (0,1)$, all $N \geq 1$, all $t \geq 1$,*

$$\Pr\Big(\Big|\mathcal{E}^N_t - \frac{\mu}{1-\Lambda p}\Big| \geq \varepsilon\Big) \leq \frac{C}{\varepsilon}\Big(\frac{1}{N} + \frac{1}{\sqrt{Nt}} + \frac{1}{t^q}\Big),$$

$$\Pr\Big(\Big|\mathcal{V}^N_t - \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big| \geq \varepsilon\Big) \leq \frac{C}{\varepsilon}\Big(\frac{\sqrt{N}}{t} + \frac{1}{\sqrt{N}}\Big),$$

$$\Pr\Big(\Big|\mathcal{W}^N_{\Delta_t,t} - \frac{\mu}{(1-\Lambda p)^3}\Big| \geq \varepsilon\Big) \leq \frac{C}{\varepsilon}\Big(\frac{1}{N} + \frac{N}{t^2} + \frac{1}{\sqrt{t^{1-4/(q+1)}}}\Big).$$

We will easily deduce the following corollary.

**Corollary 4.** *Assume $H(q)$ for some $q > 3$. For $t \geq 1$, put $\Delta_t = t/(2\lfloor t^{1-4/(q+1)}\rfloor)$. There is a constant $C$ depending only on $p$, $\mu$, $\varphi$ and $q$ such that for all $\varepsilon \in (0,1)$, all $N \geq 1$, all $t \geq 1$,*

$$\begin{aligned}\Pr\Big(\Big\|\Psi\Big(\mathcal{E}^N_t, \mathcal{V}^N_t, \mathcal{W}^N_{\Delta_t,t}\Big) - (\mu,\Lambda,p)\Big\| \geq \varepsilon\Big) \leq& \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{N}} + \frac{\sqrt{N}}{t} + \frac{1}{\sqrt{t^{1-4/(q+1)}}}\Big)\\ \leq& \frac{2C}{\varepsilon}\Big(\frac{1}{\sqrt{N}} + \frac{\sqrt{N}}{t^{1-4/(q+1)}}\Big),\end{aligned}$$

*where $\Psi = \mathbf{1}_D\Phi$ with $D = \{(u,v,w) \in \mathbb{R}^3 \ : \ w > u > 0 \text{ and } v \geq 0\}$ and $\Phi : D \mapsto \mathbb{R}^3$ defined by*

$$\Phi_1(u,v,w) = u\sqrt{\frac{u}{w}}, \quad \Phi_2(u,v,w) = \frac{v + [u - \Phi_1(u,v,w)]^2}{u[u-\Phi_1(u,v,w)]}, \quad \Phi_3(u,v,w) = \frac{1 - u^{-1}\Phi_1(u,v,w)}{\Phi_2(u,v,w)}.$$

We did not optimize the dependence in $q$: in many applications, $H(q)$ holds for all $q \geq 1$.

1.7. **Main result in the supercritical case.** For $N \geq 1$ and for $((Z^{i,N}_t)_{t\geq 0})_{i=1,\dots,N}$ the solution to (1), we set $\bar Z^N_t = N^{-1}\sum_{i=1}^N Z^{i,N}_t$. We will check later the following remark, which states that the mean number of actions per individual increases exponentially in the supercritical case.

**Remark 5.** *Assume $A$ and consider $\alpha_0 > 0$ uniquely defined by $p\int_0^\infty e^{-\alpha_0 t}\varphi(t)dt = 1$. Then*

$$\text{for all } \eta > 0, \quad \lim_{t\to\infty}\lim_{N\to\infty} \Pr\Big(\bar Z^N_t \in [e^{(\alpha_0-\eta)t}, e^{(\alpha_0+\eta)t}]\Big) = 1.$$

We next introduce

$$\mathcal{U}_t^N = \Big[ \sum_{i=1}^N \Big( \frac{Z_t^{i,N} - \bar{Z}_t^N}{\bar{Z}_t^N} \Big)^2 - \frac{N}{\bar{Z}_t^N} \Big] \mathbf{1}_{\{\bar{Z}_t^N > 0\}} \quad \text{and} \quad \mathcal{P}_t^N = \frac{1}{\mathcal{U}_t^N + 1} \mathbf{1}_{\{\mathcal{U}_t^N \geq 0\}}.$$

**Theorem 6.** *Assume $A$ and consider $\alpha_0 > 0$ defined in Remark 5. For all $\eta > 0$, there is a constant $C_\eta > 0$ (depending only on $p, \mu, \varphi, \eta$) such that for all $N \geq 1$, all $t \geq 1$, all $\varepsilon \in (0,1)$,*

$$\Pr\Big( |\mathcal{P}_t^N - p| \geq \varepsilon \Big) \leq \frac{C_\eta e^{\eta t}}{\varepsilon} \Big( \frac{\sqrt{N}}{e^{\alpha_0 t}} + \frac{1}{\sqrt{N}} \Big).$$

1.8. **Detecting subcriticality and supercriticality.** In practise, we may of course not know if we are in the subcritical or supercritical case.

**Proposition 7.** *(i) Under $H(1)$, there are some constants $0 < c < C$ depending only on $p, \mu, \varphi$ such that for all $N \geq 1$, all $t \geq 1$, $\Pr(\log(\bar{Z}_t^N) \geq (\log t)^2) \leq C(e^{-cN} + t^{1/2} e^{-(\log t)^2})$.*

*(ii) Under $A$, for all $\eta > 0$, there is a constant $C_\eta$ depending only on $p, \mu, \varphi, \eta$ such that for all $N \geq 1$, all $t \geq 1$, $\Pr(\log(\bar{Z}_t^N) \leq (\log t)^2) \leq C_\eta e^{\eta t}(N^{-1/2} + e^{-\alpha_0 t})$.*

It is then not hard to check that, with the notation of Corollary 4 and Theorem 6, under $H(q)$ (for some $q > 3$) or $A$, the estimator

$$\hat{p}_t^N = \mathbf{1}_{\{\log(\bar{Z}_t^N) < (\log t)^2\}} \Psi_3(\mathcal{E}_{t/2}^N, \mathcal{V}_{t/2}^N, \mathcal{W}_{\Delta_{t/2}, t/2}^N) + \mathbf{1}_{\{\log(\bar{Z}_t^N) > (\log t)^2\}} \mathcal{P}_t^N,$$

which is based on the observation of $(Z_s^{i,N})_{s \in [0,t], i=1,\dots,N}$, converges in probability to $p$, with the same speed of convergence as in Corollary 4 (under $H(q)$ for some $q > 3$) or as in Theorem 6 (under $A$).

1.9. **About optimality.** In Subection 2.3, we will see on a toy model that there is no real hope to find an estimator of $p$ with a better precision than $N^{-1/2} + N^{1/2} m_t^{-1}$, where $m_t$ is something like the mean number of jumps per individual during $[0,t]$. Consequently, we believe that the precision we found in Corollary 4 is almost optimal, since then $m_t \simeq t$ by Remark 2 and since we reach the precision $N^{-1/2} + N^{1/2} t^{\alpha - 1}$ for any $\alpha > 0$ (if $\varphi$ has a fast decay), so that the loss is arbitrarily small. Similarly, the precision found in Theorem 6 is rather satisfying, since then $m_t \simeq e^{\alpha_0 t}$ by Remark 5 and since we reach the precision $e^{\eta t}(N^{-1/2} + N^{1/2} e^{-\alpha_0 t})$ for any $\eta > 0$, so that the loss is, here also, arbitrarily small.

The main default of the present paper is that the constants in Corollary 4 and in Theorem 6 strongly depend on the parameters $\mu, \Lambda, p$. They also depend on $q$ in the subcritical case. In particular, it would be quite delicate to understand how they behave when approaching, from below or from above, the critical case.

1.10. **About the modeling.** There are two main limitations in our setting.

Assuming that the $\theta_{ij}$'s are i.i.d. is of course a strong assumption. What we really need is that the family $(\theta_{ij})_{i,j=1,\dots,N}$ satisfies similar properties as those shown in Subsection 4.1 (in the subcritical case) and in Subsection 5.1 (in the supercritical case). This clearly requires that the family $(\theta_{ij})_{i,j=1,\dots,N}$ is not too far from being i.i.d., and it does *not* suffice that $\lim_{N\to\infty} N^{-2} \sum_{i,j=1}^N \theta_{ij} = p$. However, we believe that all the conclusions of the present paper are still true if one assumes that $(\theta_{ij})_{1 \leq i \leq j \leq N}$ is i.i.d. and that $\theta_{ji} = \theta_{ij}$ for all $1 \leq i < j \leq N$, which might be the case in some applications where the interactions are symmetric. A rigorous proof would require some work but should not be too hard. We will study this problem numerically at the end of the paper.

Assuming that we observe all the population is also rather stringent. It would be interesting to study what happens if one observes only $(Z_s^{i,N})_{i=1,\dots,K,s\in[0,t]}$, for some $K$ large but smaller than $N$. It is not difficult to guess how to adapt the estimators to such a situation (see Section 7 for precise formulae). The theoretical analysis would require a careful and tedious study. Again, we will discuss this numerically.

1.11. **Notation.** We denote by $\Pr_\theta$ the conditional probability knowing $(\theta_{ij})_{i,j=1,\dots,N}$. We introduce $\mathbb{E}_\theta$, $\mathbb{V}\mathrm{ar}_\theta$ and $\mathbb{C}\mathrm{ov}_\theta$ accordingly.

For two functions $f, g : [0,\infty) \mapsto \mathbb{R}$, we introduce (if it exists) $(f \star g)(t) = \int_0^t f(t-s)g(s)ds$. The functions $\varphi^{\star n}$ will play an important role in the paper. Observe that, since $\int_0^\infty \varphi(s)ds = \Lambda$, $\int_0^\infty \varphi^{\star n}(s)ds = \Lambda^n$. We adopt the conventions $\varphi^{\star 0}(s)ds = \delta_0(ds)$ and $\varphi^{\star 0}(t-s)ds = \delta_t(ds)$. We also adopt the convention that $\varphi^{\star n}(s) = 0$ for $s < 0$.

All the finite constants used in the upperbounds are denoted by $C$, the positive constants used in the lowerbounds are denoted by $c$ and their values change from line to line. They are allowed to depend only on $\mu$, $p$ and $\varphi$ (and on $q$ under $H(q)$), but never on $N$ nor on $t$. Any other dependence will be indicated in subscript. For example, $C_\eta$ is a finite constant depending only on $\mu$, $p$, $\varphi$ and $\eta$ (and on $q$ under $H(q)$).

1.12. **Plan of the paper.** In the next section, we try to give the main reasons why our estimators should be convergent, which should help the reader to understand the strategies of the proofs. We also briefly and formally introduce a Gaussian toy model in Section 2.3 to show that the rates of convergence we obtain are not far from being the best we can hope for. In Section 3, we prove Proposition 1 (strong existence and uniqueness of the process) and check a few more or less explicit formulae concerning $(Z_t^{i,N})_{i=1,\dots,N,t\ge 0}$ of constant use. Section 4 is devoted to the proof of Theorem 3 and Corollary 4 (main results in the subcritical case). Theorem 6 (main result in the supercritical case) is proved in Section 5. We check Proposition 7 in Section 6. Finally, we illustrate numerically the results of the paper and some possible extensions in the last section.

## 2. Heuristics

This section is completely informal and the symbol $\simeq$ means nothing precise. For example, "$Z_t^{i,N} \simeq \mathbb{E}_\theta[Z_t^{i,N}]$ for $t$ large" should be understood as "we hope that $Z_t^{i,N}/\mathbb{E}_\theta[Z_t^{i,N}]$ tends to 1 as $t \to \infty$ in probability or in another sense."

2.1. **The subcritical case.** We assume that $\Lambda p \in [0,1)$ and try to explain the asymptotics of $(Z_t^{i,N})_{i=1,\dots,N,t\ge 0}$ and where the three estimators $\mathcal{E}_t^N$, $\mathcal{V}_t^N$ and $\mathcal{W}_{\Delta,t}^N$ come from. We introduce the matrices $A_N(i,j) = N^{-1}\theta_{ij}$ and $Q_N = (I - \Lambda A_N)^{-1}$, which exists with high probability because $\Lambda p < 1$. We also set $\ell_N(i) = \sum_{j=1}^N Q_N(i,j)$ and $c_N(i) = \sum_{j=1}^N Q_N(j,i)$.

Fixing $N$ and knowing $(\theta_{ij})_{i,j=1,\dots,N}$, we expect that $Z_t^{i,N} \simeq \mathbb{E}_\theta[Z_t^{i,N}]$ for $t$ large by a law of large numbers. Next, it is not hard to check that $\mathbb{E}_\theta[Z_t^{i,N}] = \mu t + N^{-1}\sum_{j=1}^N \theta_{ij}\int_0^t \varphi(t-s)\mathbb{E}_\theta[Z_s^{j,N}]ds$. Assume now that $\gamma_N(i) = \lim_{t\to\infty} t^{-1}\mathbb{E}_\theta[Z_t^{i,N}]$ exists for each $i = 1,\dots,N$. Then, using that $\int_0^t \varphi(t-s)sds \simeq \Lambda t$ for $t$ large, we find that the vector $\gamma_N$ must solve $\gamma_N = \mu\mathbf{1}_N + \Lambda A_N\gamma_N$, where $\mathbf{1}_N$ is the $N$-dimensional vector with all coordinates equal to 1. This implies that $\gamma_N = \mu(I-\Lambda A_N)^{-1}\mathbf{1}_N = \mu\ell_N$. We thus expect that $Z_t^{i,N} \simeq \mathbb{E}_\theta[Z_t^{i,N}] \simeq \mu\ell_N(i)t$.

Based on this and setting $\bar\ell_N = N^{-1}\sum_{i=1}^N \ell_N(i)$, we expect that $\bar Z_t^N \simeq \mu\bar\ell_N t$ for large values of $t$, whence $\tilde{\mathcal{E}}_t^N := t^{-1}\bar Z_t^N \simeq \mu\bar\ell_N$.

Knowing $(\theta_{ij})_{i,j=1,\dots,N}$, $Z^{1,N}_t$ should resemble, roughly, a Poisson process, so that it should approximately hold true that $\mathbb{V}\mathrm{ar}_\theta\,(Z^{1,N}_t) \simeq \mathbb{E}_\theta[Z^{1,N}_t]$. Thus $N^{-1}\sum_{i=1}^N (Z^{i,N}_t - \bar Z^N_t)^2$ should resemble $\mathbb{V}\mathrm{ar}\,(Z^{1,N}_t) = \mathbb{V}\mathrm{ar}\,(\mathbb{E}_\theta[Z^{1,N}_t]) + \mathbb{E}[\mathbb{V}\mathrm{ar}_\theta\,(Z^{1,N}_t)] \simeq \mathbb{V}\mathrm{ar}\,(\mathbb{E}_\theta[Z^{1,N}_t]) + \mathbb{E}[Z^{1,N}_t]$, which itself resembles $N^{-1}\sum_{i=1}^N(\mathbb{E}_\theta[Z^{i,N}_t] - \mathbb{E}_\theta[\bar Z^N_t])^2 + \bar Z^N_t \simeq N^{-1}\mu^2 t^2 \sum_{i=1}^N (\ell_N(i) - \bar\ell_N)^2 + \bar Z^N_t$. Consequently, we expect that $\tilde{\mathcal{V}}^N_t := t^{-2}[\sum_{i=1}^N (Z^{i,N}_t - \bar Z^N_t)^2 - N\bar Z^N_t] \simeq \mu^2 \sum_{i=1}^N(\ell_N(i) - \bar\ell_N)^2$ for $t$ large.

Finally, the *temporal* empirical variance $\Delta t^{-1}\sum_{k=1}^{t/\Delta}[\bar Z^N_{k\Delta} - \bar Z^N_{(k-1)\Delta} - \Delta t^{-1}\bar Z^N_t]^2$ should resemble $\mathbb{V}\mathrm{ar}_\theta\,[\bar Z^N_\Delta]$ if $1 \ll \Delta \ll t$. Thus $\tilde{\mathcal{W}}^N_{\Delta,t} := N t^{-1}\sum_{k=1}^{t/\Delta}[\bar Z^N_{k\Delta} - \bar Z^N_{(k-1)\Delta} - \Delta t^{-1}\bar Z^N_t]^2 \simeq N\Delta^{-1}\mathbb{V}\mathrm{ar}_\theta\,[\bar Z^N_\Delta]$. Introducing the martingales $M^{i,N}_t = Z^{i,N}_t - C^{i,N}_t$ (where $C^{i,N}$ is the compensator of $Z^{i,N}$), the centered processes $U^{i,N}_t = Z^{i,N}_t - \mathbb{E}_\theta[Z^{i,N}_t]$, and the $N$-dimensional vectors $\mathbf{U}^N_t$ and $\mathbf{M}^N_t$ with coordinates $U^{i,N}_t$ and $M^{i,N}_t$, we will see in Section 3 that $\mathbf{U}^N_t = \mathbf{M}^N_t + A_N\int_0^t \varphi(t-s)\mathbf{U}^N_s ds$, so that for large times, $\mathbf{U}^N_t \simeq \mathbf{M}^N_t + \Lambda A_N \mathbf{U}^N_t$ and thus $\mathbf{U}^N_t \simeq Q_N \mathbf{M}^N_t$. Consequently, we hope that $\bar U^N_t \simeq \overline{Q_N\mathbf{M}^N_t}$, where $\bar U^N_t = N^{-1}\sum_{i=1}^N U^{i,N}_t$ and $\overline{Q_N\mathbf{M}^N_t} = N^{-1}\sum_{i=1}^N (Q_N\mathbf{M}^N_t)_i$. A little study shows that the martingales $M^{j,N}_t$ are orthogonal and that $[M^{j,N}, M^{j,N}]_t = Z^{j,N}_t \simeq \mu\ell_N(j)t$, so that $\mathbb{V}\mathrm{ar}_\theta\,(\overline{Q_N\mathbf{M}^N_t}) \simeq \mu t N^{-2}\sum_{j=1}^N(\sum_{i=1}^N Q_N(i,j))^2\ell_N(j) = \mu t N^{-2}\sum_{j=1}^N (c_N(j))^2\ell_N(j)$. Finally, $\mathbb{V}\mathrm{ar}_\theta\,[\bar Z^N_t] = \mathbb{V}\mathrm{ar}_\theta\,[\bar U^N_t] \simeq \mu t N^{-2}\sum_{j=1}^N (c_N(j))^2 \ell_N(j)$ and we hope that $\tilde{\mathcal{W}}^N_{\Delta,t} \simeq N\Delta^{-1}\mathbb{V}\mathrm{ar}_\theta\,[\bar Z^N_\Delta] \simeq \mu N^{-1}\sum_{j=1}^N (c_N(j))^2\ell_N(j)$ if $1 \ll \Delta \ll t$.

We thus need to find the limits of $\bar\ell_N$, $\sum_{i=1}^N(\ell_N(i) - \bar\ell_N)^2$ and $N^{-1}\sum_{i=1}^N \ell_N(i)(c_N(i))^2$ as $N \to \infty$. It is not easy to make rigorous, but it holds true that $\ell_N(i) \simeq 1 + \Lambda(1-\Lambda p)^{-1}L_N(i)$, where $L_N(i) = \sum_{j=1}^N A_N(i,j)$. This comes from $\sum_{j=1}^N A_N^2(i,j) = \sum_{j=1}^N A_N(i,j)\sum_{k=1}^N A_N(j,k) \simeq p\sum_{j=1}^N A_N(i,j) = pL_N(i)$, $\sum_{j=1}^N A_N^3(i,j) \simeq p^2 L_N(i)$ for similar reasons, etc. It is very rough, but it will imply that $\ell_N(i) = \sum_{n\ge 0}\Lambda^n \sum_{j=1}^N A_N^n(i,j) \simeq 1 + \sum_{n\ge1}\Lambda^n p^{n-1}L_N(i) = 1 + \Lambda(1-\Lambda p)^{-1}L_N(i)$. Once this is seen (as well as a similar fact for the columns), we get convinced, $NL_N$ being a vector of $N$ i.i.d. Binomial$(N,p)$-distributed random variables, that $\bar\ell_N \simeq 1/(1-\Lambda p)$, that $\sum_{i=1}^N(\ell_N(i) - \bar\ell_N)^2 \simeq \Lambda^2 p(1-p)/(1-\Lambda p)^2$ and that $N^{-1}\sum_{i=1}^N \ell_N(i)(c_N(i))^2 \simeq 1/(1-\Lambda p)^3$.

At the end, it should be more or less true that, for $t, \Delta$ and $N$ large enough and in a suitable regime, $\tilde{\mathcal{E}}^N_t \simeq \mu/(1-\Lambda p)$, $\tilde{\mathcal{V}}^N_t \simeq \mu^2\Lambda^2 p(1-p)/(1-\Lambda p)^2$, and $\tilde{\mathcal{W}}^N_{\Delta,t} \simeq \mu/(1-\Lambda p)^3$. Of course, all this is completely informal and many points have to be clarified.

Observe that concerning $\tilde{\mathcal{V}}^N_t$, we use that $Z^{1,N}_t$ resembles a Poisson process, while concerning $\tilde{\mathcal{W}}^N_{\Delta,t}$, we use that $\bar Z^N_t$ does not resemble a Poisson process.

The three estimators $\mathcal{E}^N_t, \mathcal{V}^N_t, \mathcal{W}^N_{\Delta,t}$ we study in the paper resemble much $\tilde{\mathcal{E}}^N_t, \tilde{\mathcal{V}}^N_t, \tilde{\mathcal{W}}^N_{\Delta,t}$ and should converge to the same limits. Let us explain why we have modified the expressions. We started this subsection by the observation that $\mathbb{E}_\theta[Z^{i,N}_t] \simeq \mu\ell_N(i)t$, on which the construction of the estimators relies. A detailed study shows that, under $H(q)$, $\mathbb{E}_\theta[Z^{i,N}_t] = \mu\ell_N(i)t + \chi^N_i \pm t^{1-q}$, for some finite random variable $\chi^N_i$. As a consequence, $t^{-1}\mathbb{E}_\theta[Z^{i,N}_{2t} - Z^{i,N}_t]$ converges to $\mu\ell_N(i)$ considerably faster (with an error in $t^{-q}$) than $t^{-1}\mathbb{E}_\theta[Z^{i,N}_t]$ (for which the error is of order $t^{-1}$). This explains our modifications and why these modifications are crucial.

Let us conclude this subsection with a technical issue. If $\Lambda > 1$ (which is not forbidden even in the subcritical case), there is a positive probability that an anomalously high proportion of the $\theta_{ij}$'s equal 1, so that $I - \Lambda A_N$ is not invertible and our multivariate Hawkes process is supercritical (on this event with small probability). We will thus work on an event $\Omega_N^1$ on which such problems do not occur and show that this event has a high probability.

2.2. **The supercritical case.** We now assume that $\Lambda p > 1$ and explain the asymptotics of $(Z_t^{i,N})_{i=1,\dots,N,t\geq 0}$ and where the estimator $\mathcal{U}_t^N$ comes from. We introduce $A_N(i,j) = N^{-1}\theta_{ij}$.

Fixing $N$ and knowing $(\theta_{ij})_{i,j=1,\dots,N}$, we expect that $Z_t^{i,N} \simeq H_N \mathbb{E}_\theta[Z_t^{i,N}]$, for some random $H_N > 0$ not depending on $i$ (and with $H_N$ almost constant for $N$ large). This is typically a supercritical phenomenon, that can already be observed on Galton-Watson processes. Fortunately, we will not really need to check it nor to study $H_N$, essentially because we will use the ratios $Z_t^{i,N}/\bar{Z}_t^N$, which makes disappear $H_N$.

Next, we believe that $\mathbb{E}_\theta[Z_t^{i,N}] \simeq \gamma_N(i)e^{\alpha_N t}$ for $t$ large, for some vector $\gamma_N$ with positive entries and some exponent $\alpha_N > 0$. Inserting this into $\mathbb{E}_\theta[Z_t^{i,N}] = \mu t + N^{-1}\sum_{j=1}^N \theta_{ij}\int_0^t \varphi(t-s)\mathbb{E}_\theta[Z_s^{j,N}]ds$, we find that $\gamma_N = A_N\gamma_N \int_0^\infty e^{-\alpha_N s}\varphi(s)ds$. The vector $\gamma_N$ being positive, it is necessarily a Perron-Frobenius eigenvector of $A_N$, so that $\rho_N = (\int_0^\infty e^{-\alpha_N s}\varphi(s)ds)^{-1}$ is its Perron-Frobenius eigenvalue (i.e. its spectral radius). We now consider the normalized Perron-Frobenius eigenvector $V_N$ such that $\sum_{i=1}^N (V_N(i))^2 = N$ and conclude that $Z_t^{i,N} \simeq K_N V_N(i)e^{\alpha_N t}$ for all $i = 1,\dots,N$, where $K_N = [N^{-1}\sum_{i=1}^N(\gamma_N(i))^2]^{1/2}H_N$.

Exactly as in the subcritical case, the empirical variance $N^{-1}\sum_{i=1}^N(Z_t^{i,N} - \bar{Z}_t^N)^2$ should resemble $N^{-1}\sum_{i=1}^N(\mathbb{E}_\theta[Z_t^{i,N}] - \mathbb{E}_\theta[\bar{Z}_t^N])^2 + \bar{Z}_t^N \simeq N^{-1}K_N^2 e^{2\alpha_N t}\sum_{i=1}^N(V_N(i) - \bar{V}_N)^2 + \bar{Z}_t^N$. Since we also guess that $\bar{Z}_t^N \simeq K_N \bar{V}_N e^{\alpha_N t}$, where $\bar{V}_N = N^{-1}\sum_{i=1}^N V_N(i)$, we expect that for $t$ large, $\mathcal{U}_t^N = (\bar{Z}_t^N)^{-2}[\sum_{i=1}^N(Z_t^{i,N} - \bar{Z}_t^N)^2 - N\bar{Z}_t^N] \simeq (\bar{V}_N)^{-2}\sum_{i=1}^N(V_N(i) - \bar{V}_N)^2$.

We now search for the limit of $(\bar{V}_N)^{-2}\sum_{i=1}^N(V_N(i) - \bar{V}_N)^2$ as $N \to \infty$. Roughly, $A_N^2(i,j) \simeq p^2/N$, whence, starting from $A_N^2 V_N = \rho_N^2 V_N$, we see that $\rho_N^2 V_N \simeq p^2 \bar{V}_N \mathbf{1}_N$, where $\mathbf{1}_N$ is the $N$-dimensional vector with all coordinates equal to 1. Consequently, $V_N = (A_N V_N)/\rho_N \simeq \kappa_N A_N \mathbf{1}_N$, where $\kappa_N = (p^2/\rho_N^3)\bar{V}_N$. In other words, $V_N$ is almost colinear to $L_N := A_N\mathbf{1}_N$, and $NL_N$ is a vector of $N$ i.i.d. Binomial$(N,p)$-distributed random variables. It is thus reasonable to expect that $(\bar{V}_N)^{-2}\sum_{i=1}^N(V_N(i) - \bar{V}_N)^2 \simeq (\bar{L}_N)^{-2}\sum_{i=1}^N(L_N(i) - \bar{L}_N)^2 \simeq p^{-2}p(1-p) = 1/p - 1$.

All in all, we hope that for $N$ and $t$ large and in a suitable regime, $\mathcal{U}_t^N \simeq 1/p - 1$.

Finally, let us mention that $\alpha_N \simeq \alpha_0$ (see Remark 5) because $\int_0^\infty e^{-\alpha_N s}\varphi(s)ds = 1/\rho_N$, because $\int_0^\infty e^{-\alpha_0 s}\varphi(s)ds = 1/p$ and because $\rho_N \simeq p$. This last assertion follows from the fact that $A_N^2(i,j) \simeq p^2/N$, so that the largest eigenvalue of $A_N^2$ should resemble $p^2$, whence that of $A_N$ should resemble $p$.

Of course, all this is not clear and has to be made rigorous. Let us mention that we will use a quantified version of the Perron-Frobenius of G. Birkhoff [7]. As we will see, the projection onto the eigenvector $V_N$ will be very fast (almost immediate for $N$ very large).

As in the subcritical case, we will have to work on an event $\Omega_N^2$, of high probability, on which the $\theta_{ij}$'s behave reasonably. For example, to apply the Perron-Frobenius theorem, we have to be sure that the matrix $A_N$ is irreducible, which is not a.s. true.

2.3. **About optimality: a related toy model.** Consider $\alpha_0 \geq 0$ and two unknown parameters $\Gamma > 0$ and $p \in (0,1]$. For $N \geq 1$, consider an i.i.d. family $(\theta_{ij})_{i,j=1,\dots,N}$ of Bernoulli$(p)$-distributed random variables, put $\lambda_t^{i,N} = N^{-1}\Gamma e^{\alpha_0 t}\sum_{j=1}^N \theta_{ij}$ and, conditionally on $(\theta_{ij})_{i,j=1,\dots,N}$, consider a family $(Z^{1,N}_t)_{t\geq 0},\dots,(Z^{N,N}_t)_{t\geq 0}$ of independent inhomogeneous Poisson processes with intensities $(\lambda^{1,N}_t)_{t\geq 0},\dots,(\lambda^{N,N}_t)_{t\geq 0}$. We observe $(Z^{i,N}_s)_{s\in[0,t],i=1,\dots,N}$ and we want to estimate $p$ in the asymptotic $(N,t)\to(\infty,\infty)$.

This problem can be seen as a strongly simplified version of the one studied in the present paper, with $\alpha_0=0$ in the subcritical case and $\alpha_0>0$ in the supercritical case. Roughly, the mean number of jumps per individual resembles $m_t=\int_0^t e^{\alpha_0 s}ds$, which is of order $t$ when $\alpha_0=0$ and $e^{\alpha_0 t}$ else.

There is classically no loss of information, since $\alpha_0$ is known, if we only observe $(Z^{i,N}_t)_{i=1,\dots,N}$: after a (deterministic and known) change of time, the processes $(Z^{i,N}_t)_{i=1,\dots,N}$ become *homogeneous* Poisson processes with unkown parameters (conditionally on $(\theta_{ij})_{i,j=1,\dots,N}$), and the conditional law of a Poisson process on $[0,t]$ knowing its value at time $t$ does not depend on its parameter.

We next proceed to a Gaussian approximation: we have $\lambda^{i,N}_t\simeq\Gamma e^{\alpha_0 t}[p+\sqrt{N^{-1}p(1-p)}]G_i$ and $Z^{i,N}_t\simeq\int_0^t\lambda^{i,N}_s ds+\sqrt{\int_0^t\lambda^{i,N}_s ds}H_i$, for two independent i.i.d. families $(G_i)_{i=1,\dots,N}$, $(H_i)_{i=1,\dots,N}$ of $\mathcal{N}(0,1)$-distributed random variables. Using finally that $(m_t)^{-1}N^{-1/2}\ll(m_t)^{-1}$ in our asymptotic, we conclude that $(m_t)^{-1}Z^{i,N}_t\simeq\Gamma p+\Gamma\sqrt{N^{-1}p(1-p)}G_i+\sqrt{(m_t)^{-1}\Gamma p}H_i$, which is $\mathcal{N}(\Gamma p,N^{-1}\Gamma^2p(1-p)+(m_t)^{-1}\Gamma p)$-distributed.

Our toy problem is thus the following: estimate $p$ when observing a $N$-sample $(X^{i,N}_t)_{i=1,\dots,N}$ of the $\mathcal{N}(\Gamma p,N^{-1}\Gamma^2p(1-p)+(m_t)^{-1}\Gamma p)$-distribution. We assume that $\Gamma p$ is known, which can only make easier the estimation of $p$. As is well-known the statistic $S^N_t=N^{-1}\sum_{i=1}^N(X^{i,N}_t-\Gamma p)^2$ is then *sufficient* and is the best estimator (in all the usual senses), for $N\geq 1$ and $t\geq 1$ fixed, of $N^{-1}\Gamma^2p(1-p)+(m_t)^{-1}\Gamma p$, so that $T^N_t=N(\Gamma p)^{-2}(S^N_t-m_t^{-1}\Gamma p)$ is more or less the best estimator of $(1/p-1)$. But $\mathbb{V}\mathrm{ar}\,S^N_t=2N^{-1}(N^{-1}\Gamma^2p(1-p)+(m_t)^{-1}\Gamma p)^2$, whence $\mathbb{V}\mathrm{ar}\,T^N_t=2(\Gamma p)^{-4}(N^{-1/2}\Gamma^2p(1-p)+N^{1/2}(m_t)^{-1}\Gamma p)^2$. It is thus not possible to estimate $(1/p-1)$ with a better precision than $N^{-1/2}+N^{1/2}(m_t)^{-1}$. This of course implies that we cannot estimate $p$ with a better precision than $N^{-1/2}+N^{1/2}(m_t)^{-1}$.

## 3. Well-posedness and explicit formulae

We first give the

*Proof of Proposition 1.* Conditionally on $(\theta_{ij})_{i,j=1,\dots,N}$, we can apply directly [13, Theorem 6], of which the assumption is satisfied here, see [13, Remark 5-(i)]: conditionally on $(\theta_{ij})_{i,j=1,\dots,N}$, there is a unique solution $(Z^{i,N}_t)_{t\geq 0,i=1,\dots,N}$ to (1) such that $\sum_{i=1}^N\mathbb{E}_\theta[Z^{i,N}_t]<\infty$ for all $t\geq 0$. Since now $(\theta_{ij})_{i,j=1,\dots,N}$ can only take a finite number of values, we immediately deduce that indeed $\sum_{i=1}^N\mathbb{E}[Z^{i,N}_t]<\infty$ for all $t\geq 0$. □

We carry on with a classical lemma. Recall that $\varphi^{\star 0}(t-s)ds=\delta_t(ds)$ by convention.

**Lemma 8.** *Consider $d\geq 1$, $A\in\mathcal{M}_{d\times d}(\mathbb{R})$, $m,g:[0,\infty)\mapsto\mathbb{R}^d$ locally bounded and assume that $\varphi:[0,\infty)\mapsto[0,\infty)$ is locally integrable. If $m_t=g_t+\int_0^t\varphi(t-s)Am_sds$ for all $t\geq 0$, then $m_t=\sum_{n\geq 0}\int_0^t\varphi^{\star n}(t-s)A^ng_sds$.*

*Proof.* The equation $m_t=g_t+\int_0^t\varphi(t-s)Am_sds$ with unkown $m$ has at most one locally bounded solution. Indeed, consider two such solutions $m,\tilde m$, observe that $u=|m-\tilde m|$ satisfies $u_t\leq|A|\int_0^t\varphi(t-s)u_sds$, and conclude that $u=0$ by the generalized Gronwall lemma, see e.g. [13, Lemma 23-(i)]. We thus just have to prove that $m_t:=\sum_{n\geq 0}\int_0^t\varphi^{\star n}(t-s)A^ng_sds$ is locally bounded and solves $m = g + A\varphi \star m$. We introduce $k_t^n = |A|^n \int_0^t \varphi^{\star n}(s)ds$, which is locally bounded because $\varphi$ is locally integrable and which satisfies $k_t^{n+1} \le |A| \int_0^t k_s^n \varphi(t-s)ds$. We use [13, Lemma 23-(ii)] to conclude that $\sum_{n\ge 0} k_t^n$ is locally bounded. Consequently, $|m_t| \le \sup_{[0,t]} |g_s| \times \sum_{n\ge 0} k_t^n$ is locally bounded. Finally, we write $m = g + \sum_{n\ge 1} A^n \varphi^{\star n} \star g = g + A\varphi \star \sum_{n\ge 0} A^n \varphi^{\star n} \star g = g + A\varphi \star m$ as desired. □

We next introduce a few processes.

**Notation 9.** *Assume only that $\varphi$ is locally integrable, fix $N \ge 1$ and consider the solution $(Z_t^{i,N})_{t\ge 0, i=1,\dots,N}$ to (1). For each $i = 1, \dots, N$, we introduce the martingale (recall that $\lambda^{i,N}$ was defined in (1))*

$$M_t^{i,N} = \int_0^t \int_0^\infty \mathbf{1}_{\{z \le \lambda_s^{i,N}\}} \tilde{\pi}^i(ds, dz),$$

*where $\tilde{\pi}^i(ds, dz) = \pi^i(ds, dz) - dsdz$ is the compensated Poisson measure associated to $\pi^i$. We also introduce $M_t^{i,N,*} = \sup_{[0,t]} |M_s^{i,N}|$, as well as the (conditionally) centered process*

$$U_t^{i,N} = Z_t^{i,N} - \mathbb{E}_\theta[Z_t^{i,N}].$$

*For each $t \ge 0$, we denote by $\mathbf{Z}_t^N$ (resp. $\mathbf{M}_t^N$, $\mathbf{M}_t^{N,*}$, $\mathbf{U}_t^N$) the $N$-dimensional vector with coordinates $Z_t^{i,N}$ (resp. $M_t^{i,N}$, $M_t^{i,N,*}$, $U_t^{i,N}$). We also set $\bar{Z}_t^N = N^{-1}\sum_{i=1}^N Z_t^{i,N}$, $\bar{M}_t^N = N^{-1}\sum_{i=1}^N M_t^{i,N}$ and $\bar{U}_t^N = N^{-1}\sum_{i=1}^N U_t^{i,N}$.*

We refer to Jacod-Shiryaev [22, Chapter 1, Section 4e] for definitions and properties of pure jump martingales and of their quadratic variations.

**Remark 10.** *Since the Poisson measures $\pi^i$ are independent, the martingales $M^{i,N}$ are orthogonal. More precisely, we have $[M^{i,N}, M^{j,N}]_t = 0$ if $i \ne j$, while $[M^{i,N}, M^{i,N}]_t = Z_t^{i,N}$ (because $Z_t^{i,N}$ counts the jumps of $M^{i,N}$, which are all of size 1). Consequently, $\mathbb{E}_\theta[M_s^{i,N} M_t^{j,N}] = \mathbf{1}_{\{i=j\}} \mathbb{E}_\theta[Z_{s\wedge t}^{i,N}]$.*

We now give some more or less explicit formulas. We denote by $\mathbf{1}_N$ the $N$-dimensional vector with all entries equal to 1 and we set $A_N(i,j) = N^{-1}\theta_{ij}$ for $i, j = 1, \dots, N$.

**Lemma 11.** *Assume only that $\varphi$ is locally integrable. We have (recall that $\varphi^{\star 0}(t-s)ds = \delta_t(ds)$):*

$$\mathbf{Z}_t^N = \mathbf{M}_t^N + \mu \mathbf{1}_N t + \int_0^t \varphi(t-s) A_N \mathbf{Z}_s^N ds, \tag{2}$$

$$\mathbb{E}_\theta[\mathbf{Z}_t^N] = \mu \sum_{n\ge 0} \Big[\int_0^t s\varphi^{\star n}(t-s)ds\Big] A_N^n \mathbf{1}_N, \tag{3}$$

$$\mathbf{U}_t^N = \sum_{n\ge 0} \int_0^t \varphi^{\star n}(t-s) A_N^n \mathbf{M}_s^N ds. \tag{4}$$

*Proof.* The first expression is not difficult: starting from (1),

$$Z_t^{i,N} = M_t^{i,N} + \int_0^t \lambda_s^{i,N} ds = M_t^{i,N} + \mu t + \sum_{j=1}^N A_N(i,j) \int_0^t \int_0^s \varphi(s-u) dZ_u^{j,N} ds.$$

Using [13, Lemma 22], we see that $\int_0^t \int_0^s \varphi(s-u) dZ_u^{j,N} ds = \int_0^t \varphi(t-s) Z_s^{j,N} ds$, whence indeed,

$$Z_t^{i,N} = M_t^{i,N} + \mu t + \int_0^t \varphi(t-s) \sum_{j=1}^N A_N(i,j) Z_s^{j,N} ds,$$

which is nothing but (2). Taking conditional expectations in (2), we find that $\mathbb{E}_\theta[\mathbf{Z}^N_t] = \mu \mathbf{1}_N t + \int_0^t \varphi(t-s) A_N \mathbb{E}_\theta[\mathbf{Z}^N_s] ds$ and thus also $\mathbf{U}^N_t = \mathbf{M}^N_t + \int_0^t \varphi(t-s) A_N \mathbf{U}^N_s ds$. Since now $\varphi$ is (a.s.) locally integrable, since $\mu \mathbf{1}_N t$ and $\mathbf{M}^N_t$ are (a.s.) locally bounded, as well as $\mathbb{E}_\theta[\mathbf{Z}^N_t]$ and $\mathbf{U}^N_t$, (3) and (4) directly follow from Lemma 8. □

## 4. The subcritical case

Here we consider the subcritical case. We first study the large $N$-asymptotic of the matrix $Q_N = (I - \Lambda A_N)^{-1}$, which plays a central role in the rest of the section. In Subsection 4.2, we finely study the behavior of $\varphi^{\star n}$. In Subsection 4.3, we handle a few computations to be used several times later. Subsections 4.4, 4.5 and 4.6 are devoted to the studies of the three estimators $\mathcal{E}^N_t$, $\mathcal{V}^N_t$ and $\mathcal{W}^N_{\Delta,t}$. We conclude the proofs of Theorem 3 and Corollary 4 in Subsection 4.7.

4.1. **Study of a random matrix.** We use the following standard notation: for $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $r \in [1, \infty)$, we set $||x||_r = (\sum_{i=1}^N |x_i|^r)^{1/r}$ and $||x||_\infty = \max_{i=1,\dots,N} |x_i|$. For $r \in [1,\infty]$, we denote by $|||\cdot|||_r$ the operator norm on $\mathcal{M}_{N\times N}(\mathbb{R})$ associated to $||\cdot||_r$. We recall that

$$|||M|||_1 = \sup_{j=1,\dots,N} \sum_{i=1}^N |M_{ij}|, \quad |||M|||_\infty = \sup_{i=1,\dots,N} \sum_{j=1}^N |M_{ij}|$$

and that for all $r \in (1,\infty)$,

$$|||M|||_r \le |||M|||_1^{1/r} |||M|||_\infty^{1-1/r}. \tag{5}$$

**Notation 12.** *We assume that $\Lambda p < 1$. For each $N \ge 1$, we introduce the $N \times N$ random matrix $A_N$ defined by $A_N(i,j) = N^{-1}\theta_{ij}$, as well as the event*

$$\Omega^1_N = \Big\{ \Lambda |||A_N|||_r \le a \ \text{ for all } r \in [1,\infty] \Big\}, \ \text{where } a = \frac{1+\Lambda p}{2} \in (\Lambda p, 1). \tag{6}$$

*On $\Omega^1_N$, the $N \times N$ matrix $Q_N = \sum_{n\ge 0} \Lambda^n A_N^n = (I - \Lambda A_N)^{-1}$ is well-defined and we introduce, for each $i = 1, \dots, N$, $\ell_N(i) = \sum_{j=1}^N Q_N(i,j)$, $c_N(i) = \sum_{j=1}^N Q_N(j,i)$, as well as $\bar\ell_N = N^{-1}\sum_{i=1}^N \ell_N(i)$ and $\bar c_N = N^{-1}\sum_{i=1}^N c_N(i)$. We of course have $\bar\ell_N = \bar c_N$.*

Let us remark once for all that, with $C = 1/(1-a) < \infty$,

$$\Omega^1_N \subset \Big\{ |||Q_N|||_r \le C \ \text{ for all } r \in [1,\infty] \Big\} \subset \Big\{ \sup_{i=1,\dots,N} \max\{\ell_N(i), c_N(i)\} \le C \Big\}, \tag{7}$$

$$\Omega^1_N \subset \Big\{ \mathbf{1}_{\{i=j\}} \le Q_N(i,j) \le \mathbf{1}_{\{i=j\}} + \Lambda C N^{-1} \ \text{ for all } i,j = 1,\dots,N \Big\}. \tag{8}$$

Indeed, (7) is straightforward since $Q_N = \sum_{n\ge0} \Lambda^n A_N^n$. To check (8), we first observe that $Q_N(i,j) \ge \Lambda^0 A_N^0(i,j) = \mathbf{1}_{\{i=j\}}$. Next, we use that $A_N(i,j) \le N^{-1}$ while, for $n \ge 2$, $A_N^n(i,j) = \sum_{k=1}^N A_N(i,k) A_N^{n-1}(k,j) \le N^{-1} \sum_{k=1}^N A_N^{n-1}(k,j) \le N^{-1} |||A_N^{n-1}|||_1 \le N^{-1} |||A_N|||_1^{n-1}$. Thus $A_N^n(i,j) \le N^{-1} |||A_N|||_1^{n-1}$ for all $n \ge 1$. Hence on $\Omega^1_N$, it holds that $Q_N(i,j) \le \mathbf{1}_{\{i=j\}} + N^{-1} \sum_{n\ge1} \Lambda^n |||A_N|||_1^{n-1} \le \mathbf{1}_{\{i=j\}} + N^{-1}\Lambda/(1-a)$ as desired.

**Lemma 13.** *Assume that $\Lambda p < 1$. It holds that $\Pr(\Omega^1_N) \ge 1 - C\exp(-cN)$.*

*Proof.* By (5), it suffices to prove that $\Pr(\Lambda |||A_N|||_1 > a) \le C\exp(-cN)$ and $\Pr(\Lambda |||A_N|||_\infty > a) \le C\exp(-cN)$. Since $|||A_N|||_\infty = |||A_N^t|||_1$ and since $A_N^t$ (the transpose of $A_N$) has the same law as $A_N$, it actually suffices to verify the first inequality. First, $N|||A_N|||_1 = \max\{X_1^N, \dots, X_N^N\}$, where $X_i^N = \sum_{j=1}^N \theta_{ij}$ is Binomial$(N,p)$-distributed for each $i$. Consequently, $\Pr(\Lambda |||A_N|||_1 > a) \le$

$N\Pr(X_1^N \geq Na/\Lambda) \leq N\Pr(|X_1^N - Np| \geq N(a/\Lambda - p))$. Since $a/\Lambda > p$, we can use the Hoeffding inequality [21] to obtain $\Pr(\Lambda|||A_N|||_1 > a) \leq 2N\exp(-2N(a/\Lambda - p)^2) \leq C\exp(-N(a/\Lambda - p)^2)$ as desired. $\square$

The next result is much harder but crucial.

**Proposition 14.** *Assume that $\Lambda p < 1$. It holds that*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big|\bar{\ell}_N - \frac{1}{1-\Lambda p}\Big|^2\Big] \leq \frac{C}{N^2},$$

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big|\frac{1}{N}\sum_{i=1}^N \ell_N(i)(c_N(i))^2 - \frac{1}{(1-\Lambda p)^3}\Big|^2\Big] \leq \frac{C}{N^2},$$

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big|\sum_{i=1}^N (\ell_N(i) - \bar{\ell}_N)^2 - \frac{\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big|\Big] \leq \frac{C}{\sqrt{N}}.$$

*Proof.* Recall that $\mathbf{1}_N$ is the $N$-dimensional vector of which all the coordinates equal 1. Let $\ell_N$ (resp. $c_N$) be the vector with coordinates $\ell_N(1), \dots, \ell_N(N)$ (resp. $c_N(1), \dots, c_N(N)$). We also introduce, for all $i = 1, \dots, N$, $L_N(i) = \sum_{j=1}^N A_N(i,j)$ and $C_N(i) = \sum_{j=1}^N A_N(j,i)$, as well as the corresponding vectors $L_N$ and $C_N$. Let us observe that, with obvious notation, $\bar{\ell}_N = \bar{c}_N$ and $\bar{L}_N = \bar{C}_N$. Finally, we introduce the vectors

$$x_N = \ell_N - \bar{\ell}_N \mathbf{1}_N, \quad y_N = c_N - \bar{c}_N \mathbf{1}_N, \quad X_N = L_N - \bar{L}_N \mathbf{1}_N, \quad Y_N = C_N - \bar{C}_N \mathbf{1}_N.$$

We recall that $a = (1 + \Lambda p)/2 \in (0, 1)$ and we introduce $b = (2 + \Lambda p)/3 \in (a, 1)$.

*Step 1.* We introduce the event

$$\mathcal{A}_N = \Big\{||L_N - p\mathbf{1}_N||_2 + ||C_N - p\mathbf{1}_N||_2 \leq N^{1/4}\Big\} \subset \Big\{||X_N||_2 + ||Y_N||_2 \leq N^{1/4}\Big\}.$$

The inclusion comes from the fact that a.s., $||X_N||_2 = ||L_N - \bar{L}_N\mathbf{1}_N||_2 \leq ||L_N - x\mathbf{1}_N||_2$ for any $x \in \mathbb{R}$. Since $NL_N = (Z_1^N, \dots, Z_N^N)$ with $Z_i^N$ i.i.d. and Binomial$(N,p)$-distributed, it is very classical that for any $\alpha > 0$, $\mathbb{E}[||L_N - p\mathbf{1}_N||_2^\alpha] \leq C_\alpha$ (uniformly in $N$), we have similarly $\mathbb{E}[||C_N - p\mathbf{1}_N||_2^\alpha] \leq C_\alpha$, so that

$$\Pr(\mathcal{A}_N) \geq 1 - C_\alpha N^{-\alpha/4}.$$

*Step 2.* We now check the following points: (i) $\mathbb{E}[|\bar{L}_N - p|^2] \leq CN^{-2}$, (ii) $\mathbb{E}[||X_N||_2^4] \leq C$, (iii) $\mathbb{E}[(||X_N||_2^2 - p(1-p))^2] \leq CN^{-1}$ and (iv) $\mathbb{E}[||A_N X_N||_2^2] \leq CN^{-1}$.

Point (i) is clear, because $\bar{L}_N = N^{-2}\sum_{i,j=1}^N \theta_{ij}$ is nothing but the empirical mean of $N^2$ independent Bernoulli$(p)$-random variables. Points (ii) and (iii) are very classical, since $N||X_N||_2^2$ is the empirical variance of $N$ independent Binomial$(N,p)$-random variables. We now prove (iv):

$$\mathbb{E}[||A_N X_N||_2^2] = \sum_{i=1}^N \mathbb{E}\Big[\Big(\sum_{j=1}^N \frac{\theta_{ij}}{N}(L_N(j) - \bar{L}_N)\Big)^2\Big] = \frac{1}{N}\mathbb{E}\Big[\Big(\sum_{j=1}^N \theta_{1j}(L_N(j) - \bar{L}_N)\Big)^2\Big]$$

by symmetry. We now write $\mathbb{E}[||A_N X_N||_2^2] \leq 4N^{-1}(I_N + J_N + K_N)$, where

$$I_N = \mathbb{E}\Big[(\bar{L}_N - p)^2\Big(\sum_{j=1}^N \theta_{1j}\Big)^2\Big], \quad J_N = \mathbb{E}\Big[\Big(\theta_{11}(L_N(1) - p)\Big)^2\Big], \quad K_N = \mathbb{E}\Big[\Big(\sum_{j=2}^N \theta_{1j}(L_N(j) - p)\Big)^2\Big].$$

First, $I_N \le N^2\mathbb{E}[(\bar L_N - p)^2] \le C$ by (i). Next, it is obvious that $J_N \le 1$ (because $\theta_{11} \in \{0,1\}$ and $L_N(1) \in [0,1]$). Finally, the random variables $\theta_{1j}(L_N(j)-p)$ being i.i.d. and centered (for $j = 2,\dots,N$), we may write

$$K_N = (N-1)\mathbb{E}\Big[\Big(\theta_{12}(L_N(2)-p)\Big)^2\Big] \le (N-1)\mathbb{E}\Big[(L_N(2)-p)^2\Big] \le C,$$

since $NL_N(2)$ follows a Binomial$(N,p)$-distribution. This completes the step.

*Step 3.* We next prove that (i) $x_N = \Lambda A_N x_N - \Lambda r_N \mathbf{1}_N + \Lambda \bar\ell_N X_N$ on $\Omega_N^1$, where $r_N = N^{-2}\sum_{i,j=1}^N(\theta_{ij}-p)x_N(j)$ and that (ii) $|r_N| \le N^{-3/4}||x_N||_2$ on $\Omega_N^1 \cap \mathcal{A}_N$.

We start from $\ell_N = Q_N\mathbf{1}_N = (I-\Lambda A_N)^{-1}\mathbf{1}_N$, whence $\ell_N = \mathbf{1}_N + \Lambda A_N \ell_N$. Since $\bar\ell_N = N^{-1}(\ell_N, \mathbf{1}_N)$, we see that $\bar\ell_N = 1 + \Lambda N^{-1}(A_N\ell_N,\mathbf{1}_N)$ (here $(\cdot,\cdot)$ is the usual scalar product on $\mathbb{R}^N$) and thus

$$\begin{aligned} x_N =&\Lambda A_N\ell_N - \Lambda N^{-1}(A_N\ell_N,\mathbf{1}_N)\mathbf{1}_N\\ =&\Lambda A_N x_N - \Lambda N^{-1}(A_N x_N,\mathbf{1}_N)\mathbf{1}_N + \bar\ell_N\Lambda A_N\mathbf{1}_N - \bar\ell_N\Lambda N^{-1}(A_N\mathbf{1}_N,\mathbf{1}_N)\mathbf{1}_N. \end{aligned}$$

It only remains to check that $N^{-1}(A_N x_N, \mathbf{1}_N) = r_N$, which follows from $N^{-1}(A_N x_N,\mathbf{1}_N) = N^{-2}\sum_{i,j=1}^N \theta_{ij}x_N(j)$ and the fact that $\sum_{j=1}^N x_N(j) = 0$; and that $A_N\mathbf{1}_N - N^{-1}(A_N\mathbf{1}_N,\mathbf{1}_N)\mathbf{1}_N = X_N$, which is clear since $A_N\mathbf{1}_N = L_N$.

To verify (ii), we observe that $r_N = N^{-1}\sum_{j=1}^N(C_N(j)-p)x_N(j)$, whence, by the Cauchy-Schwarz inequality, $|r_N| \le N^{-1}||x_N||_2||C_N - p\mathbf{1}_N||_2 \le N^{-3/4}||x_N||_2$ on $\Omega_N^1\cap\mathcal{A}_N$.

*Step 4.* Let $N_0$ be the smallest integer such that $a + \Lambda N_0^{-1/4} \le b$. We check that for all $N \ge N_0$,

$$\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}||x_N||_2 \le C||X_N||_2.$$

Using Step 3 and that $||\mathbf{1}_N||_2 = N^{1/2}$, we write $||x_N||_2 \le \Lambda|||A_N|||_2||x_N||_2 + \Lambda N^{-1/4}||x_N||_2 + \Lambda|\bar\ell_N|||X_N||_2$. But on $\Omega_N^1$, $\Lambda|||A_N|||_2 \le a$ and $|\bar\ell_N| \le C$, see (6) and (7). Hence, for $N \ge N_0$, on $\Omega_N^1\cap\mathcal{A}_N$, we have $||x_N||_2 \le (a+\Lambda N^{-1/4})||x_N||_2 + C||X_N||_2 \le b||x_N||_2 + C||X_N||_2$. Since $b<1$, the conclusion follows.

*Step 5.* We now prove that for $N \ge N_0$,

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}\Big|\bar\ell_N - \frac{1}{1-\Lambda p}\Big|^2\Big] \le \frac{C}{N^2}.$$

Using Step 3, we know that on $\Omega_N^1\cap\mathcal{A}_N$, $\ell_N = \mathbf{1}_N + \Lambda A_N\ell_N$, whence

$$\bar\ell_N = 1 + \frac{\Lambda}{N}\sum_{i,j=1}^N A_N(i,j)\ell_N(j) = 1 + \frac{\Lambda}{N}\sum_{j=1}^N C_N(j)\ell_N(j) = 1 + \Lambda p\bar\ell_N + S_N,$$

where $S_N = \Lambda N^{-1}\sum_{j=1}^N(C_N(j)-p)\ell_N(j)$. Consequently, $\bar\ell_N = (1-\Lambda p)^{-1}(1+S_N)$, and we only have to prove that $\mathbb{E}[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}S_N^2] \le CN^{-2}$. To this end, we write $S_N = \Lambda N^{-1}(a_N+b_N)$, where $a_N = \sum_{j=1}^N(C_N(j)-p)x_N(j)$ and $b_N = \bar\ell_N\sum_{j=1}^N(C_N(j)-p)$. First, since $|\bar\ell_N| \le C$ on $\Omega_N^1$ by (7), we can write $\mathbb{E}[\mathbf{1}_{\Omega_N^1}b_N^2] \le C\mathbb{E}[(\sum_{j=1}^N(C_N(j)-p))^2] = CN^2\mathbb{E}[(\bar C_N - p)^2] \le C$, the last inequality coming from Step 2-(i) since $\bar C_N = \bar L_N$. Next, we use the Cauchy-Schwarz inequality: $a_N^2 \le ||C_N - p\mathbf{1}_N||_2||x_N||_2 \le C||C_N - p\mathbf{1}_N||_2||X_N||_2$ on $\Omega_N^1\cap\mathcal{A}_N$ by Step 4. Consequently, $\mathbb{E}[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}a_N^2] \le C\mathbb{E}[||X_N||_2^2]^{1/2}\mathbb{E}[||C_N - p\mathbf{1}_N||_2^2]^{1/2}$. But $\mathbb{E}[||X_N||_2^2] \le C$ by Step 2-(ii) and we have seen at the end of Step 1 that $\mathbb{E}[||C_N - p\mathbf{1}_N||_2^2] \le C$.

*Step 6.* Here we verify that, still for $N \geq N_0$,

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}\Big|\frac{1}{N}\sum_{i=1}^N \ell_N(i)(c_N(i))^2 - \frac{1}{(1-\Lambda p)^3}\Big|^2\Big] \leq \frac{C}{N^2}.$$

We write, using that $\bar{c}_N = \bar{\ell}_N$,

$$\frac{1}{N}\sum_{i=1}^N \ell_N(i)(c_N(i))^2 = \frac{1}{N}\sum_{i=1}^N \ell_N(i)(c_N(i)-\bar{c}_N)^2 + (\bar{\ell}_N)^3 + \frac{2}{N}\bar{\ell}_N\sum_{i=1}^N \ell_N(i)(c_N(i)-\bar{c}_N).$$

First, since $|\bar{\ell}_N| \leq C$ on $\Omega_N^1$, we have $|(\bar{\ell}_N)^3 - (1-\Lambda p)^{-3}| \leq C|\bar{\ell}_N - (1-\Lambda p)^{-1}|$, whence $\mathbb{E}[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}|(\bar{\ell}_N)^3-(1-\Lambda p)^{-3}|^2] \leq CN^{-2}$ by Step 5. It thus suffices to verify that $\mathbb{E}[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}((a'_N)^2+(b'_N)^2)] \leq C$, where $a'_N = \sum_{i=1}^N \ell_N(i)(c_N(i)-\bar{c}_N)^2$ and $b'_N = \sum_{i=1}^N \ell_N(i)(c_N(i)-\bar{c}_N)$.

First, it holds that $b'_N = \sum_{i=1}^N \ell_N(i)y_N(i) = \sum_{i=1}^N x_N(i)y_N(i)$ because $\sum_{i=1}^N y_N(i) = 0$. Hence $|b'_N| \leq ||x_N||_2||y_N||_2$. But on $\Omega_N^1\cap\mathcal{A}_N$, we know from Step 4 that $||x_N||_2 \leq C||X_N||_2$, and it obviously also holds true that $||y_N||_2 \leq C||Y_N||_2$. We thus conclude that $\mathbb{E}[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}(b'_N)^2] \leq C\mathbb{E}[||X_N||_2^4]^{1/2}\mathbb{E}[||Y_N||_2^4]^{1/2} = \mathbb{E}[||X_N||_2^4]$ by symmetry. Using finally Step 2-(ii), we deduce that indeed, $\mathbb{E}[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}(b'_N)^2] \leq C$. Next, since $|\ell_N(i)| \leq C$ on $\Omega_N^1$ by (7), we can write $|a'_N| \leq C||c_N - \bar{c}_N\mathbf{1}_N||_2^2 = C||y_N||_2^2$. We conclude as previously that $\mathbb{E}[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}(a'_N)^2] \leq C$.

*Step 7.* The goal of this step is to establish that, for all $N \geq N_0$,

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}\Big|||x_N||_2^2 - \frac{\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big|\Big] \leq \frac{C}{\sqrt{N}}.$$

Starting from Step 3, we write

$$x_N - \Lambda\bar{\ell}_N X_N = \Lambda A_N x_N - \Lambda r_N \mathbf{1}_N = \Lambda A_N(x_N - \Lambda\bar{\ell}_N X_N) + \Lambda^2\bar{\ell}_N A_N X_N - \Lambda r_N\mathbf{1}_N.$$

Thus

$$||x_N - \Lambda\bar{\ell}_N X_N||_2 \leq \Lambda|||A_N|||_2||x_N - \Lambda\bar{\ell}_N X_N||_2 + \Lambda^2|\bar{\ell}_N|||A_N X_N||_2 + \Lambda N^{-1/2}||C_N - p\mathbf{1}_N||_2||x_N||_2,$$

where we used that $||\mathbf{1}_N||_2 = N^{1/2}$ and that $|r_N| \leq N^{-1}||C_N - p\mathbf{1}_N||_2||x_N||_2$ on $\Omega_N^1\cap\mathcal{A}_N$, as checked at the end of Step 3. Using now that $\Lambda|||A_N|||_2 \leq a < 1$ and $|\bar{\ell}_N| \leq C$ on $\Omega_N^1$ and that $||x_N||_2 \leq C||X_N||_2$ on $\Omega_N^1\cap\mathcal{A}_N$ by Step 4, we conclude that, still on $\Omega_N^1\cap\mathcal{A}_N$,

$$||x_N - \Lambda\bar{\ell}_N X_N||_2^2 \leq C(||A_N X_N||_2^2 + CN^{-1}||C_N - p\mathbf{1}_N||_2^2||X_N||_2^2).$$

Since now $\mathbb{E}[||A_N X_N||_2^2] \leq CN^{-1}$ by Step 2-(iv), since $\mathbb{E}[||X_N||_2^4] \leq C$ by Step 2-(ii) and since $\mathbb{E}[||C_N - p\mathbf{1}_N||_2^4] \leq C$ (see the end of Step 1), we deduce that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}||x_N - \Lambda\bar{\ell}_N X_N||_2^2\Big] \leq \frac{C}{N}.$$

Next, we observe that $\big|||x_N||_2^2 - (\Lambda\bar{\ell}_N)^2||X_N||_2^2\big| \leq ||x_N - \Lambda\bar{\ell}_N X_N||_2(||x_N||_2 + \Lambda|\bar{\ell}_N|||X_N||_2) \leq C||x_N - \Lambda\bar{\ell}_N X_N||_2||X_N||_2$ on $\Omega_N^1\cap\mathcal{A}_N$ by Step 4 and since $\bar{\ell}_N$ is bounded on $\Omega_N^1$. Hence

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}\Big|||x_N||_2^2 - (\Lambda\bar{\ell}_N)^2||X_N||_2^2\Big|\Big] \leq \frac{C}{\sqrt{N}}\mathbb{E}[||X_N||_2^2]^{1/2} \leq \frac{C}{\sqrt{N}}$$

by Step 2-(ii). To complete the step, it only remains to verify that

$$d_N = \mathbb{E}\Big[\mathbf{1}_{\Omega_N^1\cap\mathcal{A}_N}\Big|(\bar{\ell}_N)^2||X_N||_2^2 - p(1-p)(1-\Lambda p)^{-2}\Big|\Big] \leq \frac{C}{\sqrt{N}}.$$

We naturally write $d_N \le a''_N + b''_N$, where

$$a''_N = \mathbb{E}\Big[\mathbf{1}_{\Omega^1_N \cap \mathcal{A}_N}\Big|(\bar\ell_N)^2 - (1-\Lambda p)^{-2}\Big| ||X_N||_2^2\Big],$$
$$b''_N = (1-\Lambda p)^{-2}\mathbb{E}\Big[\mathbf{1}_{\Omega^1_N \cap \mathcal{A}_N}\Big| ||X_N||_2^2 - p(1-p)\Big|\Big].$$

Step 2-(iii) directly implies that $b''_N \le CN^{-1/2}$. Using that $\bar\ell_N$ is bounded on $\Omega^1_N$, we deduce that $|(\bar\ell_N)^2 - (1-\Lambda p)^{-2}| \le C|\bar\ell_N - (1-\Lambda p)^{-1}|$. Thus

$$a''_N \le C\mathbb{E}\Big[\mathbf{1}_{\Omega^1_N \cap \mathcal{A}_N}\Big|\bar\ell_N - (1-\Lambda p)^{-1}\Big|^2\Big]^{1/2}\mathbb{E}[||X_N||_2^4]^{1/2}.$$

Step 2-(ii) and Step 5 imply that $a''_N \le CN^{-1} \le CN^{-1/2}$ as desired.

*Step 8.* It remains to conclude. It clearly suffices to treat the case where $N \ge N_0$, because $\ell_N(i)$ and $c_N(i)$ are uniformly bounded on $\Omega^1_N$ by (7), so that the inequalities of the statement are trivial when $N \le N_0$ (if the constant $C$ is large enough). Since $\bar\ell_N$ is (uniformly) bounded on $\Omega^1_N$, we have

$$\mathbb{E}\Big[\mathbf{1}_{\Omega^1_N}\Big|\bar\ell_N - \frac{1}{1-\Lambda p}\Big|^2\Big] \le \mathbb{E}\Big[\mathbf{1}_{\Omega^1_N \cap \mathcal{A}_N}\Big|\bar\ell_N - \frac{1}{1-\Lambda p}\Big|^2\Big] + C\,\Pr((\mathcal{A}_N)^c).$$

The first term is bounded by $CN^{-2}$ (by Step 5), as well as the second one (use the last inequality of Step 1 with $\alpha = 8$).

Similarly, using Step 6 and that $\ell_N(i)$ and $c_N(i)$ are (uniformly) bounded on $\Omega^1_N$, we see that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega^1_N}\Big|\frac{1}{N}\sum_{i=1}^N \ell_N(i)(c_N(i))^2 - \frac{1}{(1-\Lambda p)^3}\Big|^2\Big] \le \frac{C}{N^2} + C\,\Pr((\mathcal{A}_N)^c) \le \frac{C}{N^2}.$$

Finally, observe that $\sum_{i=1}^N(\ell_N(i) - \bar\ell_N)^2 = ||x_N||_2^2$ is bounded by $CN$ on $\Omega^1_N$, so that by Step 7,

$$\mathbb{E}\Big[\mathbf{1}_{\Omega^1_N}\Big|\sum_{i=1}^N(\ell_N(i) - \bar\ell_N)^2 - \frac{\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big|\Big] \le \frac{C}{\sqrt{N}} + CN\,\Pr((\mathcal{A}_N)^c) \le \frac{C}{\sqrt{N}}.$$

We used the last inequality of Step 1 with $\alpha = 6$. ☐

4.2. **Preliminary analytic estimates.** In view of (3) and (4), it will be necessary for our purpose to study very precisely the behavior of $\varphi^{\star n}$, which we now do. The following statements may seem rather tedious, but they are exactly the ones we need. Recall that $\varphi^{\star 0}(t-s)ds = \delta_t(ds)$ and that $\varphi^{\star n}(s) = 0$ for $s < 0$ by convention.

**Lemma 15.** *Recall that $\varphi : [0,\infty) \mapsto [0,\infty)$ and that $\Lambda = \int_0^\infty \varphi(s)ds$. Assume that there is $q \ge 1$ such that $\int_0^\infty s^q\varphi(s)ds < \infty$ and set $\kappa = \Lambda^{-1}\int_0^\infty s\varphi(s)ds$.*

*(i) For $n \ge 0$ and $t \ge 0$, we have $\int_0^t s\varphi^{\star n}(t-s)ds = \Lambda^n t - n\Lambda^n\kappa + \varepsilon_n(t)$, where*

$$0 \le \varepsilon_n(t) \le Cn^q\Lambda^n t^{1-q} \quad \text{and} \quad \varepsilon_n(t) \le n\Lambda^n\kappa.$$

*(ii) For $n \ge 0$, for $0 \le t \le z$ and $s \in [0,z]$, we set $\beta_n(t,z,s) = \varphi^{\star n}(z-s) - \varphi^{\star n}(t-s)$. Then $\int_0^z |\beta_n(t,z,s)|ds \le 2\Lambda^n$ and for all $0 \le \Delta \le t$ and all $z \in [t, t+\Delta]$,*

$$\Big|\int_0^z \beta_n(t,z,s)ds\Big| \le Cn^q\Lambda^n t^{-q} \quad \text{and} \quad \int_0^{t-\Delta}|\beta_n(t,z,s)|ds + \Big|\int_{t-\Delta}^z \beta_n(t,z,s)ds\Big| \le Cn^q\Lambda^n\Delta^{-q}.$$

*(iii) For $m,n\geq 0$, for $0\leq t\leq z$, we put $\gamma_{m,n}(t,z)=\int_0^z\int_0^z (s\wedge u)\beta_m(t,z,s)\beta_n(t,z,u)duds$. It holds that $0\leq \gamma_{m,n}(t,t+\Delta)\leq \Lambda^{m+n}\Delta$, for all $t\geq 0$, all $\Delta\geq 0$. Furthermore, there is a family $\kappa_{m,n}$ satisfying $0\leq\kappa_{m,n}\leq (m+n)\kappa$ such that, for all $0\leq\Delta\leq t$,*

$$\gamma_{m,n}(t,t+\Delta)=\Delta\Lambda^{m+n}-\kappa_{m,n}\Lambda^{m+n}+\varepsilon_{m,n}(t,t+\Delta),$$

*with $|\varepsilon_{m,n}(t,t+\Delta)|\leq C(m+n)^q\Lambda^{m+n}t\Delta^{-q}$.*

*Proof.* We introduce some i.i.d. random variables $X_1,X_2,\dots$ with density $\Lambda^{-1}\varphi$ and set $S_0=0$ as well as $S_n=X_1+\dots+X_n$ for all $n\geq 1$. We observe that, by the Minkowski inequality, $\mathbb{E}[S_n^q]\leq n^q\mathbb{E}[X_1^q]\leq Cn^q$, since $\mathbb{E}[X_1^q]=\Lambda^{-1}\int_0^\infty s^q\varphi(s)ds<\infty$ by assumption.

To check (i), we use that $S_n$ has for density $\Lambda^{-n}\varphi^{\star n}$, so that we can write

$$\int_0^t s\varphi^{\star n}(t-s)ds=\int_0^t (t-s)\varphi^{\star n}(s)ds=\Lambda^n\mathbb{E}[(t-S_n)_+]=\Lambda^n t-\Lambda^n\mathbb{E}[S_n]+\varepsilon_n(t),$$

where $\varepsilon_n(t)=\Lambda^n\mathbb{E}[(S_n-t)\mathbf{1}_{\{S_n\geq t\}}]$. We clearly have that $\mathbb{E}[S_n]=n\kappa$, that $\varepsilon_n(t)\geq 0$ and that $\varepsilon_n(t)\leq\Lambda^n\mathbb{E}[S_n]=n\Lambda^n\kappa$. Finally, $\varepsilon_n(t)\leq\Lambda^n\mathbb{E}[S_n\mathbf{1}_{\{S_n\geq t\}}]\leq\Lambda^n t^{1-q}\mathbb{E}[S_n^q]\leq Cn^q\Lambda^n t^{1-q}$.

To check (ii), we observe that $\int_0^z|\beta_n(t,z,s)|ds\leq 2\Lambda^n$ is obvious because $\int_0^\infty\varphi^{\star n}(s)ds=\Lambda^n$ and that, since $\mathbb{E}[S_n^q]\leq Cn^q$,

$$\int_r^\infty\varphi^{\star n}(u)du=\Lambda^n\Pr(S_n\geq r)\leq Cn^q\Lambda^n r^{-q}.$$

We write $\int_0^z\beta_n(t,z,s)ds=\int_0^z\varphi^{\star n}(z-s)ds-\int_0^t\varphi^{\star n}(t-s)ds=\int_t^z\varphi^{\star n}(u)du$, which implies that $|\int_0^z\beta_n(t,z,s)ds|\leq\int_t^\infty\varphi^{\star n}(u)du\leq Cn^q\Lambda^n t^{-q}$. Next, we see that $\int_0^{t-\Delta}|\beta_n(t,z,s)|ds\leq\int_0^{t-\Delta}\varphi^{\star n}(z-u)du+\int_0^{t-\Delta}\varphi^{\star n}(t-u)du\leq 2\int_\Delta^\infty\varphi^{\star n}(u)du\leq Cn^q\Lambda^n\Delta^{-q}$. Finally, using the two previous bounds, $|\int_{t-\Delta}^z\beta_n(t,z,s)ds|\leq|\int_0^z\beta_n(t,z,s)ds|+|\int_0^{t-\Delta}\beta_n(t,z,s)ds|\leq Cn^q\Lambda^n t^{-q}+Cn^q\Lambda^n\Delta^{-q}\leq Cn^q\Lambda^n\Delta^{-q}$ because $\Delta\in[0,t]$ by assumption.

We finally prove (iii) and thus consider $0\leq\Delta\leq t$ and $m,n\geq 0$. We start from

$$\gamma_{m,n}(t,t+\Delta)=\int_0^{t+\Delta}\int_0^{t+\Delta}(s\wedge u)\Big[\varphi^{\star m}(t+\Delta-s)\varphi^{\star n}(t+\Delta-u)+\varphi^{\star m}(t-s)\varphi^{\star n}(t-u)$$
$$-\varphi^{\star m}(t+\Delta-s)\varphi^{\star n}(t-u)-\varphi^{\star m}(t-s)\varphi^{\star n}(t+\Delta-u)\Big]duds.$$

Using another (independent) i.i.d. family $Y_1,Y_2,\dots$ of random variables with density $\Lambda^{-1}\varphi$ and setting $T_m=Y_1+\dots+Y_m$ (or $T_m=0$ if $m=0$), we may write

$$\gamma_{m,n}(t,t+\Delta)=\Lambda^{m+n}\mathbb{E}\Big[(t+\Delta-T_m)_+\wedge(t+\Delta-S_n)_++(t-T_m)_+\wedge(t-S_n)_+$$
$$-(t+\Delta-T_m)_+\wedge(t-S_n)_+-(t-T_m)_+\wedge(t+\Delta-S_n)_+\Big].$$

This precisely rewrites $\gamma_{m,n}(t,t+\Delta)=\Lambda^{m+n}\mathbb{E}[((t+\Delta-T_m\vee S_n)_+-(t-T_m\wedge S_n)_+)_+]$, which implies that $0\leq\gamma_{m,n}(t,t+\Delta)\leq\Lambda^{m+n}\Delta$. We next introduce

$$\delta_{m,n}(t,t+\Delta)=\Lambda^{m+n}\mathbb{E}[(t+\Delta-T_m\vee S_n)-(t-T_m\wedge S_n)],$$

which is nothing but $\delta_{m,n}(t,t+\Delta)=\Lambda^{m+n}(\Delta-\kappa_{m,n})$, where $\kappa_{m,n}=\mathbb{E}[|T_m-S_n|]$ obviously satisfies $0\leq\kappa_{m,n}\leq\kappa(m+n)$. Thus $\gamma_{m,n}(t,t+\Delta)=\Lambda^{m+n}(\Delta-\kappa_{m,n})+\varepsilon_{m,n}(t,t+\Delta)$, where

$\varepsilon_{m,n}(t,t+\Delta) = \gamma_{m,n}(t,t+\Delta) - \delta_{m,n}(t,t+\Delta)$. Finally, it is clear that, since $0 \le \Delta \le t$,

$$|\varepsilon_{m,n}(t,t+\Delta)| \le \Lambda^{m+n}(t+\Delta)\Pr(T_m \vee S_n \ge t+\Delta \text{ or } T_m \wedge S_n \ge t \text{ or } |T_m - S_n| \ge \Delta)$$
$$\le 2\Lambda^{m+n} t \Pr(T_m \ge \Delta \text{ or } S_n \ge \Delta).$$

This is, as usual, bounded by $C\Lambda^{m+n}t(m^q+n^q)\Delta^{-q}$. □

4.3. **Preliminary stochastic analysis.** We handle once for all a number of useful computations concerning the processes introduced in Notation 9.

**Lemma 16.** *We assume $H(q)$ for some $q \ge 1$. Recall that $\Omega_N^1$ and $\ell_N$ were defined in Notation 12 and that all the processes below have been introduced in Notation 9.*

*(i) For any $r \in [1,\infty]$, for all $t \ge 0$,*

$$\mathbf{1}_{\Omega_N^1} \|\mathbb{E}_\theta[\mathbf{Z}_t^N]\|_r \le Ct||\mathbf{1}_N||_r.$$

*(ii) For any $r \in [1,\infty]$, for all $t \ge s \ge 0$,*

$$\mathbf{1}_{\Omega_N^1} \Big\|\mathbb{E}_\theta\Big[\mathbf{Z}_t^N - \mathbf{Z}_s^N\Big] - \mu(t-s)\ell_N\Big\|_r \le C(1 \wedge s^{1-q})||\mathbf{1}_N||_r.$$

*(iii) For all $t \ge s+1 \ge 1$,*

$$\mathbf{1}_{\Omega_N^1} \sup_{i=1,\dots,N} \mathbb{E}_\theta\Big[(Z_t^{i,N} - Z_s^{i,N})^2 + \sup_{[s,t]} |M_r^{i,N} - M_s^{i,N}|^4\Big] + \mathbf{1}_{\Omega_N^1}\mathbb{E}_\theta\Big[(\bar Z_t^N - \bar Z_s^N)^2\Big] \le C(t-s)^2.$$

*Proof.* Recall (3), which asserts that $\mathbb{E}_\theta[\mathbf{Z}_t^N] = \mu\sum_{n\ge0}[\int_0^t s\varphi^{\star n}(t-s)ds]A_N^n\mathbf{1}_N$. Using that $\int_0^t s\varphi^{\star n}(t-s)ds \le t\Lambda^n$, we deduce that $||\mathbb{E}_\theta[\mathbf{Z}_t^N]||_r \le \mu t\sum_{n\ge0}\Lambda^n|||A_N|||_r^n||\mathbf{1}_N||_r$. This is clearly bounded, on $\Omega_N^1$, by $Ct||\mathbf{1}_N||_r$, which proves (i).

Using next Lemma 15-(i), $\mathbb{E}_\theta[\mathbf{Z}_t^N] = \mu\sum_{n\ge0}[\Lambda^n t - n\Lambda^n\kappa + \varepsilon_n(t)]A_N^n\mathbf{1}_N$, where $0 \le \varepsilon_n(t) \le Cn^q\Lambda^n(t^{1-q}\wedge 1)$. Hence

$$\mathbb{E}_\theta[\mathbf{Z}_t^N] - \mathbb{E}_\theta[\mathbf{Z}_s^N] = \mu(t-s)\sum_{n\ge0}\Lambda^n A_N^n\mathbf{1}_N + \mu\sum_{n\ge0}[\varepsilon_n(t)-\varepsilon_n(s)]A_N^n\mathbf{1}_N.$$

But $\sum_{n\ge0}\Lambda^n A_N^n\mathbf{1}_N = Q_N\mathbf{1}_N = \ell_N$ on $\Omega_N^1$. Thus, still on $\Omega_N^1$, since $s \le t$ and $q \ge 1$,

$$\Big\|\mathbb{E}_\theta\Big[\mathbf{Z}_t^N - \mathbf{Z}_s^N\Big] - \mu(t-s)\ell_N\Big\|_r \le C(1\wedge s^{1-q})\sum_{n\ge0} n^q\Lambda^n|||A_N|||_r^n||\mathbf{1}_N||_r \le C(1\wedge s^{1-q})||\mathbf{1}_N||_r.$$

Since $[M^{i,N},M^{i,N}]_t = Z_t^{i,N}$ by Remark 10, the Doob inequality implies that $\mathbb{E}_\theta[\sup_{[s,t]}|M_r^{i,N} - M_s^{i,N}|^4\Big] \le C\mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^2]$. Also, the Cauchy-Schwarz inequality tells us that $\mathbb{E}_\theta[(\bar Z_t^N - \bar Z_s^N)^2] \le N^{-1}\sum_{i=1}^N\mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^2] \le \sup_{i=1,\dots,N}\mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^2]$. Hence we just have to prove that $\sup_{i=1,\dots,N}\mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^2] \le C(t-s)^2$. Recalling that $Z_t^{i,N} = U_t^{i,N} + \mathbb{E}_\theta[Z_t^{i,N}]$, we have to show that, on $\Omega_N^1$, (a) $(\mathbb{E}_\theta[Z_t^{i,N}] - \mathbb{E}_\theta[Z_s^{i,N}])^2 \le C(t-s)^2$ and (b) $\mathbb{E}_\theta[(U_t^{i,N} - U_s^{i,N})^2] \le C(t-s)^2$.

To prove (a), we use (ii) with $r=\infty$ and find that, on $\Omega_N^1$, $\mathbb{E}_\theta[Z_t^{i,N}] - \mathbb{E}_\theta[Z_s^{i,N}] \le \mu(t-s)||\ell_N||_\infty + C||\mathbf{1}_N||_\infty \le C(t-s)$, since $\ell_N$ is bounded on $\Omega_N^1$ and since $t-s \ge 1$ by assumption.

To prove (b), we use (4) to write $U_t^{i,N}-U_s^{i,N}=\sum_{n\geq 0}\int_0^t \beta_n(s,t,r)\sum_{j=1}^N A_N^n(i,j)M_r^{j,N}dr$, where we have set $\beta_n(s,t,r)=\varphi^{\star n}(t-r)-\varphi^{\star n}(s-r)$ as in Lemma 15. We deduce that

$$\mathbb{E}[(U_t^{i,N}-U_s^{i,N})^2]=\sum_{m,n\geq 0}\int_0^t\int_0^t \beta_m(s,t,u)\beta_n(s,t,v)\sum_{j,k=1}^N A_N^m(i,j)A_N^n(i,k)\mathbb{E}_\theta[M_u^{j,N}M_v^{k,N}]dvdu.$$

By Remark 10, $\mathbb{E}_\theta[M_u^{j,N}M_v^{k,N}]=\mathbf{1}_{\{j=k\}}\mathbb{E}_\theta[Z_{u\wedge v}^{j,N}]$. Using now (ii) with $s=0$ and $r=\infty$, we see that $x_t^{j,N}:=\mathbb{E}_\theta[Z_t^{j,N}]-\mu t\ell_N(j)$ satisfies $\sup_{t\geq 0,j=1,\dots,N}|x_t^{j,N}|\leq C$ on $\Omega_N^1$. We thus write $\mathbb{E}_\theta[(U_t^{i,N}-U_s^{i,N})^2]=I+J$, where

$$I=\mu\sum_{m,n\geq 0}\int_0^t\int_0^t \beta_m(s,t,u)\beta_n(s,t,v)\sum_{j=1}^N A_N^m(i,j)A_N^n(i,j)(u\wedge v)\ell_N(j)dudv,$$

$$J=\sum_{m,n\geq 0}\int_0^t\int_0^t \beta_m(s,t,u)\beta_n(s,t,v)\sum_{j=1}^N A_N^m(i,j)A_N^n(i,j)x_{u\wedge v}^{j,N}dudv.$$

First, using only that $x_t^{j,N}$ is uniformly bounded on $\Omega_N^1$ and that $\int_0^t|\beta_m(s,t,u)|du\leq 2\Lambda^m$, we find $|J|\leq C\sum_{m,n\geq 0}\Lambda^{m+n}\sum_{j=1}^N A_N^m(i,j)A_N^n(i,j)=C\sum_{j=1}^N(Q_N(i,j))^2$ on $\Omega_N^1$, whence $|J|\leq C\sum_{j=1}^N(\mathbf{1}_{\{i=j\}}+N^{-1})^2$ by (8). We conclude that $|J|\leq C\leq C(t-s)^2$. Next, we realize that, with the notation of Lemma 15-(iii),

$$I=\mu\sum_{m,n\geq 0}\gamma_{m,n}(s,t)\sum_{j=1}^N A_N^m(i,j)A_N^n(i,j)\ell_N(j).$$

But we know that $0\leq\gamma_{m,n}(s,t)\leq\Lambda^{m+n}(t-s)$. Hence $I\leq\mu(t-s)\sum_{j=1}^N(Q_N(i,j))^2\ell_N(j)\leq C(t-s)$, since $\ell_N$ is bounded on $\Omega_N^1$ and since, as already seen, $\sum_{j=1}^N(Q_N(i,j))^2$ is also bounded on $\Omega_N^1$. We conclude that $\mathbb{E}_\theta[(U_t^{i,N}-U_r^{i,N})^2]\leq C(t-s)\leq C(t-s)^2$ on $\Omega_N^1$, as desired. ☐

4.4. **First estimator.** We recall that $\mathcal{E}_t^N=(\bar{Z}_{2t}^N-\bar{Z}_t^N)/t$, that the matrices $A_N$ and $Q_N$ and the event $\Omega_N^1$ were defined in Notation 12, as well as $\ell_N(i)=\sum_{j=1}^N Q_N(i,j)$ and $\bar{\ell}_N=N^{-1}\sum_{i=1}^N\ell_N(i)$. The goal of this subsection is to establish the following estimate.

**Proposition 17.** *Assume $H(q)$ for some $q\geq 1$. Then for $t\geq 1$,*

$$\mathbf{1}_{\Omega_N^1}\mathbb{E}_\theta\Big[\Big|\mathcal{E}_t^N-\mu\bar{\ell}_N\Big|^2\Big]\leq C\Big(\frac{1}{t^{2q}}+\frac{1}{Nt}\Big).$$

We start with the following lemma (recall that $\bar{U}^N$ was defined in Notation 9).

**Lemma 18.** *Assume $H(q)$ for some $q\geq 1$. Then on $\Omega_N^1$, for $t\geq 1$,*

$$\Big|\mathbb{E}_\theta[\mathcal{E}_t^N]-\mu\bar{\ell}_N\Big|\leq Ct^{-q}\quad\textit{and}\quad \mathbb{E}_\theta[|\bar{U}_t^N|^2]\leq CtN^{-1}.$$

*Proof.* Applying Lemma 16-(ii) with $r=1$, we immediately find, on $\Omega_N^1$,

$$\Big|\mathbb{E}_\theta[\mathcal{E}_t^N]-\mu\bar{\ell}_N\Big|\leq N^{-1}\Big|\Big|\mathbb{E}_\theta\Big[\frac{\mathbf{Z}_{2t}^N-\mathbf{Z}_t^N}{t}\Big]-\mu\ell_N\Big|\Big|_1\leq CN^{-1}t^{-q}||\mathbf{1}_N||_1=Ct^{-q}.$$

Next, we deduce from (4) that $\bar{U}_t^N=N^{-1}\sum_{n\geq 0}\int_0^t\varphi^{\star n}(t-s)\sum_{i,j=1}^N A_N^n(i,j)M_s^{j,N}ds$, whence

$$\mathbb{E}_\theta[|\bar{U}_t^N|^2]^{1/2}\leq N^{-1}\sum_{n\geq 0}\int_0^t\varphi^{\star n}(t-s)\mathbb{E}_\theta\Big[\Big(\sum_{i,j=1}^N A_N^n(i,j)M_s^{j,N}\Big)^2\Big]^{1/2}ds$$

by the Minkowski inequality. But recalling Remark 10, i.e. $\mathbb{E}_\theta[M_s^{j,N}M_s^{l,N}] = \mathbf{1}_{\{j=l\}}\mathbb{E}_\theta[Z_s^{j,N}]$,

$$\mathbb{E}_\theta\Big[\Big(\sum_{i,j=1}^N A_N^n(i,j)M_s^{j,N}\Big)^2\Big] = \sum_{j=1}^N \Big(\sum_{i=1}^N A_N^n(i,j)\Big)^2 \mathbb{E}_\theta[Z_s^{j,N}] \le |||A_N|||_1^{2n}\sum_{j=1}^N \mathbb{E}_\theta[Z_s^{j,N}].$$

We know from Lemma 16-(i) with $r=1$ that $\sum_{j=1}^N \mathbb{E}_\theta[Z_s^{j,N}] \le CNs$ on $\Omega_N^1$. Hence, still on $\Omega_N^1$,

$$\mathbb{E}_\theta[|\bar{U}_t^N|^2]^{1/2} \le \frac{C}{N}\sum_{n\ge 0}|||A_N|||_1^n \int_0^t \sqrt{Ns}\varphi^{\star n}(t-s)ds \le \frac{Ct^{1/2}}{N^{1/2}}\sum_{n\ge 0}\Lambda^n |||A_N|||_1^n \le \frac{Ct^{1/2}}{N^{1/2}}$$

as desired. □

We can now give the

*Proof of Proposition 17.* It suffices to write

$$\mathbb{E}_\theta\Big[\Big|\mathcal{E}_t^N - \mu\bar{\ell}_N\Big|^2\Big] \le 2\mathbb{E}_\theta\Big[\Big|\mathcal{E}_t^N - \mathbb{E}_\theta[\mathcal{E}_t^N]\Big|^2\Big] + 2\Big|\mathbb{E}_\theta[\mathcal{E}_t^N] - \mu\bar{\ell}_N\Big|^2$$

and to observe that $|\mathcal{E}_t^N - \mathbb{E}_\theta[\mathcal{E}_t^N]| = |\bar{U}_{2t}^N - \bar{U}_t^N|/t \le |\bar{U}_{2t}^N|/t + |\bar{U}_t^N|/t$, whence finally

$$\mathbb{E}_\theta\Big[\Big|\mathcal{E}_t^N - \mu\bar{\ell}_N\Big|^2\Big] \le \frac{4}{t^2}(\mathbb{E}_\theta[|\bar{U}_{2t}^N|^2] + \mathbb{E}_\theta[|\bar{U}_t^N|^2]) + 2\Big|\mathbb{E}_\theta[\mathcal{E}_t^N] - \mu\bar{\ell}_N\Big|^2.$$

Then the proposition immediately follows from Lemma 18. □

4.5. **Second estimator.** We recall that $\mathcal{V}_t^N = \sum_{i=1}^N[(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mathcal{E}_t^N]^2 - N\mathcal{E}_t^N/t$ where $\mathcal{E}_t^N = (\bar{Z}_{2t}^N - \bar{Z}_t^N)/t$, that the matrices $A_N$ and $Q_N$ and the event $\Omega_N^1$ were defined in Notation 12, as well as $\ell_N(i) = \sum_{j=1}^N Q_N(i,j)$ and $\bar{\ell}_N = N^{-1}\sum_{i=1}^N \ell_N(i)$. We also introduce $\mathcal{V}_\infty^N = \mu^2\sum_{i=1}^N[\ell_N(i) - \bar{\ell}_N]^2$.

**Proposition 19.** *Assume $H(q)$ for some $q \ge 1$. Then for $t \ge 1$, a.s.,*

$$\mathbf{1}_{\Omega_N^1}\mathbb{E}_\theta\Big[\Big|\mathcal{V}_t^N - \mathcal{V}_\infty^N\Big|\Big] \le C\Big(1 + \sum_{i=1}^N\Big[\ell_N(i) - \bar{\ell}_N\Big]^2\Big)^{1/2}\Big(\frac{N}{t^q} + \frac{\sqrt{N}}{t} + \frac{1}{\sqrt{t}}\Big).$$

Observe that the term $\sum_{i=1}^N[\ell_N(i) - \bar{\ell}_N]^2$ will not cause any problem, since its expectation (restricted to $\Omega_N^1$) is uniformly bounded, see Proposition 14.

We write $|\mathcal{V}_t^N - \mathcal{V}_\infty^N| \le \Delta_t^{N,1} + \Delta_t^{N,2} + \Delta_t^{N,3}$, where

$$\begin{aligned}
\Delta_t^{N,1} =& \Big|\sum_{i=1}^N[(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mathcal{E}_t^N]^2 - \sum_{i=1}^N[(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mu\bar{\ell}_N]^2\Big|,\\
\Delta_t^{N,2} =& \Big|\sum_{i=1}^N[(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mu\ell_N(i)]^2 - N\mathcal{E}_t^N/t\Big|,\\
\Delta_t^{N,3} =& 2\Big|\sum_{i=1}^N[(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mu\ell_N(i)][\mu\ell_N(i) - \mu\bar{\ell}_N]\Big|.
\end{aligned}$$

We next write $\Delta_t^{N,2} \leq \Delta_t^{N,21} + \Delta_t^{N,22} + \Delta_t^{N,23}$, where

$$
\begin{aligned}
\Delta_t^{N,21} =& \Big| \sum_{i=1}^N \Big[ (Z_{2t}^{i,N} - Z_t^{i,N})/t - \mathbb{E}_\theta[(Z_{2t}^{i,N} - Z_t^{i,N})/t] \Big]^2 - N\mathcal{E}_t^N/t \Big|,\\
\Delta_t^{N,22} =& \sum_{i=1}^N \Big[ \mathbb{E}_\theta[(Z_{2t}^{i,N} - Z_t^{i,N})/t] - \mu\ell_N(i) \Big]^2,\\
\Delta_t^{N,23} =& 2\Big| \sum_{i=1}^N \Big[ (Z_{2t}^{i,N} - Z_t^{i,N})/t - \mathbb{E}_\theta[(Z_{2t}^{i,N} - Z_t^{i,N})/t] \Big] \Big[ \mathbb{E}_\theta[(Z_{2t}^{i,N} - Z_t^{i,N})/t] - \mu\ell_N(i) \Big] \Big|.
\end{aligned}
$$

We will also need to write, recalling that $U_t^{i,N} = Z_t^{i,N} - \mathbb{E}_\theta[Z_t^{i,N}]$,

$$
\Delta_t^{N,21} = \Big| \sum_{i=1}^N \Big[ (U_{2t}^{i,N} - U_t^{i,N})/t \Big]^2 - N\mathcal{E}_t^N/t \Big| \leq \Delta_t^{N,211} + \Delta_t^{N,212} + \Delta_t^{N,213},
$$

where

$$
\begin{aligned}
\Delta_t^{N,211} =& \Big| \sum_{i=1}^N \Big\{ \Big( (U_{2t}^{i,N} - U_t^{i,N})/t \Big)^2 - \mathbb{E}_\theta \Big[ \Big( (U_{2t}^{i,N} - U_t^{i,N})/t \Big)^2 \Big] \Big\} \Big|,\\
\Delta_t^{N,212} =& \Big| \sum_{i=1}^N \mathbb{E}_\theta \Big[ \Big( (U_{2t}^{i,N} - U_t^{i,N})/t \Big)^2 \Big] - \mathbb{E}_\theta[N\mathcal{E}_t^N/t] \Big|,\\
\Delta_t^{N,213} =& \Big| N\mathcal{E}_t^N/t - \mathbb{E}_\theta[N\mathcal{E}_t^N/t] \Big|.
\end{aligned}
$$

Finally, we will use that $\Delta_t^{N,3} \leq \Delta_t^{N,31} + \Delta_t^{N,32}$, where

$$
\begin{aligned}
\Delta_t^{N,31} =& 2\Big| \sum_{i=1}^N \Big[ (Z_{2t}^{i,N} - Z_t^{i,N})/t - \mathbb{E}_\theta[(Z_{2t}^{i,N} - Z_t^{i,N})/t] \Big] \Big[ \mu\ell_N(i) - \mu\bar{\ell}_N \Big] \Big|,\\
\Delta_t^{N,32} =& 2\Big| \sum_{i=1}^N \Big[ \mathbb{E}_\theta[(Z_{2t}^{i,N} - Z_t^{i,N})/t] - \mu\ell_N(i) \Big] \Big[ \mu\ell_N(i) - \mu\bar{\ell}_N \Big] \Big|.
\end{aligned}
$$

To summarize, we have to bound $\Delta_t^{N,1}$, $\Delta_t^{N,211}$, $\Delta_t^{N,212}$, $\Delta_t^{N,213}$, $\Delta_t^{N,22}$, $\Delta_t^{N,23}$, $\Delta_t^{N,31}$ and $\Delta_t^{N,32}$. Only the term $\Delta_t^{N,211}$ is really difficult.

In the following lemma, we treat the easy terms. We do not try to be optimal when not useful: for example in (iv) below, some sharper estimate could probably be obtained with more work, but since we already have a term in $N^{1/2}t^{-1}$ (see Lemma 24), this would be useless. We also recall that we do not really try to optimize the dependence in $q$: it is likely that $t^{-q}$ could be replaced by $t^{-2q}$ here and there.

**Lemma 20.** *Assume $H(q)$ for some $q \geq 1$. Then a.s. on $\Omega_N^1$, for $t \geq 1$,*
*(i)* $\mathbb{E}_\theta[\Delta_t^{N,1}] \leq C(Nt^{-2q} + t^{-1})$,
*(ii)* $\mathbb{E}_\theta[\Delta_t^{N,22}] \leq CNt^{-2q}$,
*(iii)* $\mathbb{E}_\theta[\Delta_t^{N,23}] \leq CNt^{-q}$,
*(iv)* $\mathbb{E}_\theta[\Delta_t^{N,213}] \leq CN^{1/2}t^{-3/2}$,
*(v)* $\mathbb{E}_\theta[\Delta_t^{N,32}] \leq CNt^{-q}$.

*Proof.* We work on $\Omega_N^1$ during the whole proof.

Using that $\mathcal{E}_t^N = N^{-1}\sum_{i=1}^N [(Z_{2t}^{i,N} - Z_t^{i,N})/t]$, one easily checks that $\Delta_t^{N,1} = N|\mathcal{E}_t^N - \mu\bar\ell_N|^2$. Thus point (i) follows from Proposition 17.

Next, we observe that $\Delta_t^{N,22} = ||\mathbb{E}_\theta[(\mathbf{Z}_{2t}^N - \mathbf{Z}_t^N)/t] - \mu\ell_N||_2^2$. Applying Lemma 16-(ii) with $r=2$, we conclude that indeed, $\Delta_t^{N,22} \le Ct^{-2q}||\mathbf{1}_N||_2^2 = CNt^{-2q}$.

We write

$$\Delta_t^{N,23} \le 2\Big\|(\mathbf{Z}_{2t}^N - \mathbf{Z}_t^N)/t - \mathbb{E}_\theta[(\mathbf{Z}_{2t}^N - \mathbf{Z}_t^N)/t]\Big\|_1 \Big\|\mathbb{E}_\theta[(\mathbf{Z}_{2t}^N - \mathbf{Z}_t^N)/t] - \mu\ell_N\Big\|_\infty.$$

Applying Lemma 16-(ii) with $r=\infty$, we deduce that $\|\mathbb{E}_\theta[(\mathbf{Z}_{2t}^N - \mathbf{Z}_t^N)/t] - \mu\ell_N\|_\infty \le Ct^{-q}$. Lemma 16-(i) with $r=1$ gives us that $\mathbb{E}_\theta[\|(\mathbf{Z}_{2t}^N - \mathbf{Z}_t^N)/t - \mathbb{E}_\theta[(\mathbf{Z}_{2t}^N - \mathbf{Z}_t^N)/t]\|_1] \le 2t^{-1}\|\mathbb{E}_\theta[\mathbf{Z}_{2t}^N + \mathbf{Z}_t^N]\|_1 \le CN$. We thus find that indeed, $\mathbb{E}_\theta[\Delta_t^{N,23}] \le CNt^{-q}$.

Since $\Delta_t^{N,213} = (N/t)|\mathcal{E}_t^N - \mathbb{E}_\theta[\mathcal{E}_t^N]| = Nt^{-2}|\bar U_{2t}^N - \bar U_t^N| \le Nt^{-2}(|\bar U_{2t}^N| + |\bar U_t^N|)$, we deduce from Lemma 18 that $\mathbb{E}_\theta[\Delta_t^{N,213}] \le CNt^{-2}\sqrt{t/N} = CN^{1/2}t^{-3/2}$.

Finally, starting from $\Delta_t^{N,32} \le 2\mu||\mathbb{E}_\theta[(\mathbf{Z}_{2t}^N - \mathbf{Z}_t^N)/t] - \mu\ell_N||_\infty ||\ell_N - \bar\ell_N\mathbf{1}_N||_1$ and using that, as already seen when studying $\Delta_t^{N,23}$, $\|\mathbb{E}_\theta[(\mathbf{Z}_{2t}^N - \mathbf{Z}_t^N)/t] - \mu\ell_N\|_\infty \le Ct^{-q}$, we conclude that $\Delta_t^{N,32} \le Ct^{-q}||\ell_N - \bar\ell_N\mathbf{1}_N||_1 \le CNt^{-q}$, since $\ell_N$ is bounded (see (7)) on $\Omega_N^1$. ☐

Next, we treat the term $\Delta_t^{N,212}$.

**Lemma 21.** *Assume $H(q)$ for some $q \ge 1$. Then a.s. on $\Omega_N^1$, for $t \ge 1$, $\mathbb{E}_\theta[\Delta_t^{N,212}] \le Ct^{-1}$.*

*Proof.* We work on $\Omega_N^1$. Recalling that $N\mathcal{E}_t^N = t^{-1}\sum_{i=1}^N (Z_{2t}^{i,N} - Z_t^{i,N})$, we may write $\mathbb{E}_\theta[\Delta_t^{N,212}] \le t^{-2}\sum_{i=1}^N a_i$, where $a_i = |\mathbb{E}_\theta[(U_{2t}^{i,N} - U_t^{i,N})^2 - (Z_{2t}^{i,N} - Z_t^{i,N})]|$. Now we infer from (4) that $U_t^{i,N} = M_t^{i,N} + \sum_{n\ge1}\int_0^t \varphi^{\star n}(t-s)\sum_{j=1}^N A_N^n(i,j)M_s^{j,N}ds$, so that $U_{2t}^{i,N} - U_t^{i,N} = M_{2t}^{i,N} - M_t^{i,N} + R_t^{i,N}$, where

$$R_t^{i,N} = \sum_{n\ge1}\int_0^{2t}\beta_n(t,2t,s)\sum_{j=1}^N A_N^n(i,j)M_s^{j,N}ds.$$

We have set $\beta_n(t,2t,s) = \varphi^{\star n}(2t-s) - \varphi^{\star n}(t-s)$ as in Lemma 15 and the only thing we will use is that $\int_0^{2t}|\beta_n(t,2t,s)|ds \le 2\Lambda^n$. Recalling that $M^{i,N}$ is a martingale with quadratic variation $[M^{i,N}, M^{i,N}]_t = Z_t^{i,N}$, see Remark 10, we deduce that $\mathbb{E}_\theta[(M_{2t}^{i,N} - M_t^{i,N})^2] = \mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]$. Hence

$$a_i = \mathbb{E}_\theta[(R_t^{i,N})^2] + 2\mathbb{E}_\theta[(M_{2t}^{i,N} - M_t^{i,N})R_t^{i,N}] = b_i + d_i,$$

the last equality standing for a definition. We first write

$$b_i = \sum_{m,n\ge1}\int_0^{2t}\int_0^{2t}\beta_m(t,2t,s)\beta_n(t,2t,u)\sum_{j,k=1}^N A_N^m(i,j)A_N^n(i,k)\mathbb{E}_\theta[M_s^{j,N}M_u^{k,N}]duds.$$

But we know that $\mathbb{E}_\theta[M_s^{j,N}M_u^{k,N}] = \mathbf{1}_{\{j=k\}}\mathbb{E}_\theta[Z_{s\wedge u}^{j,N}]$ by Remark 10 and that $\mathbb{E}_\theta[Z_{s\wedge u}^{j,N}] \le Ct$ on $\Omega_N^1$ by Lemma 16-(i) (with $r=\infty$). Hence

$$b_i \le Ct\sum_{m,n\ge1}\Lambda^{m+n}\sum_{j=1}^N A_N^m(i,j)A_N^n(i,j) = Ct\sum_{j=1}^N\Big(\sum_{n\ge1}\Lambda^n A_N^n(i,j)\Big)^2.$$

But $\sum_{n\ge1}\Lambda^n A_N^n(i,j) = Q_N(i,j) - \mathbf{1}_{\{i=j\}} \le CN^{-1}$ on $\Omega_N^1$ by (8), so that $b_i \le CtN^{-1}$.

Next, we start from

$$d_i = 2\sum_{n\geq 1}\int_0^{2t}\beta_n(t,2t,s)\sum_{j=1}^N A_N^n(i,j)\mathbb{E}_\theta[(M_{2t}^{i,N}-M_t^{i,N})M_s^{j,N}]ds.$$

As previously, we see that $\mathbb{E}_\theta[(M_{2t}^{i,N}-M_t^{i,N})M_s^{j,N}]=0$ if $i\neq j$ and that $\mathbb{E}_\theta[(M_{2t}^{i,N}-M_t^{i,N})M_s^{i,N}]=\mathbb{E}_\theta[Z_{2t\wedge s}^{i,N}-Z_{t\wedge s}^{i,N}]\leq Ct$ on $\Omega_N^1$ (by Lemma 16-(i)), whence

$$d_i\leq Ct\sum_{n\geq 1}\Lambda^n A_N^n(i,i)=Ct(Q_N(i,i)-1)\leq CtN^{-1}$$

on $\Omega_N^1$ by (8) again. Finally, $a_i\leq CtN^{-1}$, so that $\mathbb{E}_\theta[\Delta_t^{N,212}]\leq t^{-2}\sum_{i=1}^N a_i\leq Ct^{-1}$ on $\Omega_N^1$. □

We next compute some covariances in the following tedious lemma.

**Lemma 22.** *Assume $H(q)$ for some $q\geq 1$. Then a.s., on $\Omega_N^1$, for all $t\geq 1$, all $k,l,a,b\in\{1,\dots,N\}$, all $r,s,u,v\in[0,t]$,*

*(i)* $|\mathrm{Cov}_\theta(Z_r^{k,N},Z_s^{l,N})|=|\mathrm{Cov}_\theta(U_r^{k,N},U_s^{l,N})|\leq Ct(N^{-1}+\mathbf{1}_{\{k=l\}})$,

*(ii)* $|\mathrm{Cov}_\theta(Z_r^{k,N},M_s^{l,N})|=|\mathrm{Cov}_\theta(U_r^{k,N},M_s^{l,N})|\leq Ct(N^{-1}+\mathbf{1}_{\{k=l\}})$,

*(iii)* $|\mathrm{Cov}_\theta(Z_r^{k,N},\int_0^s M_{\tau-}^{l,N}dM_\tau^{l,N})|=|\mathrm{Cov}_\theta(U_r^{k,N},\int_0^s M_{\tau-}^{l,N}dM_\tau^{l,N})|\leq Ct^{3/2}(N^{-1}+\mathbf{1}_{\{k=l\}})$,

*(iv)* $|\mathbb{E}_\theta[M_r^{k,N}M_s^{k,N}M_u^{l,N}]|\leq CN^{-1}t$ *if* $\#\{k,l\}=2$,

*(v)* $|\mathrm{Cov}_\theta(M_r^{k,N}M_s^{l,N},M_u^{a,N}M_v^{b,N})|=0$ *if* $\#\{k,l,a,b\}=4$,

*(vi)* $|\mathrm{Cov}_\theta(M_r^{k,N}M_s^{k,N},M_u^{a,N}M_v^{b,N})|\leq CN^{-2}t$ *if* $\#\{k,a,b\}=3$,

*(vii)* $|\mathrm{Cov}_\theta(M_r^{k,N}M_s^{k,N},M_u^{a,N}M_v^{a,N})|\leq CN^{-1}t^{3/2}$ *if* $\#\{k,a\}=2$,

*(viii)* $|\mathrm{Cov}_\theta(M_r^{k,N}M_s^{l,N},M_u^{a,N}M_v^{b,N})|\leq Ct^2$ *without condition.*

*Proof.* We work on $\Omega_N^1$ and start with point (i). First, it is clear, since $U_t^{k,N}=Z_t^{k,N}-\mathbb{E}_\theta[Z_t^{k,N}]$, that $\mathrm{Cov}_\theta(Z_r^{k,N},Z_s^{l,N})=\mathrm{Cov}_\theta(U_r^{k,N},U_s^{l,N})$. Then we infer from (4) that

$$\mathrm{Cov}_\theta(U_r^{k,N},U_s^{l,N})=\sum_{m,n\geq 0}\int_0^r\int_0^s\varphi^{\star m}(r-x)\varphi^{\star n}(s-y)\sum_{i,j=1}^N A_N^m(k,i)A_N^n(l,j)\mathrm{Cov}_\theta(M_x^{i,N},M_y^{j,N})dydx.$$

But we know (see Remark 10) that $\mathrm{Cov}_\theta(M_x^{i,N},M_y^{j,N})=\mathbf{1}_{\{i=j\}}\mathbb{E}_\theta[Z_{x\wedge y}^{i,N}]\leq C\mathbf{1}_{\{i=j\}}t$ by Lemma 16-(i) (with $r=\infty$). Thus

$$|\mathrm{Cov}_\theta(U_r^{k,N},U_s^{l,N})|\leq Ct\sum_{m,n\geq 0}\Lambda^{m+n}\sum_{i=1}^N A_N^m(k,i)A_N^n(l,i)=Ct\sum_{i=1}^N Q_N(k,i)Q_N(l,i).$$

Recalling (8), $\sum_{i=1}^N Q_N(k,i)Q_N(l,i)\leq C\sum_{i=1}^N(N^{-1}+\mathbf{1}_{\{k=i\}})(N^{-1}+\mathbf{1}_{\{l=i\}})\leq C(N^{-1}+\mathbf{1}_{\{k=l\}})$. Point (i) is checked.

For point (ii), we again have $\mathrm{Cov}_\theta(Z_r^{k,N},M_s^{l,N})=\mathrm{Cov}_\theta(U_r^{k,N},M_s^{l,N})$ and, using again (4),

$$\mathrm{Cov}_\theta(U_r^{k,N},M_s^{l,N})=\sum_{n\geq 0}\int_0^r\varphi^{\star n}(r-x)\sum_{i=1}^N A_N^n(k,i)\mathrm{Cov}_\theta(M_x^{i,N},M_s^{l,N})dx.$$

Since $|\mathbb{C}\mathrm{ov}_\theta\,(M^{i,N}_x, M^{l,N}_s)| \le C\mathbf{1}_{\{i=l\}}t$ as in (i), we conclude that

$$|\mathbb{C}\mathrm{ov}_\theta\,(U^{k,N}_r, M^{l,N}_s)| \le Ct\sum_{n\ge 0}\Lambda^n A^n_N(k,l) = CtQ_N(k,l) \le Ct(N^{-1}+\mathbf{1}_{\{k=l\}}).$$

Point (iii) is checked similarly as (ii), provided we verify that $|\mathbb{C}\mathrm{ov}_\theta\,(M^{i,N}_x, \int_0^s M^{l,N}_{\tau-}dM^{l,N}_\tau)| \le C\mathbf{1}_{\{i=l\}}t^{3/2}$. This is obvious if $i\ne l$ because the martingales $M^{i,N}$ and $\int_0^\cdot M^{l,N}_{\tau-}dM^{l,N}_\tau$ are orthogonal, and relies on the fact, if $i=l$, that

$$\Big|\mathbb{C}\mathrm{ov}_\theta\,\Big(M^{i,N}_x, \int_0^s M^{i,N}_{\tau-}dM^{i,N}_\tau\Big)\Big| \le \mathbb{E}_\theta[|M^{i,N}_x|^2]^{1/2}\mathbb{E}_\theta\Big[\Big|\int_0^s M^{i,N}_{\tau-}dM^{i,N}_\tau\Big|^2\Big]^{1/2} \le Ct^{3/2}.$$

The last inequality uses that $\mathbb{E}_\theta[|M^{i,N}_x|^2] = \mathbb{E}_\theta[Z^{i,N}_x] \le Ct$ by Remark 10 and Lemma 16-(i) and that $\mathbb{E}_\theta[|\int_0^s M^{i,N}_{\tau-}dM^{i,N}_\tau|^2] \le Ct^2$. Indeed, we have $[\int_0^\cdot M^{i,N}_{\tau-}dM^{i,N}_\tau, \int_0^\cdot M^{i,N}_{\tau-}dM^{i,N}_\tau]_s = \int_0^s (M^{i,N}_{\tau-})^2 dZ^{i,N}_{\tau-} \le (M^{i,N,*}_s)^2 Z^{i,N}_s$, whence

$$\mathbb{E}_\theta\Big[\Big|\int_0^s M^{i,N}_{\tau-}dM^{i,N}_\tau\Big|^2\Big] \le \mathbb{E}_\theta[(M^{i,N,*}_s)^2 Z^{i,N}_s] \le \mathbb{E}_\theta[(M^{i,N,*}_s)^4]^{1/2}\mathbb{E}_\theta[(Z^{i,N}_s)^2]^{1/2},$$

which is bounded by $Ct^2$ by Lemma 16-(iii).

For point (iv), we assume e.g. that $r\le s$ and first note that

$$\mathbb{E}_\theta[M^{k,N}_r M^{k,N}_s M^{l,N}_u] = \mathbb{E}_\theta[M^{k,N}_r \mathbb{E}_\theta[M^{k,N}_s M^{l,N}_u|\mathcal{F}_r]] = \mathbb{E}_\theta[(M^{k,N}_r)^2 M^{l,N}_{u\wedge r}]$$

because the martingales $M^{k,N}$ and $M^{l,N}$ are orthogonal. Since $[M^{k,N},M^{k,N}]_r = Z^{k,N}_r$, it holds that $(M^{k,N}_r)^2 = 2\int_0^r M^{k,N}_{\tau-}dM^{k,N}_\tau + Z^{k,N}_r$. Using that $\int_0^\cdot M^{k,N}_{\tau-}dM^{k,N}_\tau$ and $M^{l,N}$ are orthogonal, we conclude that $\mathbb{E}_\theta[(M^{k,N}_r)^2 M^{l,N}_{u\wedge r}] = \mathbb{E}_\theta[Z^{k,N}_r M^{l,N}_{u\wedge r}] = \mathbb{C}\mathrm{ov}_\theta\,(Z^{k,N}_r, M^{l,N}_{u\wedge r})$. Since $k\ne l$, we conclude using point (ii).

Point (v) is obvious, since when $k,l,a,b$ are pairwise different, the martingales $M^{k,N}$, $M^{l,N}$, $M^{a,N}$ and $M^{b,N}$ are orthogonal.

Point (vi) is harder. Recall that $\#\{k,a,b\}=3$, so that clearly, $\mathbb{C}\mathrm{ov}_\theta\,(M^{k,N}_r M^{k,N}_s, M^{a,N}_u M^{b,N}_v) = \mathbb{E}_\theta[M^{k,N}_r M^{k,N}_s M^{a,N}_u M^{b,N}_v]$. We assume e.g. that $r\le s$ and we observe that

$$\mathbb{E}_\theta[M^{k,N}_r M^{k,N}_s M^{a,N}_u M^{b,N}_v] = \mathbb{E}_\theta[M^{k,N}_r \mathbb{E}_\theta[M^{k,N}_s M^{a,N}_u M^{b,N}_v|\mathcal{F}_r]] = \mathbb{E}_\theta[(M^{k,N}_r)^2 M^{a,N}_{u\wedge r} M^{b,N}_{v\wedge r}]$$

because $M^{k,N}$, $M^{a,N}$ and $M^{b,N}$ are orthogonal. We thus have to prove that for all $r,u,v\in[0,t]$ with $u,v\le r$, $|\mathbb{E}_\theta[(M^{k,N}_r)^2 M^{a,N}_u M^{b,N}_v]| \le CN^{-2}t$. We write $(M^{k,N}_r)^2 = 2\int_0^r M^{k,N}_{\tau-}dM^{k,N}_\tau + Z^{k,N}_r$ as in the proof of (iv). The three martingales $\int_0^\cdot M^{k,N}_{\tau-}dM^{k,N}_\tau$, $M^{a,N}$ and $M^{b,N}$ being orthogonal, we find $\mathbb{E}_\theta[(M^{k,N}_r)^2 M^{a,N}_u M^{b,N}_v] = \mathbb{E}_\theta[Z^{k,N}_r M^{a,N}_u M^{b,N}_v] = \mathbb{E}_\theta[U^{k,N}_r M^{a,N}_u M^{b,N}_v]$. We next write, starting again from (4),

$$\mathbb{E}_\theta[U^{k,N}_r M^{a,N}_u M^{b,N}_v] = \sum_{n\ge 0}\int_0^r \varphi^{\star n}(r-x)\sum_{j=1}^N A^n_N(k,j)\mathbb{E}_\theta[M^{j,N}_x M^{a,N}_u M^{b,N}_v]dx.$$

But $|\mathbb{E}_\theta[M^{j,N}_x M^{a,N}_u M^{b,N}_v]|$ is zero if $j\notin\{a,b\}$ because the martingales $M^{j,N}$, $M^{a,N}$ and $M^{b,N}$ are orthogonal, and is bounded by $CN^{-1}t$ else by point (iv). As a consequence,

$$|\mathbb{E}_\theta[U^{k,N}_r M^{a,N}_u M^{b,N}_v]| \le CN^{-1}t\sum_{n\ge 0}\Lambda^n(A^n_N(k,a)+A^n_N(k,b)) = CN^{-1}t(Q_N(k,a)+Q_N(k,b)).$$

Since $k\ne a$ and $k\ne b$, this is bounded by $CN^{-2}t$ by (8).

For (vii), we assume e.g. that $r \le s$ and $u \le v$ and we recall that $k \ne a$. We have

$$\begin{aligned}&\mathbb{C}\mathrm{ov}_\theta\,(M^{k,N}_r M^{k,N}_s, M^{a,N}_u M^{a,N}_v)\\ =&\mathbb{C}\mathrm{ov}_\theta\,((M^{k,N}_r)^2,(M^{a,N}_u)^2)+\mathbb{C}\mathrm{ov}_\theta\,(M^{k,N}_r(M^{k,N}_s-M^{k,N}_r),(M^{a,N}_u)^2)\\ &+\mathbb{C}\mathrm{ov}_\theta\,((M^{k,N}_r)^2,M^{a,N}_u(M^{a,N}_v-M^{a,N}_u))+\mathbb{C}\mathrm{ov}_\theta\,(M^{k,N}_r(M^{k,N}_s-M^{k,N}_r),M^{a,N}_u(M^{a,N}_v-M^{a,N}_u))\\ =&I+J+K+L.\end{aligned}$$

First, $L=0$. Indeed, assuming e.g. that $r \ge u$, we have

$$\begin{aligned}L=&\mathbb{E}_\theta[M^{k,N}_r(M^{k,N}_s-M^{k,N}_r)M^{a,N}_u(M^{a,N}_v-M^{a,N}_r+M^{a,N}_r-M^{a,N}_u)]\\ =&\mathbb{E}_\theta[M^{k,N}_r M^{a,N}_u \mathbb{E}_\theta[(M^{k,N}_s-M^{k,N}_r)(M^{a,N}_v-M^{a,N}_r)|\mathcal{F}_r]]\\ &+\mathbb{E}_\theta[M^{k,N}_r M^{a,N}_u(M^{a,N}_r-M^{a,N}_u)\mathbb{E}_\theta[M^{k,N}_s-M^{k,N}_r|\mathcal{F}_r]]\end{aligned}$$

and in both terms, the conditional expectation vanishes. Next, we write as usual $(M^{k,N}_r)^2 = 2\int_0^r M^{k,N}_{\tau-}dM^{k,N}_\tau + Z^{k,N}_r$ and $(M^{a,N}_u)^2 = 2\int_0^u M^{a,N}_{\tau-}dM^{a,N}_\tau + Z^{a,N}_u$. By orthogonality of the martingales $\int_0^\cdot M^{k,N}_{\tau-}dM^{k,N}_\tau$ and $\int_0^\cdot M^{a,N}_{\tau-}dM^{a,N}_\tau$, we find

$$I=\mathbb{C}\mathrm{ov}_\theta\,(Z^{k,N}_r,Z^{a,N}_u)+2\mathbb{C}\mathrm{ov}_\theta\Big(Z^{k,N}_r,\int_0^u M^{a,N}_{\tau-}dM^{a,N}_\tau\Big)+2\mathbb{C}\mathrm{ov}_\theta\Big(\int_0^r M^{k,N}_{\tau-}dM^{k,N}_\tau,Z^{a,N}_u\Big).$$

We deduce from points (i) and (iii), since $k \ne a$, that $|I| \le C(N^{-1}t+N^{-1}t^{3/2}) \le CN^{-1}t^{3/2}$. We now treat $K$. It vanishes if $u \ge r$, because $\mathbb{E}_\theta[M^{a,N}_v - M^{a,N}_u|\mathcal{F}_u]=0$. We thus assume that $u<r$. We write as usual $(M^{k,N}_r)^2=(M^{k,N}_u)^2+2\int_u^r M^{k,N}_{\tau-}dM^{k,N}_\tau + Z^{k,N}_r - Z^{k,N}_u$ and

$$\begin{aligned}K=&\mathbb{E}_\theta[(M^{k,N}_u)^2 M^{a,N}_u(M^{a,N}_v-M^{a,N}_u)]+2\mathbb{E}\Big[\Big(\int_u^r M^{k,N}_{\tau-}dM^{k,N}_\tau\Big)M^{a,N}_u(M^{a,N}_v-M^{a,N}_u)\Big]\\ &+\mathbb{E}_\theta[(Z^{k,N}_r-Z^{k,N}_u)M^{a,N}_u(M^{a,N}_v-M^{a,N}_u)].\end{aligned}$$

The first term vanishes (because $\mathbb{E}_\theta[M^{a,N}_v-M^{a,N}_u|\mathcal{F}_u]=0$), as well as the second one (because $\mathbb{E}_\theta[(\int_u^r M^{k,N}_{\tau-}dM^{k,N}_\tau)(M^{a,N}_v-M^{a,N}_u)|\mathcal{F}_u]=0$ by orthogonality of the involved martingales). Consequently,

$$K=\mathbb{E}_\theta[(Z^{k,N}_r-Z^{k,N}_u)M^{a,N}_u(M^{a,N}_v-M^{a,N}_u)]=\mathbb{E}_\theta[(U^{k,N}_r-U^{k,N}_u)M^{a,N}_u(M^{a,N}_v-M^{a,N}_u)].$$

Using (4) and recalling that $\beta_n(u,r,x)=\varphi^{\star n}(r-x)-\varphi^{\star n}(u-x)$, we find

$$K=\sum_{n\ge 0}\int_0^r \beta_n(u,r,x)\sum_{j=1}^N A^n_N(k,j)\mathbb{E}_\theta[M^{j,N}_x M^{a,N}_u(M^{a,N}_v-M^{a,N}_u)]dx.$$

But $|\mathbb{E}_\theta[M^{j,N}_x M^{a,N}_u(M^{a,N}_v-M^{a,N}_u)]| \le CN^{-1}t$ if $a \ne j$ by (iv), while $|\mathbb{E}_\theta[M^{j,N}_x M^{a,N}_u(M^{a,N}_v-M^{a,N}_u)]| \le Ct^{3/2}$ if $a=j$ by Lemma 16-(iii). Thus

$$|K| \le C\sum_{n\ge 0}\Lambda^n\Big[A^n_N(k,a)t^{3/2}+\sum_{j=1}^N A^n_N(k,j)N^{-1}t\Big] \le C\Big[Q_N(k,a)t^{3/2}+N^{-1}\sum_{j=1}^N Q_N(k,j)t\Big].$$

But $k \ne a$ implies that $Q_N(k,a) \le CN^{-1}$ by (8), while $N^{-1}\sum_{j=1}^N Q_N(k,j) \le CN^{-1}|||Q_N|||_\infty \le CN^{-1}$. As a conclusion, $|K| \le CN^{-1}(t^{3/2}+t) \le CN^{-1}t^{3/2}$. Of course, $J$ is treated similarly, and this completes the proof of point (vii).

Point (viii) is obvious: it suffices to use the Hölder inequality to find

$$|\mathbb{C}\mathrm{ov}_\theta\,(M^{k,N}_r M^{l,N}_s, M^{a,N}_u M^{b,N}_v)| \le \mathbb{E}_\theta[(M^{k,N}_r)^4]^{1/4}\mathbb{E}_\theta[(M^{l,N}_s)^4]^{1/4}\mathbb{E}_\theta[(M^{a,N}_u)^4]^{1/4}\mathbb{E}_\theta[(M^{b,N}_v)^4]^{1/4},$$

which is bounded by $Ct^2$ by Lemma 16-(iii). $\square$

We can now easily bound $\Delta_t^{N,31}$.

**Lemma 23.** *Assume $H(q)$ for some $q \geq 1$. Then a.s., on $\Omega_N^1$, for $t \geq 1$,*

$$\mathbb{E}_\theta[(\Delta_t^{N,31})^2] \leq Ct^{-1} \sum_{i=1}^N \Big[\ell_N(i) - \bar{\ell}_N\Big]^2.$$

*Proof.* We first note that

$$\Delta_t^{N,31} = 2\mu t^{-1} \Big| \sum_{i=1}^N \Big[U_{2t}^{i,N} - U_t^{i,N}\Big]\Big[\ell_N(i) - \bar{\ell}_N\Big]\Big|.$$

Since $U_{2t}^{i,N} - U_t^{i,N}$ is centered (its conditional expectation $\mathbb{E}_\theta$ vanishes),

$$\mathbb{E}_\theta[(\Delta_t^{N,31})^2] = 4\mu^2 t^{-2} \sum_{i,j=1}^N \Big[\ell_N(i) - \bar{\ell}_N\Big]\Big[\ell_N(j) - \bar{\ell}_N\Big]\mathbb{C}\mathrm{ov}_\theta\,(U_{2t}^{i,N} - U_t^{i,N}, U_{2t}^{j,N} - U_t^{j,N}).$$

Using now Lemma 22-(i), we deduce that $|\mathbb{C}\mathrm{ov}_\theta\,(U_{2t}^{i,N} - U_t^{i,N}, U_{2t}^{j,N} - U_t^{j,N})| \leq Ct(\mathbf{1}_{\{i=j\}} + N^{-1})$ on $\Omega_N^1$. Using furthermore that $[\ell_N(i) - \bar{\ell}_N][\ell_N(j) - \bar{\ell}_N] \leq [\ell_N(i) - \bar{\ell}_N]^2 + [\ell_N(j) - \bar{\ell}_N]^2$ and a symmetry argument, we conclude that

$$\mathbb{E}_\theta[(\Delta_t^{N,31})^2] \leq Ct^{-1} \sum_{i,j=1}^N \Big[\ell_N(i) - \bar{\ell}_N\Big]^2(\mathbf{1}_{\{i=j\}} + N^{-1}) = Ct^{-1} \sum_{i=1}^N \Big[\ell_N(i) - \bar{\ell}_N\Big]^2,$$

which was our goal. $\square$

We can finally estimate $\Delta_t^{N,211}$.

**Lemma 24.** *Assume $H(q)$ for some $q \geq 1$. Then a.s., on $\Omega_N^1$, for $t \geq 1$, $\mathbb{E}_\theta[(\Delta_t^{N,211})^2] \leq CNt^{-2}$.*

*Proof.* We as usual work on $\Omega_N^1$. We first note that $\mathbb{E}_\theta[(\Delta_t^{N,211})^2] = t^{-4}\sum_{i,j=1}^N a_{ij}$, where

$$a_{ij} = \mathbb{C}\mathrm{ov}_\theta\,((U_{2t}^{i,N} - U_t^{i,N})^2, (U_{2t}^{j,N} - U_t^{j,N})^2).$$

But recalling (4) and setting $\alpha_N(s,t,i,k) = \sum_{n\geq 0} A_N^n(i,k)[\varphi^{\star n}(2t-s) - \varphi^{\star n}(t-s)]$ for all $0 \leq s \leq 2t$ and $i,k \in \{1,\dots,N\}$,

$$U_{2t}^{i,N} - U_t^{i,N} = \int_0^{2t} \sum_{k=1}^N \alpha_N(s,t,i,k) M_s^{k,N} ds. \tag{9}$$

Concerning $\alpha_N$, we will only use that, on $\Omega_N^1$,

$$\int_0^{2t} |\alpha_N(s,t,i,k)| ds \leq 2\sum_{n\geq 0} \Lambda^n A_N^n(i,k) = 2Q_N(i,k) \leq C(\mathbf{1}_{\{i=k\}} + N^{-1}), \tag{10}$$

the last inequality coming from (8). A direct computation starting from (9) shows that

$$a_{ij} = \sum_{k,l,a,b=1}^N \int_0^{2t}\int_0^{2t}\int_0^{2t}\int_0^{2t} \alpha_N(r,t,i,k)\alpha_N(s,t,i,l)\alpha_N(u,t,j,a)\alpha_N(v,t,j,b)$$
$$\mathbb{C}\mathrm{ov}_\theta\,(M_r^{k,N} M_s^{l,N}, M_u^{a,N} M_v^{b,N}) dv du ds dr.$$

Let us now denote by $\Gamma_{k,l,a,b}(t) = \sup_{r,s,u,v\in[0,2t]} |\mathbb{C}\mathrm{ov}_\theta\,(M^{k,N}_r M^{l,N}_s, M^{a,N}_u M^{b,N}_v)|$. We can write, recalling (10),

$$\sum_{i,j=1}^N a_{ij} \le C \sum_{i,j,k,l,a,b=1}^N (\mathbf{1}_{\{i=k\}} + N^{-1})(\mathbf{1}_{\{i=l\}} + N^{-1})(\mathbf{1}_{\{j=a\}} + N^{-1})(\mathbf{1}_{\{j=b\}} + N^{-1})\Gamma_{k,l,a,b}(t).$$

Using some symmetry arguments, we find that $\sum_{i,j=1}^N a_{ij} \le C[R_1 + \cdots + R_6]$, where

$$R_1 = N^{-4} \sum_{i,j,k,l,a,b=1}^N \Gamma_{k,l,a,b}(t) = N^{-2} \sum_{k,l,a,b=1}^N \Gamma_{k,l,a,b}(t),$$
$$R_2 = N^{-3} \sum_{i,j,k,l,a,b=1}^N \mathbf{1}_{\{i=k\}}\Gamma_{k,l,a,b}(t) = N^{-2} \sum_{k,l,a,b=1}^N \Gamma_{k,l,a,b}(t),$$
$$R_3 = N^{-2} \sum_{i,j,k,l,a,b=1}^N \mathbf{1}_{\{i=k\}}\mathbf{1}_{\{j=a\}}\Gamma_{k,l,a,b}(t) = N^{-2} \sum_{k,l,a,b=1}^N \Gamma_{k,l,a,b}(t),$$
$$R_4 = N^{-2} \sum_{i,j,k,l,a,b=1}^N \mathbf{1}_{\{i=k\}}\mathbf{1}_{\{i=l\}}\Gamma_{k,l,a,b}(t) = N^{-1} \sum_{k,a,b=1}^N \Gamma_{k,k,a,b}(t),$$
$$R_5 = N^{-1} \sum_{i,j,k,l,a,b=1}^N \mathbf{1}_{\{i=k\}}\mathbf{1}_{\{i=l\}}\mathbf{1}_{\{j=a\}}\Gamma_{k,l,a,b}(t) = N^{-1} \sum_{k,a,b=1}^N \Gamma_{k,k,a,b}(t),$$
$$R_6 = \sum_{i,j,k,l,a,b=1}^N \mathbf{1}_{\{i=k\}}\mathbf{1}_{\{i=l\}}\mathbf{1}_{\{j=a\}}\mathbf{1}_{\{j=b\}}\Gamma_{k,l,a,b}(t) = \sum_{k,a=1}^N \Gamma_{k,k,a,a}(t).$$

Using Lemma 22-(v)-(viii), from which $\Gamma_{k,l,a,b}(t) \le Ct^2\mathbf{1}_{\{\#\{k,l,a,b\}<4\}}$, we deduce that $R_1 = R_2 = R_3 \le CNt^2$. Next we use Lemma 22-(vi)-(viii), that is $\Gamma_{k,k,a,b}(t) \le C(\mathbf{1}_{\{\#\{k,a,b\}=3\}}N^{-2}t + \mathbf{1}_{\{\#\{k,a,b\}<3\}}t^2)$, whence $R_4 = R_5 \le Ct + CNt^2 \le CNt^2$. Finally, we use Lemma 22-(vii)-(viii), i.e. $\Gamma_{k,k,a,a}(t) \le C(\mathbf{1}_{\{\#\{k,a\}=2\}}N^{-1}t^{3/2} + \mathbf{1}_{\{\#\{k,a\}=1\}}t^2)$ and find that $R_6 \le CNt^{3/2} + CNt^2 \le CNt^2$. All in all, we have proved that $\sum_{i,j=1}^N a_{ij} \le CNt^2$, which completes the proof. □

We can finally give the

*Proof of Proposition 19.* It suffices to recall that $|\mathcal{V}^N_t - \mathcal{V}^N_\infty| \le \Delta^{N,1}_t + \Delta^{N,211}_t + \Delta^{N,212}_t + \Delta^{N,213}_t + \Delta^{N,22}_t + \Delta^{N,23}_t + \Delta^{N,31}_t + \Delta^{N,32}_t$ and to use Lemmas 20, 21, 23 and 24: this gives, on $\Omega^1_N$,

$$\mathbb{E}_\theta[|\mathcal{V}^N_t - \mathcal{V}^N_\infty|] \le C\Big(\frac{N}{t^{2q}} + \frac{1}{t} + \frac{N}{t^q} + \frac{N^{1/2}}{t^{3/2}} + \Big[\sum_{i=1}^N [\ell_N(i) - \bar\ell_N]^2\Big]^{1/2}\frac{1}{t^{1/2}} + \frac{N^{1/2}}{t}\Big).$$

Recalling that $t \ge 1$, the conclusion immediately follows. □

4.6. **Third estimator.** We recall that, for $\Delta > 0$ such that $t/(2\Delta)$ is an integer, we have set $\mathcal{E}^N_t = (\bar Z^N_{2t} - \bar Z^N_t)/t$, $\mathcal{Z}^N_{\Delta,t} = (N/t)\sum_{a=t/\Delta+1}^{2t/\Delta}[\bar Z^N_{a\Delta} - \bar Z^N_{(a-1)\Delta} - \Delta\mathcal{E}^N_t]^2$ and $\mathcal{W}^N_{\Delta,t} = 2\mathcal{Z}^N_{2\Delta,t} - \mathcal{Z}^N_{\Delta,t}$. The matrices $A_N$ and $Q_N$ and the event $\Omega^1_N$ were defined in Notation 12, as well as $\ell_N(i) = \sum_{j=1}^N Q_N(i,j)$ and $c_N(i) = \sum_{j=1}^N Q_N(j,i)$. We finally introduce $\mathcal{W}^N_{\infty,\infty} = \mu N^{-1}\sum_{i=1}^N \ell_N(i)(c_N(i))^2$. The aim of the subsection is to verify the following result.

**Proposition 25.** *Assume $H(q)$ for some $q \geq 3$. Then a.s., for $t \geq 4$ and $\Delta \in [1, t/4]$ such that $t/(2\Delta)$ is a positive integer,*

$$\mathbf{1}_{\Omega_N^1} \mathbb{E}_\theta\Big[\Big|\mathcal{W}^N_{\Delta,t} - \mathcal{W}^N_{\infty,\infty}\Big|\Big] \leq C\Big(\sqrt{\frac{\Delta}{t}} + \frac{N}{\Delta^{(q+1)/2}} + \frac{t}{\Delta^{q/2+1}}\Big).$$

Recall that we do not try to optimize the dependence in $q$. We first write

$$|\mathcal{W}^N_{\Delta,t} - \mathcal{W}^N_{\infty,\infty}| \leq D^{N,1}_{\Delta,t} + 2D^{N,1}_{2\Delta,t} + D^{N,2}_{\Delta,t} + 2D^{N,2}_{2\Delta,t} + D^{N,3}_{\Delta,t} + 2D^{N,3}_{2\Delta,t} + D^{N,4}_{\Delta,t},$$

where

$$D^{N,1}_{\Delta,t} = \frac{N}{t}\Big|\sum_{a=t/\Delta+1}^{2t/\Delta}\Big[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta} - \Delta\mathcal{E}^N_t\Big]^2 - \sum_{a=t/\Delta+1}^{2t/\Delta}\Big[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta} - \Delta\mu\bar{\ell}_N\Big]^2\Big|,$$

$$D^{N,2}_{\Delta,t} = \frac{N}{t}\Big|\sum_{a=t/\Delta+1}^{2t/\Delta}\Big[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta} - \Delta\mu\bar{\ell}_N\Big]^2 - \sum_{a=t/\Delta+1}^{2t/\Delta}\Big[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta}]\Big]^2\Big|,$$

$$\begin{aligned}D^{N,3}_{\Delta,t} =& \frac{N}{t}\Big|\sum_{a=t/\Delta+1}^{2t/\Delta}\Big[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta}]\Big]^2 \\ &- \mathbb{E}_\theta\Big[\sum_{a=t/\Delta+1}^{2t/\Delta}\Big[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta}]\Big]^2\Big]\Big|,\end{aligned}$$

$$\begin{aligned}D^{N,4}_{\Delta,t} =& \Big|\frac{2N}{t}\mathbb{E}_\theta\Big[\sum_{a=t/(2\Delta)+1}^{t/\Delta}\Big[\bar{Z}^N_{2a\Delta} - \bar{Z}^N_{2(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^N_{2a\Delta} - \bar{Z}^N_{2(a-1)\Delta}]\Big]^2\Big] \\ &- \frac{N}{t}\mathbb{E}_\theta\Big[\sum_{a=t/\Delta+1}^{2t/\Delta}\Big[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta}]\Big]^2\Big] - \mathcal{W}^N_{\infty,\infty}\Big|.\end{aligned}$$

We treat these four terms one by one.

**Lemma 26.** *Assume $H(q)$ for some $q \geq 1$. Then a.s. on $\Omega_N^1$, for $1 \leq \Delta \leq t$, $\mathbb{E}_\theta[D^{N,1}_{\Delta,t}] \leq C\Delta[t^{-1} + Nt^{-2q}]$.*

*Proof.* Using that $(\Delta/t)\sum_{a=t/\Delta+1}^{2t/\Delta}(\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta}) = \Delta\mathcal{E}^N_t$, we find that

$$D^{N,1}_{\Delta,t} = \frac{N}{t}\frac{t}{\Delta}(\Delta\mu\bar{\ell}_N - \Delta\mathcal{E}^N_t)^2 = N\Delta(\mu\bar{\ell}_N - \mathcal{E}^N_t)^2,$$

whence, on $\Omega_N^1$, see Proposition 17, $\mathbb{E}_\theta[D^{N,1}_{\Delta,t}] \leq CN\Delta(t^{-2q} + (Nt)^{-1}) \leq C\Delta(Nt^{-2q} + t^{-1})$. □

The second term is also easy.

**Lemma 27.** *Assume $H(q)$ for some $q \geq 1$. Then on $\Omega_N^1$, for $1 \leq \Delta \leq t$, $\mathbb{E}_\theta[D^{N,2}_{\Delta,t}] \leq CNt^{1-q}$.*

*Proof.* Using that $|(A-x)^2 - (A-y)^2| \leq |x-y|(|x| + |y| + 2|A|)$,

$$D^{N,2}_{\Delta,t} \leq \frac{N}{t}\sum_{a=t/\Delta+1}^{2t/\Delta}\Big|\Delta\mu\bar{\ell}_N - \mathbb{E}_\theta[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta}]\Big|\Big[\Delta\mu\bar{\ell}_N + \mathbb{E}_\theta[\bar{Z}^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta}] + 2(Z^N_{a\Delta} - \bar{Z}^N_{(a-1)\Delta})\Big],$$

whence

$$\mathbb{E}_\theta[D^{N,2}_{\Delta,t}] \leq \frac{N}{t}\sum_{a=t/\Delta+1}^{2t/\Delta}\Big|\Delta\mu\bar\ell_N - \mathbb{E}_\theta[\bar Z^N_{a\Delta}-\bar Z^N_{(a-1)\Delta}]\Big|\Big[\Delta\mu\bar\ell_N + 3\mathbb{E}_\theta[\bar Z^N_{a\Delta}-\bar Z^N_{(a-1)\Delta}]\Big].$$

But we deduce from Lemma 16-(ii) with $r=1$ that, since $(a-1)\Delta\geq t$,

$$\Big|\Delta\mu\bar\ell_N - \mathbb{E}_\theta[\bar Z^N_{a\Delta}-\bar Z^N_{(a-1)\Delta}]\Big| \leq Ct^{1-q},$$

whence also $\mathbb{E}_\theta[\bar Z^N_{a\Delta}-\bar Z^N_{(a-1)\Delta}]\leq \Delta\mu\bar\ell_N + Ct^{1-q}\leq \Delta\mu\bar\ell_N + C$. We conclude that

$$\mathbb{E}_\theta[D^{N,2}_{\Delta,t}] \leq C\frac{N}{t}\sum_{a=t/\Delta+1}^{2t/\Delta} t^{1-q}\Big[4\Delta\mu\bar\ell_N + C\Big].$$

Since $\bar\ell_N$ is bounded on $\Omega^1_N$ and since $\Delta\geq 1\geq t^{1-q}$, we find $\mathbb{E}_\theta[D^{N,2}_{\Delta,t}]\leq C(N/t)(t/\Delta)t^{1-q}\Delta\leq CNt^{1-q}$. ☐

To treat $D^{N,4}_{\Delta,t}$, we need the following lemma.

**Lemma 28.** *Assume $H(q)$ for some $q\geq 1$. Almost surely on $\Omega^1_N$, for all $1\leq\Delta\leq x/2$,*

$$\mathbb{V}\mathrm{ar}_\theta\,(\bar U^N_{x+\Delta}-\bar U^N_x) = \frac{\Delta}{N}\mathcal{W}^N_{\infty,\infty} - \mathcal{X}^N + r_N(x,\Delta),$$

*where $\mathcal{X}_N$ is a $\sigma((\theta_{ij})_{i,j=1,\dots,N})$-measurable finite random variable and where $r_N$ satisfies, for some deterministic constant $C$, the inequality $|r_N(x,\Delta)|\leq Cx\Delta^{-q}N^{-1}$.*

*Proof.* We set $V^N_{x,\Delta}=\mathbb{V}\mathrm{ar}_\theta\,(\bar U^N_{x+\Delta}-\bar U^N_x)$.

*Step 1.* Recalling (4) and setting $\beta_n(x,x+\Delta,s)=\varphi^{\star n}(x+\Delta-s)-\varphi^{\star n}(x-s)$ as in Lemma 15, we get

$$\bar U^N_{x+\Delta}-\bar U^N_x = \sum_{n\geq 0}\int_0^{x+\Delta}\beta_n(x,x+\Delta,s)N^{-1}\sum_{i,j=1}^N A^n_N(i,j)M^{j,N}_s ds.$$

Hence

$$V^N_{x,\Delta} = \sum_{m,n\geq 0}\int_0^{x+\Delta}\int_0^{x+\Delta}\beta_m(x,x+\Delta,r)\beta_n(x,x+\Delta,s)N^{-2}\sum_{i,j,k,l=1}^N A^m_N(i,j)A^n_N(k,l)\,\mathbb{C}\mathrm{ov}_\theta\,(M^{j,N}_r,M^{l,N}_s)drds.$$

Using Remark 10, we find

$$V^N_{x,\Delta} = \sum_{m,n\geq 0}\int_0^{x+\Delta}\int_0^{x+\Delta}\beta_m(x,x+\Delta,r)\beta_n(x,x+\Delta,s)N^{-2}\sum_{i,j,k=1}^N A^m_N(i,j)A^n_N(k,j)\mathbb{E}_\theta[Z^{j,N}_{r\wedge s}]drds.$$

*Step 2.* Here we show that $\mathbb{E}_\theta[Z^{j,N}_s]=\mu\ell_N(j)s - X^N_j + R^N_j(s)$, with, for some constant $C$, for all $j=1,\dots,N$,

$$0\leq X^N_j\leq C \quad\text{and}\quad |R^N_j(s)|\leq C(s^{1-q}\wedge 1).$$

By (3), we have $\mathbb{E}_\theta[Z^{j,N}_s]=\mu\sum_{n\geq 0}(\int_0^s r\varphi^{\star n}(s-r)dr)\sum_{l=1}^N A^n_N(j,l)$, whence by Lemma 15-(i),

$$\mathbb{E}_\theta[Z^{j,N}_s]=\mu\sum_{n\geq 0}(\Lambda^n s - n\Lambda^n\kappa + \varepsilon_n(s))\sum_{l=1}^N A^n_N(j,l) = \mu\ell_N(j)s - X^N_j + R^N_j(s).$$

We have used that $\sum_{n\geq 0}\Lambda^n\sum_{l=1}^N A_N^n(j,l) = \sum_{l=1}^N Q_N(j,l) = \ell_N(j)$ and we have set $X_j^N = \mu\kappa\sum_{n\geq 0} n\Lambda^n\sum_{l=1}^N A_N^n(j,l)$ and $R_j^N(s) = \mu\sum_{n\geq 0}\varepsilon_n(s)\sum_{l=1}^N A_N^n(j,l)$. We obviously have $0 \leq X_j^N \leq \mu\kappa\sum_{n\geq 0} n\Lambda^n|||A_N|||_\infty^n \leq C$ on $\Omega_N^1$ and, since $\varepsilon_n(s) \leq Cn^q\Lambda^n(s^{1-q}\wedge 1)$ by Lemma 15-(i), $|R_j^N(s)| \leq C(s^{1-q}\wedge 1)\sum_{n\geq 0} n^q\Lambda^n|||A_N|||_\infty^n \leq C(s^{1-q}\wedge 1)$, still on $\Omega_N^1$.

*Step 3.* Gathering Steps 1 and 2, we now write $V_{x,\Delta}^N = I - J + K$, where

$$I = \sum_{m,n\geq 0}\int_0^{x+\Delta}\int_0^{x+\Delta}\beta_m(x,x+\Delta,r)\beta_n(x,x+\Delta,s)N^{-2}\sum_{i,j,k=1}^N A_N^m(i,j)A_N^n(k,j)\mu\ell_N(j)(r\wedge s)drds,$$
$$J = \sum_{m,n\geq 0}\int_0^{x+\Delta}\int_0^{x+\Delta}\beta_m(x,x+\Delta,r)\beta_n(x,x+\Delta,s)N^{-2}\sum_{i,j,k=1}^N A_N^m(i,j)A_N^n(k,j)X_j^N drds,$$
$$K = \sum_{m,n\geq 0}\int_0^{x+\Delta}\int_0^{x+\Delta}\beta_m(x,x+\Delta,r)\beta_n(x,x+\Delta,s)N^{-2}\sum_{i,j,k=1}^N A_N^m(i,j)A_N^n(k,j)R_j^N(r\wedge s)drds.$$

*Step 4.* Here we verify that $|J| \leq Cx^{-2q}N^{-1}$ on $\Omega_N^1$. Using that $|\int_0^{x+\Delta}\beta_m(x,x+\Delta,r)dr| \leq Cn^q\Lambda^n x^{-q}$ by Lemma 15-(ii) and that $X_j^N$ is bounded by Step 2 (and does not depend on time),

$$\begin{aligned}|J| \leq& C\sum_{m,n\geq 0} m^q n^q\Lambda^{m+n}x^{-2q}N^{-2}\sum_{i,j,k=1}^N A_N^m(i,j)A_N^n(k,j)\\ \leq& Cx^{-2q}N^{-1}\sum_{m,n\geq 0} m^q n^q\Lambda^{m+n}|||A_N|||_1^{m+n}.\end{aligned}$$

The conclusion follows, since $\Lambda|||A_N|||_1 \leq a < 1$ on $\Omega_N^1$.

*Step 5.* We next check that $|K| \leq Cx\Delta^{-q}N^{-1}$ on $\Omega_N^1$. Using the bound on $R_j^N$ (see Step 2), we start from

$$\begin{aligned}|K| \leq& C\sum_{m,n\geq 0}\int_0^{x+\Delta}\int_0^{x+\Delta}|\beta_m(x,x+\Delta,r)||\beta_n(x,x+\Delta,s)|N^{-1}|||A_N|||_1^{m+n}[(r\wedge s)^{1-q}\wedge 1]drds\\ \leq& C(K_1+K_2),\end{aligned}$$

where, using that $x-\Delta \geq x/2$ (whence $(r\wedge s)^{1-q} \leq Cx^{1-q}$ if $r\wedge s \geq x-\Delta$) and a symmetry argument,

$$\begin{aligned}K_1 =& x^{1-q}\sum_{m,n\geq 0}\int_{x-\Delta}^{x+\Delta}\int_{x-\Delta}^{x+\Delta}|\beta_m(x,x+\Delta,r)||\beta_n(x,x+\Delta,s)|N^{-1}|||A_N|||_1^{m+n}drds,\\ K_2 =& \sum_{m,n\geq 0}\int_0^{x-\Delta}\int_0^{x+\Delta}|\beta_m(x,x+\Delta,r)||\beta_n(x,x+\Delta,s)|N^{-1}|||A_N|||_1^{m+n}drds.\end{aligned}$$

First, on $\Omega_N^1$,

$$K_1 \leq Cx^{1-q}\sum_{m,n\geq 0}\Lambda^{m+n}N^{-1}|||A_N|||_1^{m+n} \leq CN^{-1}x^{1-q} \leq Cx\Delta^{-q}N^{-1}$$

since $x \geq \Delta$. Next, using that $\int_0^{x-\Delta} |\beta_m(x, x+\Delta, r)|dr \leq C m^q \Lambda^m \Delta^{-q}$ by Lemma 15-(ii) and that $\int_0^{x+\Delta} |\beta_n(x, x+\Delta, s)|ds \leq 2\Lambda^n$, still on $\Omega_N^1$,

$$K_2 \leq C\Delta^{-q} \sum_{m,n\geq 0} m^q \Lambda^{m+n} N^{-1} |||A_N|||_1^{m+n} \leq C\Delta^{-q} N^{-1} \leq Cx\Delta^{-q} N^{-1},$$

since $x \geq 1$ by assumption.

*Step 6.* Finally recall that $\gamma_{m,n}(x, x+\Delta) = \int_0^{x+\Delta}\int_0^{x+\Delta}(s\wedge u)\beta_m(x, x+\Delta, s)\beta_n(x, x+\Delta, u)duds = \Delta\Lambda^{m+n} - \kappa_{m,n}\Lambda^{m+n} + \varepsilon_{m,n}(x, x+\Delta)$ with the notation of Lemma 15-(iii). We thus may write

$$I = \mu \sum_{m,n\geq 0} \gamma_{m,n}(x, x+\Delta) N^{-2} \sum_{i,j,k=1}^N A_N^m(i,j) A_N^n(k,j) \ell_N(j) = I_1 - I_2 + I_3,$$

where

$$\begin{aligned} I_1 =& \mu\Delta \sum_{m,n\geq 0} \Lambda^{m+n} N^{-2} \sum_{i,j,k=1}^N A_N^m(i,j) A_N^n(k,j) \ell_N(j), \\ I_2 =& \mu \sum_{m,n\geq 0} \kappa_{m,n}\Lambda^{m+n} N^{-2} \sum_{i,j,k=1}^N A_N^m(i,j) A_N^n(k,j) \ell_N(j), \\ I_3 =& \mu \sum_{m,n\geq 0} \varepsilon_{m,n}(x, x+\Delta) N^{-2} \sum_{i,j,k=1}^N A_N^m(i,j) A_N^n(k,j) \ell_N(j). \end{aligned}$$

First, we clearly have

$$I_1 = \mu\Delta N^{-2} \sum_{i,j,k=1}^N Q_N(i,j) Q_N(k,j) \ell_N(j) = \mu\Delta N^{-2} \sum_{j=1}^N (c_N(j))^2 \ell_N(j) = \Delta N^{-1} \mathcal{W}_{\infty,\infty}^N.$$

We next simply set $\mathcal{X}^N = I_2$, which is clearly $\sigma((\theta_{i,j})_{i,j=1,\dots,N})$-measurable and well-defined on $\Omega_N^1$. Finally, since $\varepsilon_{m,n}(x, x+\Delta) \leq C(m+n)^q \Lambda^{m+n} x \Delta^{-q}$ by Lemma 15-(iii), since $\ell_N$ is bounded on $\Omega_N^1$ and since, as already seen, $\sum_{i,j,k=1}^N A_N^m(i,j) A_N^n(k,j) \leq N |||A_N|||_1^{m+n}$,

$$|I_3| \leq Cx\Delta^{-q} N^{-1} \sum_{m,n\geq 0} (n+m)^q \Lambda^{m+n} |||A_N|||_1^{m+n} \leq Cx\Delta^{-q} N^{-1}.$$

All this implies that $|I - \Delta N^{-1}\mathcal{W}_{\infty,\infty}^N + \mathcal{X}^N| \leq Cx\Delta^{-q}N^{-1}$. Since $V_{x,\Delta}^N = I - J + K$ by Step 3 and since we have seen in Steps 4 and 5 that $|J| \leq Cx^{-2q}N^{-1} \leq Cx\Delta^{-q}N^{-1}$ and $|K| \leq Cx\Delta^{-q}N^{-1}$, we conclude that, on $\Omega_N^1$, $|V_{x,\Delta}^N - \Delta N^{-1}\mathcal{W}_{\infty,\infty}^N + \mathcal{X}^N| \leq Cx\Delta^{-q}N^{-1}$ as desired. □

We can now study the term $D_{\Delta,t}^{N,4}$.

**Lemma 29.** *Assume $H(q)$ for some $q \geq 1$. Then a.s. on $\Omega_N^1$, for $1 \leq \Delta \leq t/4$, $D_{\Delta,t}^{N,4} \leq Ct\Delta^{-1-q}$.*

*Proof.* We clearly have

$$D_{\Delta,t}^{N,4} = \Big|\frac{2N}{t} \sum_{a=t/(2\Delta)+1}^{t/\Delta} \mathbb{V}\mathrm{ar}_\theta\,(\bar U_{2a\Delta}^N - \bar U_{2(a-1)\Delta}^N) - \frac{N}{t} \sum_{a=t/\Delta+1}^{2t/\Delta} \mathbb{V}\mathrm{ar}_\theta\,(\bar U_{a\Delta}^N - \bar U_{(a-1)\Delta}^N) - \mathcal{W}_{\infty,\infty}^N\Big|.$$

Using Lemma 28, (observe that for $a \in \{t/(2\Delta)+1, \dots, t/\Delta\}$, $x = 2(a-1)\Delta \geq t$ satisfies $2\Delta \leq x/2$ and, for $a \in \{t/\Delta + 1, \dots, 2t/\Delta\}$, $x = (a-1)\Delta \geq t$ satisfies $\Delta \leq x/2$), we get

$$D^{N,4}_{\Delta,t} = \Big| \frac{2N}{t} \sum_{a=t/(2\Delta)+1}^{t/\Delta} \Big[ \frac{2\Delta}{N} \mathcal{W}^N_{\infty,\infty} - \mathcal{X}^N + r_N(2(a-1)\Delta, 2\Delta) \Big] - \frac{N}{t} \sum_{a=t/\Delta+1}^{2t/\Delta} \Big[ \frac{\Delta}{N} \mathcal{W}^N_{\infty,\infty} - \mathcal{X}^N + r_N((a-1)\Delta, \Delta) \Big] - \mathcal{W}^N_{\infty,\infty} \Big|.$$

This rewrites

$$D^{N,4}_{\Delta,t} = \Big| \frac{2N}{t} \sum_{a=t/(2\Delta)+1}^{t/\Delta} r_N(2(a-1)\Delta, 2\Delta) - \frac{N}{t} \sum_{a=t/\Delta+1}^{2t/\Delta} r_N((a-1)\Delta, \Delta) \Big|.$$

Since $r_N(x,\Delta) \leq Cx\Delta^{-q}N^{-1}$, we find that $D^{N,4}_{\Delta,t} \leq C(N/t)(t/\Delta)(t\Delta^{-q}N^{-1}) = Ct\Delta^{-1-q}$. □

The following tedious lemma will allow us to treat the last term $D^{N,3}_{\Delta,t}$.

**Lemma 30.** *Assume $H(q)$ for some $q \geq 1$. On $\Omega^1_N$, for all $t, x, \Delta \geq 1$ with $t/2 \leq x - \Delta \leq x + \Delta \leq 2t$,*

$$\mathbb{V}\mathrm{ar}_\theta\,((\bar{U}^N_{x+\Delta} - \bar{U}^N_x)^2) \leq C\Big( \frac{\Delta^2}{N^2} + \frac{t^2}{N^2\Delta^{4q}} \Big)$$

*and, if $t/2 \leq y - \Delta \leq y + \Delta \leq x - 2\Delta \leq x + \Delta \leq 2t$,*

$$\mathbb{C}\mathrm{ov}_\theta\,((\bar{U}^N_{x+\Delta} - \bar{U}^N_x)^2, (\bar{U}^N_{y+\Delta} - \bar{U}^N_y)^2) \leq C\Big( \frac{t^{1/2}}{N\Delta^{q-1}} + \frac{t^2}{N^2\Delta^{4q}} + \frac{t^{1/2}}{N^2\Delta^{q-3/2}} \Big).$$

*Proof.* We divide the proof in several steps. We work on $\Omega^1_N$.

*Step 1.* For $i = 1, \dots, N$ and $z \in [x, x+\Delta]$, we can write, recalling (4) and that $\beta_n(x,z,r) = \varphi^{\star n}(z-r) - \varphi^{\star n}(x-r)$,

$$U^{i,N}_z - U^{i,N}_x = \sum_{n\geq 0} \int_0^z \beta_n(x,z,r) \sum_{j=1}^N A^n_N(i,j) M^{j,N}_r dr = \Gamma^{i,N}_{x,z} + X^{i,N}_{x,z},$$

where

$$\Gamma^{i,N}_{x,z} = \sum_{n\geq 0} \int_{x-\Delta}^z \beta_n(x,z,r) \sum_{j=1}^N A^n_N(i,j)(M^{j,N}_r - M^{j,N}_{x-\Delta}) dr,$$

$$X^{i,N}_{x,z} = \sum_{n\geq 0} \Big( \int_{x-\Delta}^z \beta_n(x,z,r) dr \Big) \sum_{j=1}^N A^n_N(i,j) M^{j,N}_{x-\Delta} + \sum_{n\geq 0} \int_0^{x-\Delta} \beta_n(x,z,r) \sum_{j=1}^N A^n_N(i,j) M^{j,N}_r dr,$$

and we set as usual $\bar{\Gamma}^N_{x,z} = N^{-1}\sum_{i=1}^N \Gamma^{i,N}_{x,z}$ and $\bar{X}^N_{x,z} = N^{-1}\sum_{i=1}^N X^{i,N}_{x,z}$.

*Step 2.* We now show that, on $\Omega^1_N$, for $z \in [x, x+\Delta]$,

$$\sup_{i=1,\dots,N} \mathbb{E}_\theta[(X^{i,N}_{x,z})^4] \leq Ct^2\Delta^{-4q} \quad \text{and} \quad \mathbb{E}_\theta[(\bar{X}^N_{x,z})^4] \leq Ct^2N^{-2}\Delta^{-4q}.$$

Using that $|\int_{x-\Delta}^z \beta_n(x,z,r)dr| + \int_0^{x-\Delta} |\beta_n(x,z,r)|dr \leq Cn^q\Lambda^n\Delta^{-q}$ by Lemma 15-(ii)

$$|X^{i,N}_{x,z}| \leq C\sum_{n\geq 0} n^q\Lambda^n\Delta^{-q}\sum_{j=1}^N A_N^n(i,j)\sup_{[0,2t]}|M_r^{j,N}|.$$

But we now from Lemma 16-(iii) that $\sup_{j=1,\dots,N}\mathbb{E}_\theta[\sup_{[0,2t]}|M_r^{j,N}|^4] \leq Ct^2$. We thus deduce from the Minkowski inequality that, still on $\Omega_N^1$,

$$\mathbb{E}_\theta[(X^{i,N}_{x,z})^4]^{1/4} \leq Ct^{1/2}\Delta^{-q}\sum_{n\geq 0} n^q\Lambda^n\sum_{j=1}^N A_N^n(i,j) \leq Ct^{1/2}\Delta^{-q}\sum_{n\geq 0} n^q\Lambda^n|||A_N|||_\infty^n \leq Ct^{1/2}\Delta^{-q}.$$

We next observe that

$$\bar X^N_{x,z} = \sum_{n\geq 0}\Big(\int_{x-\Delta}^z \beta_n(x,z,r)dr\Big)O^{N,n}_{x-\Delta} + \sum_{n\geq 0}\int_0^{x-\Delta}\beta_n(x,z,r)O_r^{N,n}dr,$$

where the martingale

$$O_r^{N,n} = N^{-1}\sum_{i,j=1}^N A_N^n(i,j)M_r^{j,N}$$

has for quadratic variation $[O^{N,n},O^{N,n}]_r = N^{-2}\sum_{j=1}^N(\sum_{i=1}^N A_N^n(i,j))^2 Z_r^{j,N} \leq N^{-1}|||A_N|||_1^{2n}\bar Z_r^N$ by Remark 10. By Lemma 16-(iii), we conclude that, on $\Omega_N^1$,

$$\mathbb{E}_\theta\Big[\sup_{[0,2t]}(O_r^{N,n})^4\Big] \leq CN^{-2}|||A_N|||_1^{4n}\mathbb{E}_\theta[(\bar Z_{2t}^N)^2] \leq CN^{-2}|||A_N|||_1^{4n}t^2.$$

Using again that $|\int_{x-\Delta}^z \beta_n(x,z,r)dr| + \int_0^{x-\Delta} |\beta_n(x,z,r)|dr \leq Cn^q\Lambda^n\Delta^{-q}$ by Lemma 15-(ii),

$$|\bar X^N_{x,z}| \leq C\sum_{n\geq 0} n^q\Lambda^n\Delta^{-q}\sup_{[0,2t]}|O_r^{N,n}|.$$

Thus, we infer from the Minkowski inequality that, still on $\Omega_N^1$,

$$\mathbb{E}[(\bar X^N_{x,z})^4]^{1/4} \leq C\sum_{n\geq 0} n^q\Lambda^n\Delta^{-q}N^{-1/2}|||A_N|||_1^n t^{1/2} \leq C\Delta^{-q}N^{-1/2}t^{1/2}.$$

*Step 3.* We next check that $\mathbb{E}_\theta[(\bar\Gamma^N_{x,z})^4] \leq C\Delta^2N^{-2}$ for any $z\in[x,x+\Delta]$, on $\Omega_1^N$. Using the same martingale $O^{N,n}$ as in Step 2,

$$\bar\Gamma^N_{x,z} = \sum_{n\geq 0}\int_{x-\Delta}^z \beta_n(x,z,r)[O_r^{N,n} - O^{N,n}_{x-\Delta}]dr.$$

Recalling that $[O^{N,n},O^{N,n}]_r = N^{-2}\sum_{j=1}^N(\sum_{i=1}^N A_N^n(i,j))^2Z_r^{j,N}$ with $\sum_{i=1}^N A_N^n(i,j) \leq |||A_N|||_1^n$,

$$\begin{aligned}\mathbb{E}_\theta\Big[\sup_{[x-\Delta,z]}(O_r^{N,n} - O^{N,n}_{x-\Delta})^4\Big] \leq& CN^{-4}|||A_N|||_1^{4n}\mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N(Z_z^{j,N} - Z^{j,N}_{x-\Delta})\Big)^2\Big]\\ =& CN^{-2}|||A_N|||_1^{4n}\mathbb{E}_\theta\Big[\Big(\bar Z_z^N - \bar Z^N_{x-\Delta})\Big)^2\Big].\end{aligned}$$

We conclude from Lemma 16-(iii) that (recall that $z\in[x,x+\Delta]$)

$$\mathbb{E}_\theta\Big[\sup_{[x-\Delta,z]}(O_r^{N,n} - O^{N,n}_{x-\Delta})^4\Big] \leq C\Delta^2N^{-2}|||A_N|||_1^{4n}.$$

Using that $\int_{x-\Delta}^z |\beta_n(x,z,r)|dr \le 2\Lambda^n$ and the Minkowski inequality,

$$\mathbb{E}[(\bar\Gamma^N_{x,z})^4]^{1/4} \le C\sum_{n\ge 0}\Lambda^n \Delta^{1/2}N^{-1/2}|||A_N|||_1^n \le C\Delta^{1/2}N^{-1/2}.$$

*Step 4.* Recalling Step 1, $(\bar U^N_{x+\Delta}-\bar U^N_x)^4 = (\bar\Gamma^N_{x,x+\Delta}+\bar X^N_{x,x+\Delta})^4 \le 8(\bar\Gamma^N_{x,x+\Delta})^4+8(\bar X^N_{x,x+\Delta})^4$. We deduce from Steps 2 and 3 that $\mathbb{V}\mathrm{ar}_\theta\left((\bar U^N_{x+\Delta}-\bar U^N_x)^2\right) \le C(\Delta^2N^{-2}+t^2N^{-2}\Delta^{-4q})$.

*Step 5.* Here we show that

$$\Big|\mathbb{C}\mathrm{ov}_\theta\left((\bar U^N_{x+\Delta}-\bar U^N_x)^2,(\bar U^N_{y+\Delta}-\bar U^N_y)^2\right)\Big| \le \Big|\mathbb{C}\mathrm{ov}_\theta\left((\bar\Gamma^N_{x,x+\Delta})^2,(\bar\Gamma^N_{y,y+\Delta})^2\right)\Big| + \frac{C}{N^2}\Big(\frac{t^2}{\Delta^{4q}}+\frac{t^{1/2}}{\Delta^{q-3/2}}\Big).$$

It suffices to write that $(\bar U^N_{x+\Delta}-\bar U^N_x)^2 = (\bar\Gamma^N_{x,x+\Delta})^2+(\bar X^N_{x,x+\Delta})^2+2\bar\Gamma^N_{x,x+\Delta}\bar X^N_{x,x+\Delta}$, the same formula with $y$ instead of $x$, and to use the bilinearity of the covariance: we have the term $\mathbb{C}\mathrm{ov}_\theta\left((\bar\Gamma^N_{x,x+\Delta})^2,(\bar\Gamma^N_{y,y+\Delta})^2\right)$, and the other ones are bounded by

$$\begin{aligned}\mathbb{E}_\theta\Big[&(\bar\Gamma^N_{x,x+\Delta})^2(\bar X^N_{y,y+\Delta})^2+2(\bar\Gamma^N_{x,x+\Delta})^2|\bar\Gamma^N_{y,y+\Delta}\bar X^N_{y,y+\Delta}|+(\bar X^N_{x,x+\Delta})^2(\bar\Gamma^N_{y,y+\Delta})^2\\ &+(\bar X^N_{x,x+\Delta})^2(\bar X^N_{y,y+\Delta})^2+2(\bar X^N_{x,x+\Delta})^2|\bar\Gamma^N_{y,y+\Delta}\bar X^N_{y,y+\Delta}|+2|\bar\Gamma^N_{x,x+\Delta}\bar X^N_{x,x+\Delta}|(\bar\Gamma^N_{y,y+\Delta})^2\\ &+2|\bar\Gamma^N_{x,x+\Delta}\bar X^N_{x,x+\Delta}|(\bar X^N_{y,y+\Delta})^2+4|\bar\Gamma^N_{x,x+\Delta}\bar X^N_{x,x+\Delta}\bar\Gamma^N_{y,y+\Delta}\bar X^N_{y,y+\Delta}|\Big].\end{aligned}$$

We bound all these terms, using only the Hölder inequality and recalling that $\mathbb{E}[(\bar\Gamma^N_{x,x+z})^4] \le C\Delta^2N^{-2}$ and $\mathbb{E}[(\bar X^N_{x,x+z})^4] \le Ct^2N^{-2}\Delta^{-4q}$ and that the same bounds hold with $y$ instead of $x$. We finally remove a few terms using the inequality $a+a^{3/4}b^{1/4}+a^{1/2}b^{1/2}+a^{1/4}b^{3/4} \le 4(a+a^{1/4}b^{3/4})$ with $a=t^2N^{-2}\Delta^{-4q}$ and $b=\Delta^2N^{-2}$.

*Step 6.* Recall that $y+\Delta \le x-2\Delta$. We check here that for any $r,s\in[x-\Delta,x+\Delta]$, any $u,v\in[y-\Delta,y+\Delta]$, any $i,j,k,l\in\{1,\dots,N\}$,

$$\Big|\mathbb{C}\mathrm{ov}_\theta\left((M^{i,N}_r-M^{i,N}_{x-\Delta})(M^{j,N}_s-M^{j,N}_{x-\Delta}),(M^{k,N}_u-M^{k,N}_{y-\Delta})(M^{l,N}_v-M^{l,N}_{y-\Delta})\right)\Big| \le C\mathbf{1}_{\{i=j\}}t^{1/2}\Delta^{1-q}.$$

First, $i\neq j$ implies that the covariance vanishes, since $\mathbb{E}_\theta[(M^{i,N}_r-M^{i,N}_{x-\Delta})(M^{j,N}_s-M^{j,N}_{x-\Delta})|\mathcal{F}_{x-\Delta}]=0$ and since $u,v\le y+\Delta\le x-\Delta$. We next assume that $i=j$ and w.l.o.g. that $r\le s$. Conditioning with respect to $\mathcal{F}_r$, we easily find, since $u,v\le x-\Delta\le r$,

$$\begin{aligned}K:=&\mathbb{C}\mathrm{ov}_\theta\left((M^{i,N}_r-M^{i,N}_{x-\Delta})(M^{i,N}_s-M^{i,N}_{x-\Delta}),(M^{k,N}_u-M^{k,N}_{y-\Delta})(M^{l,N}_v-M^{l,N}_{y-\Delta})\right)\\ =&\mathbb{C}\mathrm{ov}_\theta\left((M^{i,N}_r-M^{i,N}_{x-\Delta})^2,(M^{k,N}_u-M^{k,N}_{y-\Delta})(M^{l,N}_v-M^{l,N}_{y-\Delta})\right).\end{aligned}$$

We write as usual $(M^{i,N}_r-M^{i,N}_{x-\Delta})^2 = 2\int_{x-\Delta}^r M^{i,N}_{\tau-}dM^{i,N}_\tau+Z^{i,N}_r-Z^{i,N}_{x-\Delta}$, because $[M^{i,N},M^{i,N}]_\tau = Z^{i,N}_\tau$ by Remark 10. Since $\mathbb{E}[\int_{x-\Delta}^r M^{i,N}_{\tau-}dM^{i,N}_\tau|\mathcal{F}_{x-\Delta}]=0$ and since $u,v\le x-\Delta$, we find that

$$\begin{aligned}K=&\mathbb{C}\mathrm{ov}_\theta\left(Z^{i,N}_r-Z^{i,N}_{x-\Delta},(M^{k,N}_u-M^{k,N}_{y-\Delta})(M^{l,N}_v-M^{l,N}_{y-\Delta})\right)\\ =&\mathbb{C}\mathrm{ov}_\theta\left(U^{i,N}_r-U^{i,N}_{x-\Delta},(M^{k,N}_u-M^{k,N}_{y-\Delta})(M^{l,N}_v-M^{l,N}_{y-\Delta})\right)\\ =&\mathbb{C}\mathrm{ov}_\theta\left(\Gamma^{i,N}_{x-\Delta,r}+X^{i,N}_{x-\Delta,r},(M^{k,N}_u-M^{k,N}_{y-\Delta})(M^{l,N}_v-M^{l,N}_{y-\Delta})\right)\end{aligned}$$

with the notation of Step 1. But $\Gamma^{i,N}_{x-\Delta,r}$ involves only increments of martingales of the form $M^{j,N}_\tau-M^{j,N}_{x-2\Delta}$, of which the conditional expectation knowing $\mathcal{F}_{x-2\Delta}$ vanishes. Since now $u,v\le$

$y+\Delta \le x-2\Delta$, we deduce that

$$K = \mathbb{C}\mathrm{ov}_\theta\Big(X^{i,N}_{x-\Delta,r}, (M^{k,N}_u - M^{k,N}_{y-\Delta})(M^{l,N}_v - M^{l,N}_{y-\Delta})\Big),$$

whence

$$|K| \le \mathbb{E}_\theta[(X^{i,N}_{x-\Delta,r})^2]^{1/2}\mathbb{E}[(M^{k,N}_u - M^{k,N}_{y-\Delta})^4]^{1/4}\mathbb{E}[(M^{l,N}_v - M^{l,N}_{y-\Delta})^4]^{1/4}.$$

Using Step 2, Lemma 16-(iii) and that $u-(y-\Delta)\le 2\Delta$ and $v-(y-\Delta)\le 2\Delta$, we easily conclude that indeed, $|K| \le Ct^{1/2}\Delta^{-q}\Delta$.

*Step 7.* We now show, recalling that $y+\Delta \le x-2\Delta$, that

$$\Big|\mathbb{C}\mathrm{ov}_\theta\left((\bar\Gamma^N_{x,x+\Delta})^2, (\bar\Gamma^N_{y,y+\Delta})^2\right)\Big| \le CN^{-1}t^{1/2}\Delta^{1-q}.$$

We denote by $|I|$ the left hand side and we start from

$$\bar\Gamma^N_{x,x+\Delta} = \sum_{n\ge 0}\int_{x-\Delta}^{x+\Delta}\beta_n(x,x+\Delta,r)N^{-1}\sum_{i,j=1}^N A^n_N(i,j)(M^{j,N}_r - M^{j,N}_{x-\Delta})dr,$$

whence

$$I = \sum_{m,n,a,b\ge 0}\int_{x-\Delta}^{x+\Delta}\int_{x-\Delta}^{x+\Delta}\int_{y-\Delta}^{y+\Delta}\int_{y-\Delta}^{y+\Delta}\beta_m(x,x+\Delta,r)\beta_n(x,x+\Delta,s)\beta_a(y,y+\Delta,u)\beta_b(y,y+\Delta,v)$$
$$N^{-4}\sum_{i,j,k,l=1}^N\sum_{\alpha,\delta,\gamma,\zeta=1}^N A^m_N(i,j)A^n_N(k,l)A^a_N(\alpha,\delta)A^b_N(\gamma,\zeta)$$
$$\mathbb{C}\mathrm{ov}_\theta\Big((M^{j,N}_r - M^{j,N}_{x-\Delta})(M^{l,N}_s - M^{l,N}_{x-\Delta}), (M^{\delta,N}_u - M^{\delta,N}_{y-\Delta})(M^{\zeta,N}_v - M^{\zeta,N}_{y-\Delta})\Big)dvdudsdr.$$

Using that $\int_{x-\Delta}^{x+\Delta}|\beta_m(x,x+\Delta,r)|dr \le 2\Lambda^m$ (and the same formula for the three other integrals), Step 6 and that $\sum_{i=1}^N A^m_N(i,j) \le |||A_N|||_1^m$ (and the same formula for the sums in $k,\alpha,\gamma$), we find that, still on $\Omega^1_N$,

$$|I| \le C\sum_{m,n,a,b\ge 0}\Lambda^{m+n+a+b}|||A_N|||_1^{m+n+a+b}N^{-4}\sum_{j,l,\delta,\zeta=1}^N t^{1/2}\Delta^{1-q}\mathbf{1}_{\{j=l\}} \le CN^{-1}t^{1/2}\Delta^{1-q}.$$

*Step 8.* Gathering Steps 5 and 7, we find that

$$\Big|\mathbb{C}\mathrm{ov}_\theta\left((\bar U_{x+\Delta} - \bar U^N_x)^2, (\bar U_{y+\Delta} - \bar U^N_y)^2\right)\Big| \le C(N^{-1}t^{1/2}\Delta^{1-q} + N^{-2}t^2\Delta^{-4q} + N^{-2}t^{1/2}\Delta^{3/2-q}),$$

which completes the proof. □

We can finally treat the last term.

**Lemma 31.** *Assume $H(q)$ for some $q\ge 1$. On $\Omega^1_N$, for all $1\le\Delta\le t/2$,*

$$\mathbb{E}_\theta[(D^{N,3}_{\Delta,t})^2] \le C\Big(\frac{\Delta}{t} + \frac{t}{\Delta^{4q+1}} + \frac{Nt^{1/2}}{\Delta^{q+1}} + \frac{t^2}{\Delta^{4q+2}} + \frac{t^{1/2}}{\Delta^{q+1/2}}\Big).$$

*Proof.* First note that by definition of $D^{N,3}_{\Delta,t}$ and since $\bar U^N_r = \bar Z^N_r - \mathbb{E}_\theta[\bar Z^N_r]$,

$$\mathbb{E}_\theta[(D^{N,3}_{\Delta,t})^2] = \frac{N^2}{t^2}\mathbb{V}\mathrm{ar}_\theta\Big(\sum_{a=t/\Delta+1}^{2t/\Delta}(\bar U^N_{a\Delta} - \bar U^N_{(a-1)\Delta})^2\Big) = \frac{N^2}{t^2}\sum_{a,b=t/\Delta+1}^{2t/\Delta}K_{a,b},$$

where $K_{a,b} = \mathbb{C}\mathrm{ov}_\theta\,((\bar U^N_{a\Delta} - \bar U^N_{(a-1)\Delta})^2, (\bar U^N_{b\Delta} - \bar U^N_{(b-1)\Delta})^2)$. If $|a-b|\le 2$, we only use that

$$|K_{a,b}| \le \Big(\mathbb{V}\mathrm{ar}_\theta\,\Big((\bar U^N_{a\Delta} - \bar U^N_{(a-1)\Delta})^2\Big)\mathbb{V}\mathrm{ar}_\theta\,\Big((\bar U^N_{b\Delta} - \bar U^N_{(b-1)\Delta})^2\Big)\Big)^{1/2} \le C\Big(\frac{\Delta^2}{N^2} + \frac{t^2}{N^2\Delta^{4q}}\Big).$$

We finally used the first estimate of Lemma 30, which is valid since $x=(a-1)\Delta$ satisfies $x\ge t$ and thus $t/2 \le x-\Delta \le x+\Delta \le 2t$ and $x=(b-1)\Delta$ satisfies the same conditions. If now $|a-b|\ge 3$ and w.l.o.g. $a>b$, we use the second estimate of Lemma 30, which is valid since $x=(a-1)\Delta$ and $y=(b-1)\Delta$ satisfy the required conditions (in particular, $y+\Delta\le x-2\Delta$). This gives

$$|K_{a,b}| \le C\Big(\frac{t^{1/2}}{N\Delta^{q-1}} + \frac{t^2}{N^2\Delta^{4q}} + \frac{t^{1/2}}{N^2\Delta^{q-3/2}}\Big).$$

We end with

$$\mathbb{E}_\theta[(D^{N,3}_{\Delta,t})^2] \le C\frac{N^2}{t^2}\frac{t}{\Delta}\Big(\frac{\Delta^2}{N^2} + \frac{t^2}{N^2\Delta^{4q}}\Big) + C\frac{N^2}{t^2}\frac{t^2}{\Delta^2}\Big(\frac{t^{1/2}}{N\Delta^{q-1}} + \frac{t^2}{N^2\Delta^{4q}} + \frac{t^{1/2}}{N^2\Delta^{q-3/2}}\Big).$$

The conclusion follows. □

We can at last give the

*Proof of Proposition 25.* Gathering Lemmas 26, 27, 29 and 31, we see that, on $\Omega^1_N$, if $1\le\Delta\le t/4$,

$$\begin{aligned}\mathbb{E}_\theta[|\mathcal{W}^N_{\Delta,t} - \mathcal{W}^N_{\infty,\infty}|] \le& \mathbb{E}_\theta[D^{N,1}_{\Delta,t} + 2D^{N,1}_{2\Delta,t} + D^{N,2}_{\Delta,t} + 2D^{N,2}_{2\Delta,t} + D^{N,3}_{\Delta,t} + 2D^{N,3}_{2\Delta,t} + D^{N,4}_{\Delta,t}]\\ \le& C\Big(\frac{\Delta}{t} + \frac{N\Delta}{t^{2q}} + \frac{N}{t^{q-1}} + \frac{t}{\Delta^{q+1}} + \sqrt{\frac{\Delta}{t} + \frac{t}{\Delta^{4q+1}} + \frac{Nt^{1/2}}{\Delta^{q+1}} + \frac{t^2}{\Delta^{4q+2}} + \frac{t^{1/2}}{\Delta^{q+1/2}}}\Big).\end{aligned}$$

Using that $q\ge 3$ (whence in particular $2q-1\ge q-1\ge (q+1)/2$) and that $1\le\Delta\le t$, we easily deduce that $\Delta/t \le (\Delta/t)^{1/2}$, that $N\Delta t^{-2q} \le N\Delta^{1-2q} \le N\Delta^{-(q+1)/2}$, that $Nt^{1-q} \le N\Delta^{1-q} \le N\Delta^{-(q+1)/2}$, that $t\Delta^{-q-1} \le t\Delta^{-q/2-1}$, that $t^{1/2}\Delta^{-2q-1/2} \le t\Delta^{-2q-1} \le t\Delta^{-q/2-1}$, that $N^{1/2}t^{1/4}\Delta^{-(q+1)/2} \le N\Delta^{-(q+1)/2} + t^{1/2}\Delta^{-(q+1)/2} \le N\Delta^{-(q+1)/2} + t\Delta^{-q/2-1}$, that $t\Delta^{-2q-1} \le t\Delta^{-q/2-1}$ and that $t^{1/4}\Delta^{-q/2-1/4} \le t\Delta^{-q/2-1}$. This gives, still on $\Omega^1_N$,

$$\mathbb{E}_\theta[|\mathcal{W}^N_{\Delta,t} - \mathcal{W}^N_{\infty,\infty}|] \le C\Big(\sqrt{\frac{\Delta}{t}} + \frac{N}{\Delta^{(q+1)/2}} + \frac{t}{\Delta^{q/2+1}}\Big)$$

as desired. □

## 4.7. **Conclusion.** We now have all the weapons to check our main result.

*Proof of Theorem 3.* Recall that we assume $H(q)$ for some $q>3$ and that $\Delta_t = t/(2\lfloor t^{1-4/(q+1)}\rfloor) \sim t^{4/(q+1)}/2$ (for $t$ large). We can of course assume that $t\ge 4$ is large enough so that $\Delta_t\in[1,t/4]$, because else the inequalities of the statement are trivial. Using Propositions 14 and 17, we find

$$\mathbb{E}\Big[\mathbf{1}_{\Omega^1_N}\Big|\mathcal{E}^N_t - \frac{\mu}{1-\Lambda p}\Big|\Big] \le \mathbb{E}\Big[\mathbf{1}_{\Omega^1_N}\Big|\mathcal{E}^N_t - \mu\bar\ell_N\Big|\Big] + \mu\mathbb{E}\Big[\mathbf{1}_{\Omega^1_N}\Big|\bar\ell_N - \frac{1}{1-\Lambda p}\Big|\Big] \le C\Big(\frac{1}{N} + \frac{1}{\sqrt{Nt}} + \frac{1}{t^q}\Big).$$

Since now $\Pr((\Omega^1_N)^c) \le Ce^{-cN}$ by Lemma 13, we conclude that for any $\varepsilon\in(0,1)$,

$$\Pr\Big(\Big|\mathcal{E}^N_t - \frac{\mu}{1-\Lambda p}\Big| \ge \varepsilon\Big) \le Ce^{-cN} + \frac{C}{\varepsilon}\Big(\frac{1}{N} + \frac{1}{\sqrt{Nt}} + \frac{1}{t^q}\Big) \le \frac{C}{\varepsilon}\Big(\frac{1}{N} + \frac{1}{\sqrt{Nt}} + \frac{1}{t^q}\Big).$$

Similarly, Propositions 14 and 19 imply, since $\mathcal{V}_\infty^N = \mu^2 \sum_{i=1}^N |\ell_N(i) - \bar\ell_N|^2$,

$$\begin{aligned}\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big|\mathcal{V}_t^N - \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big|\Big] \leq & \mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big|\mathcal{V}_t^N - \mathcal{V}_\infty^N\Big|\Big] + \mu^2\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big|\sum_{i=1}^N |\ell_N(i)-\bar\ell_N|^2 - \frac{\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big|\Big]\\ \leq & \frac{C}{\sqrt{N}} + C\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big(1+\sum_{i=1}^N \Big[\ell_N(i)-\bar\ell_N\Big]^2\Big)^{1/2}\Big]\Big(\frac{N}{t^q}+\frac{\sqrt{N}}{t}+\frac{1}{\sqrt{t}}\Big)\\ \leq & C\Big(\frac{1}{\sqrt{N}}+\frac{N}{t^q}+\frac{\sqrt{N}}{t}+\frac{1}{\sqrt{t}}\Big).\end{aligned}$$

The last inequality uses a second time Proposition 14. We conclude, using Lemma 13 as previously, that for any $\varepsilon \in (0,1)$,

$$\Pr\Big(\Big|\mathcal{V}_t^N - \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big| \geq \varepsilon\Big) \leq Ce^{-cN} + \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{N}}+\frac{N}{t^q}+\frac{\sqrt{N}}{t}+\frac{1}{\sqrt{t}}\Big) \leq \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{N}}+\frac{N}{t^q}+\frac{\sqrt{N}}{t}\Big)$$

because $t^{-1/2} = (N^{1/4}t^{-1/2})N^{-1/4} \leq N^{1/2}t^{-1} + N^{-1/2}$. This implies that

$$\Pr\Big(\Big|\mathcal{V}_t^N - \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big| \geq \varepsilon\Big) \leq \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{N}}+\frac{\sqrt{N}}{t}\Big).$$

Indeed, either $\sqrt{N} > t$ and the inequality is trivial or $\sqrt{N} \leq t$ and then $Nt^{-q} \leq Nt^{-2} \leq N^{1/2}t^{-1}$.

Finally, we infer from Propositions 14 and 25, since $\mathcal{W}_{\infty,\infty}^N = \mu N^{-1}\sum_{i=1}^N \ell_N(i)(c_N(i))^2$, that

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big|\mathcal{W}_{\Delta_t,t}^N - \frac{\mu}{(1-\Lambda p)^3}\Big|\Big]\\ \leq & \mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big|\mathcal{W}_{\Delta_t,t}^N - \mathcal{W}_{\infty,\infty}^N\Big|\Big] + \mu\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big|N^{-1}\sum_{i=1}^N \ell_N(i)(c_N(i))^2 - \frac{1}{(1-\Lambda p)^3}\Big|\Big]\\ \leq & C\Big(\frac{1}{N}+\sqrt{\frac{\Delta_t}{t}}+\frac{N}{\Delta_t^{(q+1)/2}}+\frac{t}{\Delta_t^{q/2+1}}\Big),\end{aligned}$$

whence as usual by Lemma 13, for $\varepsilon \in (0,1)$,

$$\begin{aligned}\Pr\Big(\Big|\mathcal{W}_{\Delta_t,t}^N - \frac{\mu}{(1-\Lambda p)^3}\Big| \geq \varepsilon\Big) \leq & Ce^{-cN} + \frac{C}{\varepsilon}\Big(\frac{1}{N}+\sqrt{\frac{\Delta_t}{t}}+\frac{N}{\Delta_t^{(q+1)/2}}+\frac{t}{\Delta_t^{q/2+1}}\Big)\\ \leq & \frac{C}{\varepsilon}\Big(\frac{1}{N}+\frac{1}{\sqrt{t^{1-4/(q+1)}}}+\frac{N}{t^2}\Big).\end{aligned}$$

We finally used that $\Delta_t \sim t^{4/(q+1)}/2$, which implies that $\sqrt{\Delta_t/t} \sim \sqrt{1/(2t^{1-4/(q+1)})}$, that $N/\Delta_t^{(q+1)/2} \sim 2^{(q+1)/2}Nt^{-2}$ and $t/\Delta_t^{q/2+1} \sim 2^{q/2+1}t^{-(q+3)/(q+1)} \leq 2^{q/2+1}/\sqrt{t^{1-4/(q+1)}}$. □

*Proof of Corollary 4.* Recall that we assume $H(q)$ for some $q > 3$. We fix $\mu > 0$, $\Lambda > 0$ and $p \in (0,1]$ such that $\Lambda p \in (0,1)$. We define $u = \mu/(1-\Lambda p)$, $v = \mu^2\Lambda^2 p(1-p)/(1-\Lambda p)^2$ and $w = \mu/(1-\Lambda p)^3$. It holds that $(u,v,w) \in D$ (which would not be the case if $\Lambda p = 0$) and $\Psi(u,v,w) = (\mu,\Lambda,p)$. Furthermore, $\Psi$ is obviously of class $C^\infty$ on $D$, it is in particular locally Lipschitz continuous. As a consequence, there is a constant $c \in (0,1)$ (depending on $\mu,\Lambda,p$) such that for any $N \geq 1$, any $t \geq 1$, any $\varepsilon \in (0, 1/c)$,

$$\Pr\Big(\Big\|\Psi(\mathcal{E}_t^N, \mathcal{V}_t^N, \mathcal{W}_{\Delta_t,t}^N) - (\mu, \Lambda, p)\Big\| \geq \varepsilon\Big) \leq \Pr\Big(|\mathcal{E}_t^N - u| + |\mathcal{V}_t^N - v| + |\mathcal{W}_{\Delta_t,t}^N - w| \geq c\varepsilon\Big)$$
$$\leq \frac{C}{\varepsilon}\Big(\frac{1}{N} + \frac{1}{\sqrt{Nt}} + \frac{1}{t^q} + \frac{\sqrt{N}}{t} + \frac{1}{\sqrt{N}} + \frac{1}{N} + \frac{N}{t^2} + \frac{1}{\sqrt{t^{1-4/(q+1)}}}\Big)$$

by Theorem 3. Using next that $q > 3$, that $t \geq 1$ and $N \geq 1$, and that either $N^{1/2}t^{-1} \geq 1$ (whence the inequality below is trivial) or $N^{1/2}t^{-1} < 1$ (whence $Nt^{-2} \leq N^{1/2}t^{-1}$), we find

$$\Pr\Big(\Big\|\Psi(\mathcal{E}_t^N, \mathcal{V}_t^N, \mathcal{W}_{\Delta_t,t}^N) - (\mu, \Lambda, p)\Big\| \geq \varepsilon\Big) \leq \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{N}} + \frac{\sqrt{N}}{t} + \frac{1}{\sqrt{t^{1-4/(q+1)}}}\Big).$$

Noting that $t^{-(1-4/(q+1))/2} = [N^{1/4}t^{-(1-4/(q+1))/2}]N^{-1/4} \leq N^{-1/2} + N^{1/2}t^{-(1-4/(q+1))}$ concludes the proof. $\square$

We finally give the

*Proof of Remark 2.* Lemma 16-(ii) with $r = 1$ and $s = 0$ tells us that on $\Omega_N^1$, $|\mathbb{E}_\theta[\bar{Z}_t^N] - \mu\bar{\ell}_N t| \leq C$. By Lemma 18, we know that $\mathbb{E}_\theta[|\bar{Z}_t^N - \mathbb{E}_\theta[\bar{Z}_t^N]|] = \mathbb{E}_\theta[|\bar{U}_t^N|] \leq C(t/N)^{1/2}$, still on $\Omega_N^1$ and, by Proposition 14, $\mathbb{E}[\mathbf{1}_{\Omega_N^1}|\bar{\ell}_N - 1/(1-\Lambda p)|] \leq CN^{-1}$. We easily deduce that

$$\mathbb{E}[\mathbf{1}_{\Omega_N^1}|\bar{Z}_t^N - \mu(1-\Lambda p)^{-1}t|] \leq C(1 + (t/N)^{1/2} + t/N) \leq C(1 + t/N).$$

Since $\Pr(\Omega_N^1) \geq 1 - Ce^{-cN}$ by Lemma 13, we find that for any $\varepsilon > 0$,

$$\Pr\Big(\Big|\frac{\bar{Z}_t^N}{t} - \frac{\mu}{1-\Lambda p}\Big| \geq \varepsilon\Big) \leq Ce^{-cN} + \frac{C}{\varepsilon}\Big(\frac{1}{t} + \frac{1}{N}\Big).$$

The conclusion follows. $\square$

## 5. The supercritical case

The goal of this section is to prove Theorem 6. In Subsection 5.1, we study precisely the Perron-Frobenius eigenvalue and eigenvector of the matrix with nonnegative entries $A_N(i,j) = N^{-1}\theta_{ij}$. In Subsection 5.2, we state and prove a few results on some series involving $\varphi^{\star n}$. A few preliminary stochastic analysis is handled in Subsection 5.3. We finally conclude the proof in Subsection 5.4.

### 5.1. Perron-Frobenius analysis of the random matrix $A_N$.

We recall that the norms $||\cdot||_r$ on $\mathbb{R}^N$ and $|||\cdot|||_r$ on $\mathcal{M}_{N\times N}(\mathbb{R})$ were defined in Subsection 4.1. We denote by $(e_1, \dots, e_N)$ the canonical basis of $\mathbb{R}^N$ and by $\mathbf{1}_N = \sum_{i=1}^N e_i$ the vector will all entries equal to 1.

**Notation 32.** *We consider the matrix $A_N(i,j) = N^{-1}\theta_{ij}$ and the event*

$$\Omega_N^2 = \Big\{\frac{1}{N}\sum_{i,j=1}^N A_N(i,j) > \frac{p}{2} \text{ and for all } i,j = 1, \dots, N,\ |NA_N^2(i,j) - p^2| < \frac{p^2}{2N^{3/8}}\Big\}.$$

Actually, 3/8 could be replaced by any other exponent in $[3/8, 1/2)$. We first show that $\Omega_N^2$ has a high probability.

**Lemma 33.** *Assume that $p \in (0,1]$. It holds that* $\Pr(\Omega_N^2) \geq 1 - Ce^{-cN^{1/4}}$.

*Proof.* We recall the Hoeffding inequality [21] for a Binomial$(n,q)$-distributed random variable $X$: for all $x \geq 0$, it holds that $\Pr(|X-nq| \geq x) \leq 2\exp(-2x^2/n)$.

Since $N\sum_{i,j=1}^N A_N(i,j) = \sum_{i,j=1}^N \theta_{ij} \sim$ Binomial$(N^2,p)$, $\Pr(N^{-1}\sum_{i,j=1}^N A_N(i,j) \leq p/2) \leq \Pr(|N\sum_{i,j=1}^N A_N(i,j) - N^2p| \geq N^2p/2) \leq 2\exp(-N^2p^2/2)$.

For each $i \neq j$, we write $N^2A_N^2(i,j) = \sum_{k=1}^N \theta_{ik}\theta_{kj} = Z_{ij}^N + \theta_{ii}\theta_{ij} + \theta_{ij}\theta_{jj}$, where $Z_{ij}^N$ follows a Binomial$(N-2,p^2)$ distribution. We thus have $|N^2A_N^2(i,j) - Z_{ij}^N| \leq 2$. This obviously extends to the case where $i=j$. Hence for any $i,j$, $|NA_N^2(i,j)-p^2| \geq p^2/(2N^{3/8})$ implies that $|Z_{ij}^N - (N-2)p^2| \geq p^2N^{5/8}/2 - 4$ and thus, if $N \geq (16/p^2)^{8/5}$, that $|Z_{ij}^N - (N-2)p^2| \geq p^2N^{5/8}/4$. By the Hoeffding inequality, $\Pr(|NA_N^2(i,j)-p^2| \geq p^2/(2N^{3/8})) \leq 2\exp(-p^4N^{5/4}/(8(N-2)) \leq 2\exp(-p^4N^{1/4}/8)$.

All this shows that $\Pr((\Omega_N^2)^c) \leq 2\exp(-N^2p^2/2) + 2N^2\exp(-p^4N^{1/4}/8)$ for all $N \geq (16/p^2)^{8/5}$. The conclusion easily follows: we can find $0<c<C$ depending only on $p$ such that for all $N \geq 1$, $\Pr((\Omega_N^2)^c) \leq Ce^{-cN^{1/4}}$. ☐

Next, we apply the Perron-Frobenius theorem.

**Lemma 34.** *Assume that $p \in (0,1]$. On $\Omega_N^2$, the spectral radius $\rho_N$ of $A_N$ is a simple eigenvalue of $A_N$ and $\rho_N \in [p(1-1/(2N^{3/8})), p(1+1/(2N^{3/8}))]$. There is a unique eigenvector $V_N \in (\mathbb{R}_+)^N$ of $A_N$ for the eigenvalue $\rho_N$ such that $||V_N||_2 = \sqrt{N}$. We also have $V_N(i) > 0$ for all $i=1,\dots,N$.*

*Proof.* The matrix $A_N$ has nonnegative entries and is irreducible on $\Omega_N^2$ since $A_N^2$ has positive entries. We thus infer from the Perron-Frobenius theorem that on $\Omega_N^2$, $\rho_N$ is a simple eigenvalue of $A_N$, that there is a unique corresponding eigenvector $V_N$ with nonnegative entries such that $||V_N||_2 = \sqrt{N}$ and that $V_N(i) > 0$ for all $i=1,\dots,N$.

Since $NA_N^2(i,j) \in [p^2(1-1/(2N^{3/8})), p^2(1+1/(2N^{3/8}))]$ for all $i,j=1,\dots,N$ on $\Omega_N^2$ we deduce from $\rho_N^2 V_N = A_N^2 V_N$ that $\rho_N^2||V_N||_1 = \sum_{i,j=1}^N A_N^2(i,j)V_N(j) \leq p^2(1+1/(2N^{3/8}))||V_N||_1$, whence $\rho_N^2 \leq p^2(1+1/(2N^{3/8}))$ and thus $\rho_N \leq p(1+1/(2N^{3/8}))$. Similarly, $\rho_N^2||V_N||_1 = \sum_{i,j=1}^N A_N^2(i,j)V_N(j) \geq p^2(1-1/(2N^{3/8}))||V_N||_1$, whence $\rho_N^2 \geq p^2(1-1/(2N^{3/8}))$ and thus $\rho_N \geq p(1-1/(2N^{3/8}))$. ☐

We now gather a number of important facts.

**Lemma 35.** *Assume that $p \in (0,1]$. There is $N_0 \geq 1$ (depending only on $p$) such that for all $N \geq N_0$, on $\Omega_N^2$, the following properties hold true for all $i,j,k,l = 1,\dots,N$:*

*(i) for all $n \geq 2$, $A_N^n(i,j) \leq (3/2)A_N^n(k,l)$,*

*(ii) $V_N(i) \in [1/2,2]$,*

*(iii) for all $n \geq 0$, $||A_N^n\mathbf{1}_N||_2 \in [\sqrt{N}\rho_N^n/2, 2\sqrt{N}\rho_N^n]$,*

*(iv) for all $n \geq 2$, $A_N^n(i,j) \in [\rho_N^n/(3N), 3\rho_N^n/N]$,*

*(v) for all $n \geq 0$, all $r \in [1,\infty]$, $\big|\big| ||A_N^n\mathbf{1}_N||_r^{-1}A_N^n\mathbf{1}_N - ||V_N||_r^{-1}V_N\big|\big|_r \leq 3(2N^{-3/8})^{\lfloor n/2\rfloor+1}$,*

*(vi) for all $n \geq 0$, all $r \in [1,\infty]$, $\big|\big| ||A_N^n e_j||_r^{-1}A_N^n e_j - ||V_N||_r^{-1}V_N\big|\big|_r \leq 12(2N^{-3/8})^{\lfloor n/2\rfloor}$,*

*(vii) for all $n \geq 1$, $||A_N^n e_j||_2 \leq 3\rho_N^n/(p\sqrt{N})$ and for all $n \geq 0$, $||A_N^n\mathbf{1}_N||_\infty \leq 3\rho_N^n/p$.*

The proof requires a quantitative version of the Perron-Frobenius theorem due to G. Birkhoff [7]. It is based on the use of the Hilbert projective distance.

**Notation 36.** *For $x = (x_i)_{i=1,\dots,N}$ and $y = (y_i)_{i=1,\dots,N}$ in $(0,\infty)^N$, we set*

$$d_N(x,y) = \log\Big(\frac{\max_{i=1,\dots,N}(x_i/y_i)}{\min_{i=1,\dots,N}(x_i/y_i)}\Big).$$

*We have $d_N(x,y) = d_N(y,x) = d_N(x,\lambda y)$ for all $\lambda > 0$ and $d_N(x,y) \le d_N(x,z) + d_N(z,y)$. Finally, $d_N(x,y) = 0$ if and only if $x$ and $y$ are colinear.*

The result of Birkhoff quantifies the projection on the Perron-Frobenius vector.

**Theorem 37** (Birkhoff [7], Cavazos-Cadena [11])**.** *For any $A \in \mathcal{M}_{N\times N}(\mathbb{R})$ with positive entries and any $x$ and $y$ in $(0,\infty)^N$, we have $d_N(Ax,Ay) \le k_A d_N(x,y)$, where*

$$\Gamma_A = \max_{i,j,k,l=1,\dots,N} \frac{A(i,k)A(j,l)}{A(i,l)A(j,k)} \ge 1 \quad \textit{and} \quad k_A = \frac{\sqrt{\Gamma_A}-1}{\sqrt{\Gamma_A}+1} \le \frac{\Gamma_A - 1}{4}.$$

In our context, this gives the following estimates.

**Remark 38.** *Assume that $p \in (0,1]$. Then on $\Omega_N^2$, it holds that for all $x,y \in (0,\infty)^N$, we have (i) $d_N(A_N x, A_N y) \le d_N(x,y)$ and (ii) $d_N(A_N^2 x, A_N^2 y) \le 2N^{-3/8} d_N(x,y)$.*

*Proof.* On $\Omega_N^2$, we have

$$A_N^2(i,j) \in [p^2 N^{-1}(1 - 1/(2N^{3/8})), p^2 N^{-1}(1 + 1/(2N^{3/8}))]. \tag{11}$$

This implies that for each $i = 1,\dots,N$, $\sum_{k=1}^N A_N(i,k) > 0$ (because else, $A_N^2(i,j)$ would vanish for all $j = 1,\dots,N$). Thus for $x,y \in (0,\infty)^N$, we have $A_N x, A_N y \in (0,\infty)^N$ so that $d_N(A_N x, A_N y)$ is well-defined. We put $m = \min_i(x_i/y_i)$ and $M = \max_i(x_i/y_i)$. We then have $m(A_N y)_i \le (A_N x)_i \le M(A_N y)_i$ for all $i$, whence $d_N(A_N x, A_N y) \le \log(M/m) = d_N(x,y)$, which proves (i). For point (ii), it suffices to use Theorem 37, and to note that, by (11),

$$\Gamma_{A_N^2} = \max_{i,j,k,l=1,\dots,N} \frac{A_N^2(i,k)A_N^2(j,l)}{A_N^2(i,l)A_N^2(j,k)} \le \frac{(1+1/(2N^{3/8}))^2}{(1-1/(2N^{3/8}))^2} \le 1 + 8N^{-3/8},$$

whence $k_{A_N^2} \le (\Gamma_{A_N^2} - 1)/4 \le 2N^{-3/8}$. □

We will also use the following easy remark.

**Lemma 39.** *For all $r \in [1,\infty]$ and all $x,y \in (0,\infty)^N$ such that $d_N(x,y) \le 1$, we have the inequality $\big|\big| ||x||_r^{-1}x - ||y||_r^{-1}y \big|\big|_r \le 3d_N(x,y)$.*

*Proof.* We fix $r \in [1,\infty]$ and assume without loss of generality that $||x||_r = ||y||_r = 1$. We set $m = \min_i(x_i/y_i)$ and $M = \max_i(x_i/y_i)$. Since $||x||_r = ||y||_r$, it holds that $m \le 1 \le M$. Using that $1 \ge d_N(x,y) = \log(1 + (M-m)/m)$, we deduce that $(M-m)/m \le e - 1 \le 2$. Since $\log(1+u) \ge u/3$ on $[0,2]$, we conclude that $d_N(x,y) \ge (M-m)/(3m) \ge (M-m)/3$. But for all $i$, we have $x_i \in [my_i, My_i]$, whence $|x_i - y_i| \le (M-m)y_i$. Thus $||x-y||_r \le (M-m)||y||_r = (M-m) \le 3d_N(x,y)$. □

We can now give the

*Proof of Lemma 35.* We work on $\Omega_N^2$ during the whole proof.

*Step 1.* We first check that $d_N(\mathbf{1}_N, V_N) \le 2N^{-3/8}$. We start from $A_N^2 V_N = \rho_N V_N$, so that for all $i$, $V_N(i) = \rho_N^{-2} \sum_{j=1}^N A_N^2(i,j) V_N(j)$. But using (11) and setting $\kappa_N = p^2 \rho_N^{-2} N^{-1} \sum_{j=1}^N V_N(j)$, we find that $V_N(i) \in [\kappa_N(1-1/(2N^{3/8})), \kappa_N(1+1/(2N^{3/8}))]$. Consequently, $\max_i V_N(i)/\min_i V_N(i) \leq (1+1/(2N^{3/8}))/(1-1/(2N^{3/8})) \leq 1+2N^{-3/8}$. Hence

$$d_N(\mathbf{1}_N, V_N) \leq \log[(1+1/(2N^{3/8}))/(1-1/(2N^{3/8}))] \leq \log(1+2N^{-3/8}) \leq 2N^{-3/8}.$$

*Step 2.* Here we show that for all $i$, $V_N(i) \in [(1+2N^{-3/8})^{-1}, (1+2N^{-3/8})]$. This will imply point (ii) (for $N$ large enough so that $2N^{-3/8} \leq 1$). We introduce $m = \min_i V_N(i)$ and $M = \max_i V_N(i)$. We have seen in Step 1 that $M/m \leq 1+2N^{-3/8}$. Recalling that $||V_N||_2 = \sqrt{N}$ by definition, we deduce that $N = \sum_{i=1}^N (V_N(i))^2 \leq NM^2 \leq N(1+2N^{-3/8})^2 m^2$, whence $m \geq (1+2N^{-3/8})^{-1}$. Similarly, $N = \sum_{i=1}^N (V_N(i))^2 \geq Nm^2 \geq N(1+2N^{-3/8})^{-2} M^2$, whence $M \leq (1+2N^{-3/8})$.

*Step 3.* We verify that for all $n \geq 0$, $d_N(A_N^n \mathbf{1}_N, V_N) \leq (2N^{-3/8})^{\lfloor n/2 \rfloor + 1}$. By Lemma 39, this will imply point (v) for all $N$ large enough so that $2N^{-3/8} \leq 1$. Using that $A_N^n V_N = \rho_N^n V_N$, we deduce that $d_N(A_N^n \mathbf{1}_N, V_N) = d_N(A_N^n \mathbf{1}_N, A_N^n V_N)$. Hence for all $n$ even, we deduce from Remark 38-(ii) and Step 1 that $d_N(A_N^n \mathbf{1}_N, V_N) \leq (2N^{-3/8})^{n/2} d_N(\mathbf{1}_N, V_N) \leq (2N^{-3/8})^{n/2+1}$. When $n$ is odd, we simply use that $d_N(A_N^n \mathbf{1}_N, V_N) = d_N(A_N^n \mathbf{1}_N, A_N V_N) \leq d_N(A_N^{n-1} \mathbf{1}_N, V_N)$ by Remark 38-(i).

*Step 4.* We now prove (vi). We fix $r \in [1, \infty]$ and $j \in \{1, \ldots, N\}$. The result is obvious if $n = 0$ or $n = 1$ because then $\big|\big| ||A_N^n e_j||_r^{-1} A_N^n e_j - ||V_N \mathbf{1}_N||_r^{-1} V_N \big|\big|_r \leq 2 \leq 12(2N^{-3/8})^{\lfloor n/2 \rfloor}$.

By Remark 38-(ii), $d_N(A_N^{2k} e_j, V_N) = d_N(A_N^{2k} e_j, A_N^{2k} V_N) \leq (2N^{-3/8})^{k-1} d_N(A_N^2 e_j, V_N)$ for all $k \geq 1$.

We next write $d_N(A_N^2 e_j, V_N) \leq d_N(A_N^2 e_j, \mathbf{1}_N) + d_N(\mathbf{1}_N, V_N)$. By Step 1, we have $d_N(\mathbf{1}_N, V_N) \leq \log[(1+N^{-3/8}/2)/(1+N^{-3/8}/2)]$. Furthermore, we deduce from (11) that $d_N(A_N^2 e_j, \mathbf{1}_N) = \log[\max_i(A_N^2(i,j))/\min_i(A_N^2(i,j))] \leq \log[(1+N^{-3/8}/2)/(1+N^{-3/8}/2)]$. All in all, we find that $d_N(A_N^2 e_j, V_N) \leq \log[(1+N^{-3/8}/2)^2/(1-N^{-3/8}/2)^2] \leq \log(1+8N^{-3/2}) \leq 8N^{-3/2}$.

Hence for all $k \geq 1$, $d_N(A_N^{2k} e_j, V_N) \leq 8N^{-3/8}(2N^{-3/8})^{k-1} = 4(2N^{-3/8})^k$. We also have, by Remark 38-(i), $d_N(A_N^{2k+1} e_j, V_N) = d_N(A_N^{2k+1} e_j, A_N V_N) \leq d_N(A_N^{2k} e_j, V_N)$. Thus for all $n \geq 2$, $d_N(A_N^n e_j, V_N) \leq 4(2N^{-3/8})^{\lfloor n/2 \rfloor}$. This implies that indeed, $\big|\big| ||A_N^n e_j||_r^{-1} A_N^n e_j - ||V_N||_r^{-1} V_N \big|\big|_r \leq 12(2N^{-3/8})^{\lfloor n/2 \rfloor}$ by Lemma 39, if $N$ is large enough so that $2N^{-3/8} \leq 1/4$.

*Step 5.* We check (i). Using Step 2, we see that for all $j = 1, \ldots, N$, all $n \geq 2$,

$$d_N(A_N^n e_j, V_N) = \log\Big(\frac{\max_i(A_N^n(i,j)/V_N(i))}{\min_i(A_N^n(i,j)/V_N(i))}\Big) \geq \log\Big(\frac{\max_i A_N^n(i,j)}{\min_i A_N^n(i,j)} \times (1+2N^{-3/8})^{-2}\Big).$$

But for all $n \geq 2$, using Remark 38-(i), we see that $d_N(A_N^n e_j, V_N) = d_N(A_N^n e_j, A_N^{n-2} V_N) \leq d_N(A_N^2 e_j, V_N) \leq \log(1+8N^{-3/8})$ as seen in Step 4. We conclude that

$$\frac{\max_i A_N^n(i,j)}{\min_i A_N^n(i,j)} \leq (1+2N^{-3/8})^2(1+8N^{-3/8}).$$

Using the same arguments with the transpose matrix $A_N^t$ (which satisfies exactly the same assumptions as $A_N$ on $\Omega_N^2$), we see that for all $i = 1, \ldots, N$,

$$\frac{\max_j A_N^n(i,j)}{\min_j A_N^n(i,j)} \leq (1+2N^{-3/8})^2(1+8N^{-3/8}).$$

Finally, we conclude that for all $n \geq 2$,

$$\frac{\max_{i,j} A_N^n(i,j)}{\min_{i,j} A_N^n(i,j)} \leq (1+2N^{-3/8})^4(1+8N^{-3/8})^2.$$

This is indeed smaller than $3/2$ if $N$ is large enough.

*Step 6.* We now verify (iii). We write $A_N^n \mathbf{1}_N = ||A_N^n \mathbf{1}_N||_2 (N^{-1/2} V_N + Z_{N,n})$, where $Z_{N,n} = ||A_N^n \mathbf{1}_N||_2^{-1} A_N^n \mathbf{1}_N - N^{-1/2} V_N$. We know by (v) (with $r = 2$) that $||Z_{N,n}||_2 \leq 3(2N^{-3/8})^{\lfloor n/2 \rfloor + 1}$. We next write, for each $n \geq 0$, $A_N^{n+1} \mathbf{1}_N = ||A_N^n \mathbf{1}_N||_2 (N^{-1/2} \rho_N V_N + A_N Z_{N,n})$. Using that $||V_N||_2 = \sqrt{N}$ and that $|||A_N|||_2 \leq 1$ (which immediately follows from the fact that $0 \leq A_N(i,j) \leq 1/N$), we conclude that $\big| ||A_N^{n+1} \mathbf{1}_N||_2 - \rho_N ||A_N^n \mathbf{1}_N||_2 \big| \leq 3 ||A_N^n \mathbf{1}_N||_2 (2N^{-3/8})^{\lfloor n/2 \rfloor + 1}$.

We now set $x_n = ||A_N^n \mathbf{1}_N||_2 / (\sqrt{N} \rho_N^n)$. For all $n \geq 0$, we have

$$|x_{n+1} - x_n| \leq 3 x_n (2N^{-3/8})^{\lfloor n/2 \rfloor + 1} / \rho_N \leq 6 x_n (2N^{-3/8})^{\lfloor n/2 \rfloor + 1} / p,$$

because $\rho_N \geq p/2$ on $\Omega_N^2$, see Lemma 34. If now $N$ is large enough so that $6(2N^{-3/8})^{1/2}/p \leq 1/2$, we easily conclude, using that $x_0 = 1$, that, for all $n \geq 1$,

$$x_n \in \Big[ \prod_{k=1}^n (1 - 6(2N^{-3/8})^{\lfloor k/2 \rfloor + 1}/p), \prod_{k=1}^n (1 + 6(2N^{-3/8})^{\lfloor k/2 \rfloor + 1}/p) \Big],$$

which is included in $[1/2, 2]$ if $N$ is large enough (depending only on $p$). Since $x_0 = 1$, we thus have $x_n \in [1/2, 2]$ for all $n \geq 0$, and thus $||A_N^n \mathbf{1}_N||_2 \in [\sqrt{N} \rho_N^n / 2, 2\sqrt{N} \rho_N^n]$ for all $n \geq 0$.

*Step 7.* Here we prove (iv). We fix $n \geq 2$ and set $m = \min_{i,j} A_N^n(i,j)$ and $M = \max_{i,j} A_N^n(i,j)$. We know from (i) that $M/m \leq 3/2$. Starting from point (iii), we write $\sqrt{N} \rho_N^n / 2 \leq ||A_N^n \mathbf{1}_N||_2 = (\sum_{i=1}^N (\sum_{j=1}^N A_N^n(i,j))^2)^{1/2} \leq N^{3/2} M \leq 3N^{3/2} m/2$, whence $m \geq \rho_N^n / (3N)$. By the same way, $2\sqrt{N} \rho_N^n \geq ||A_N^n \mathbf{1}_N||_2 \geq N^{3/2} m \geq 2N^{3/2} M/3$, whence $M \leq 3\rho_N^n / N$.

*Step 8.* It only remains to check (vii). We know from (iv) that for all $n \geq 2$, $A_N^n(i,j) \leq 3\rho_N^n / N \leq 3\rho_N^n / (pN)$. And for $n = 1$, $A_N(i,j) \leq 1/N \leq 3\rho_N / (pN)$ because $\rho_N \geq p/3$ on $\Omega_N^2$, see Lemma 34. We conclude that for all $n \geq 1$, $A_N^n(i,j) \leq 3\rho_N^n / (pN)$. This immediately implies that for all $n \geq 1$, $||A_N^n e_j||_2 = (\sum_{i=1}^N (A_N^n(i,j))^2)^{1/2} \leq 3\rho_N^n / (p\sqrt{N})$ and $||A_N^n \mathbf{1}_N||_\infty = \max_i \sum_{j=1}^N A_N^n(i,j) \leq 3\rho_N^n / p$. Finally, for $n = 0$, we of course have $||A_N^0 \mathbf{1}_N||_\infty = 1 \leq 3\rho_N^0 / p$. $\square$

Finally, the following tedious result is crucial for our estimation method.

**Proposition 40.** *We assume that $p \in (0,1]$ and we introduce, on $\Omega_N^2$, $\bar{V}_N = N^{-1} \sum_{i=1}^N V_N(i)$ and*

$$\mathcal{U}_\infty^N = \sum_{i=1}^N \Big( \frac{V_N(i) - \bar{V}_N}{\bar{V}_N} \Big)^2.$$

*There is $N_0 \geq 1$ and $C > 0$ (depending only on $p$) such that for all $N \geq N_0$,*

$$\mathbb{E}\Big[ \mathbf{1}_{\Omega_N^2} \Big| \mathcal{U}_\infty^N - \Big( \frac{1}{p} - 1 \Big) \Big| \Big] \leq \frac{C}{\sqrt{N}} \quad \textit{and} \quad \mathbb{E}[\mathbf{1}_{\Omega_N^2} ||V_N - \bar{V}_N \mathbf{1}_N||_2^2] \leq C.$$

*Proof.* We work with $N$ large enough so that we can apply Lemma 35. We introduce the vectors $L_N = A_N \mathbf{1}_N$ and $\mathcal{L}_N = A_N^6 \mathbf{1}_N$, we set $\bar{L}_N = N^{-1} \sum_{i=1}^N L_N(i)$, $\bar{\mathcal{L}}_N = N^{-1} \sum_{i=1}^N \mathcal{L}_N(i)$,

$$H_N = \sum_{i=1}^N \Big( \frac{L_N(i) - \bar{L}_N}{\bar{L}_N} \Big)^2 \quad \text{and} \quad \mathcal{H}_N = \sum_{i=1}^N \Big( \frac{\mathcal{L}_N(i) - \bar{\mathcal{L}}_N}{\bar{\mathcal{L}}_N} \Big)^2.$$

We checked in the proof of Proposition 14-Step 2 that (i) $\mathbb{E}[|\bar{L}_N - p|^2] \leq CN^{-2}$, (ii) $\mathbb{E}[||L_N - \bar{L}_N \mathbf{1}_N||_2^4] \leq C$, (iii) $\mathbb{E}[(||L_N - \bar{L}_N \mathbf{1}_N||_2^2 - p(1-p))^2] \leq CN^{-1}$, (iv) $\mathbb{E}[||A_N L_N - \bar{L}_N L_N||_2^2] \leq CN^{-1}$.

We also recall that $\bar L_N \le 1$ and $|||A_N|||_2 \le 1$ (simply because $0 \le A_N(i,j) \le 1/N$). Furthermore, on $\Omega_N^2$, it holds that $\bar L_N = N^{-1}\sum_{i,j=1}^N A_N(i,j) \ge p/2$, that $\bar{\mathcal{L}}_N = N^{-1}\sum_{i,j=1}^N A_N^6(i,j) \ge \rho_N^6/3 \ge p^6/192$ (by Lemma 35-(iv) and because $\rho_N \ge p/2$ by Lemma 34) and that $\bar V_N \ge 1/2$ (by Lemma 35-(ii)).

*Step 1.* We show that on $\Omega_N^2$, $\Delta_N = |\mathcal{U}_\infty^N - \mathcal{H}_N| \le CN^{-1/2}$. A simple computation shows that

$$\Delta_N = \Big|\sum_{i=1}^N \Big[\Big(\frac{V_N(i)}{\bar V_N}\Big)^2 - \Big(\frac{\mathcal{L}_N(i)}{\bar{\mathcal{L}}_N}\Big)^2\Big]\Big| \le \Big(\sum_{i=1}^N \Big|\frac{V_N(i)}{\bar V_N} - \frac{\mathcal{L}_N(i)}{\bar{\mathcal{L}}_N}\Big|\Big)\Big(\max_i \Big(\frac{V_N(i)}{\bar V_N} + \frac{\mathcal{L}_N(i)}{\bar{\mathcal{L}}_N}\Big)\Big) = S_N T_N,$$

the last equality being a definition.

Lemma 35-(ii) implies that $\max_i(V_N(i)/\bar V_N) \le \max_i V_N(i)/\min_i V_N(i) \le 4$ and Lemma 35-(i) implies that $\max_i(\mathcal{L}_N(i)/\bar{\mathcal{L}}_N) \le \max_i \mathcal{L}_N(i)/\min_i \mathcal{L}_N(i) \le 3/2$ because $\mathcal{L}_N = A_N^6 \mathbf{1}_N$. Thus $T_N \le 4 + 3/2 \le 6$. Next, it holds that $S_N = N\Big|\Big| ||A_N^6\mathbf{1}_N||_1^{-1} A_N^6 \mathbf{1}_N - ||V_N\mathbf{1}_N||_1^{-1} V_N\Big|\Big|_1$. We thus infer from Lemma 35-(v) that $S_N \le 3N(2N^{-3/8})^4 = 48N^{-1/2}$. The conclusion follows.

*Step 2.* We next prove that $\mathbb{E}[\mathbf{1}_{\Omega_N^2}|\mathcal{H}_N - H_N|] \le CN^{-1/2}$. We first write

$$||\mathcal{L}_N - (\bar L_N)^5 L_N||_2 = ||A_N^6\mathbf{1}_N - (\bar L_N)^5 A_N\mathbf{1}_N||_2 \le \sum_{k=1}^5 ||(\bar L_N)^{5-k} A_N^{k+1}\mathbf{1}_N - (\bar L_N)^{6-k} A_N^k \mathbf{1}_N||_2.$$

Using that $\bar L_N \le 1$ and $|||A_N|||_2 \le 1$, we deduce that

$$||\mathcal{L}_N - (\bar L_N)^5 L_N||_2 \le 5||A_N^2\mathbf{1}_N - \bar L_N A_N \mathbf{1}_N||_2 = 5||A_N L_N - \bar L_N L_N||_2.$$

We thus deduce from point (iv) recalled above that $\mathbb{E}[||\mathcal{L}_N - (\bar L_N)^5 L_N||_2^2] \le CN^{-1}$. But it holds that $||\mathcal{L}_N - (\bar L_N)^5 L_N||_2 = I_N + J_N$, where $I_N = ||(\mathcal{L}_N - \bar{\mathcal{L}}_N\mathbf{1}_N) - (\bar L_N)^5(L_N - \bar L_N \mathbf{1}_N)||_2$ and $J_N = ||\bar{\mathcal{L}}_N\mathbf{1}_N - (\bar L_N)^6\mathbf{1}_N||_2 = \sqrt{N}|\bar{\mathcal{L}}_N - (\bar L_N)^6|$. Consequently, $\mathbb{E}[I_N^2] + \mathbb{E}[J_N^2] \le CN^{-1}$. Using now that

$$H_N = \frac{||L_N - \bar L_N\mathbf{1}_N||_2^2}{(\bar L_N)^2} = \frac{||(\bar L_N)^5(L_N - \bar L_N\mathbf{1}_N)||_2^2}{(\bar L_N)^{12}} \quad \text{and} \quad \mathcal{H}_N = \frac{||\mathcal{L}_N - \bar{\mathcal{L}}_N\mathbf{1}_N||_2^2}{(\bar{\mathcal{L}}_N)^2},$$

the facts that $\bar{\mathcal{L}}_N \ge p^6/192$ and $(\bar L_N)^6 \ge p^6/64$ on $\Omega_N^2$ and that the map $x \mapsto x^{-2}$ is globally Lipschitz and bounded on $[p^6/192, \infty)$, we conclude that, still on $\Omega_N^2$,

$$\begin{aligned}|H_N - \mathcal{H}_N| \le & C\Big(||(\bar L_N)^5(L_N - \bar L_N\mathbf{1}_N)||_2^2|(\bar L_N)^6 - \bar{\mathcal{L}}_N| \\ & + \Big|||\mathcal{L}_N - \bar{\mathcal{L}}_N\mathbf{1}_N||_2^2 - ||(\bar L_N)^5(L_N - \bar L_N\mathbf{1}_N)||_2^2\Big|\Big).\end{aligned}$$

Using now the inequality $|a^2 - b^2| \le (a-b)^2 + 2a|a-b|$ for $a, b \ge 0$, we deduce that

$$\begin{aligned}|H_N - \mathcal{H}_N| \le & C\Big(||(\bar L_N)^5(L_N - \bar L_N\mathbf{1}_N)||_2^2 N^{-1/2} J_N + I_N^2 + ||(\bar L_N)^5(L_N - \bar L_N\mathbf{1}_N)||_2^2 I_N\Big) \\ \le & C\Big(||L_N - \bar L_N\mathbf{1}_N||_2^2 N^{-1/2} J_N + I_N^2 + ||L_N - \bar L_N\mathbf{1}_N||_2^2 I_N\Big)\end{aligned}$$

because $\bar L_N \le 1$. Using the Cauchy-Schwarz inequality, that $\mathbb{E}[I_N^2] + \mathbb{E}[J_N^2] \le CN^{-1}$ and that $\mathbb{E}[||L_N - \bar L_N\mathbf{1}_N||_2^4] \le C$ by point (ii) recalled above, we conclude that $\mathbb{E}[\mathbf{1}_{\Omega_N^2}|\mathcal{H}^N - H_N|] \le CN^{-1/2}$.

*Step 3.* Here we check that $\mathbb{E}[\mathbf{1}_{\Omega_N^2}|H_N - (1/p-1)|] \leq CN^{-1/2}$. Since $\bar{L}_N \geq p/2$ on $\Omega_N^2$ and since $x \mapsto x^{-2}$ is bounded and globally Lipschitz continuous on $[p/2, \infty)$, we can write

$$\Big|H_N - \Big(\frac{1}{p}-1\Big)\Big| = \Big|\frac{||L_N - \bar{L}_N \mathbf{1}_N||_2^2}{(\bar{L}_N)^2} - \frac{p(1-p)}{p^2}\Big|$$
$$\leq C\Big(|\bar{L}_N - p|p(1-p) + \Big|||L_N - \bar{L}_N\mathbf{1}_N||_2^2 - p(1-p)\Big|\Big).$$

The conclusion follows, since as recalled in points (i) and (iii) above, $\mathbb{E}[|\bar{L}_N - p|] \leq CN^{-1}$ and $\mathbb{E}[\big|||L_N - \bar{L}_N\mathbf{1}_N||_2^2 - p(1-p)\big|] \leq CN^{-1/2}$.

*Step 4.* Gathering Steps 1, 2 and 3, we immediately deduce that $\mathbb{E}[\mathbf{1}_{\Omega_N^2}|\mathcal{U}_\infty^N - (1/p-1)|] \leq CN^{-1/2}$. Since now $\bar{V}_N \leq 2$ on $\Omega_N^2$ by Lemma 35-(ii), $||V_N - \bar{V}_N\mathbf{1}_N||_2^2 = (\bar{V}_N)^2\mathcal{U}_\infty^N \leq 4\mathcal{U}_\infty^N$, whence of course, $\mathbb{E}[\mathbf{1}_{\Omega_N^2}||V_N - \bar{V}_N\mathbf{1}_N||_2^2] \leq C$. ☐

5.2. **Preliminary analytic estimates.** We recall the following lemma, relying on some results of Feller [14] on convolution equations, that can be found in [13, Lemma 26-(b)].

**Lemma 41.** *Let $\psi : [0,\infty) \mapsto [0,\infty)$ be integrable and such that $\int_0^\infty \psi(t)dt > 1$. Assume also that $t \mapsto \int_0^t |d\psi(s)|$ has at most polynomial growth and set $\Gamma_t = \sum_{n\geq 0} \psi^{\star n}(t)$. Consider $\alpha > 0$ such that $\int_0^\infty e^{-\alpha t}\psi(t)dt = 1$. There are $0 < c < C$ such that for all $t \geq 0$, $1 + \Gamma_t \in [ce^{\alpha t}, Ce^{\alpha t}]$.*

Based on this, it is not hard to verify the following result.

**Lemma 42.** *Assume A. Recall that $\alpha_0$ was defined in Remark 5 such that $p\int_0^\infty e^{-\alpha_0 t}\varphi(t)dt = 1$ and that $\rho_N$ was defined, for each $N \geq 1$, in Lemma 34. We now set $\Gamma_t^N = \sum_{n\geq 0}\rho_N^n \varphi^{\star n}(t)$. For any $\eta > 0$, we can find $N_\eta \geq 1$ and $0 < c_\eta < C_\eta$ (depending only on $p, \varphi$ and $\eta$) such that for all $N \geq N_\eta$, on $\Omega_N^2$, for all $t \geq 0$, $1 + \Gamma_t^N \in [c_\eta e^{(\alpha_0-\eta)t}, C_\eta e^{(\alpha_0+\eta)t}]$.*

*Proof.* We only prove the result when $\eta \in (0, \alpha_0)$, which of course suffices. We consider $\rho_\eta^+ > p > \rho_\eta^-$ defined by $\int_0^\infty e^{-(\alpha_0+\eta)t}\varphi(t)dt = 1/\rho_\eta^+$ and $\int_0^\infty e^{-(\alpha_0-\eta)t}\varphi(t)dt = 1/\rho_\eta^-$. We put $\Gamma_t^{\eta,+} = \sum_{n\geq 0}(\rho_\eta^+)^n\varphi^{\star n}(t)$ and $\Gamma_t^{\eta,-} = \sum_{n\geq 0}(\rho_\eta^-)^n\varphi^{\star n}(t)$. Applying Lemma 41 with $\psi = \rho_\eta^+\varphi$ and with $\psi = \rho_\eta^-\varphi$, we deduce that there are some constants $0 < c_\eta < C_\eta$ such that for all $t \geq 0$, $c_\eta e^{(\alpha_0-\eta)t} \leq 1 + \Gamma_t^{\eta,-} \leq 1 + \Gamma_t^{\eta,+} \leq C_\eta e^{(\alpha_0+\eta)t}$. But on $\Omega_N^2$, we know from Lemma 34 that $\rho_N \in [p(1 - N^{-3/8}/2), p(1 + N^{-3/8}/2)]$. Thus for $N$ large enough, we clearly have $\rho_N \in [\rho_\eta^-, \rho_\eta^+]$, so that $\Gamma_t^N \in [\Gamma_t^{\eta,-}, \Gamma_t^{\eta,+}]$. The conclusion follows. ☐

We next gather a number of consequences of the above estimate that we will use later.

**Lemma 43.** *Assume A. Recall that $\alpha_0$ was defined in Remark 5, that $\rho_N$ was defined in Lemma 34 and that $\Gamma_t^N = \sum_{n\geq 0}\rho_N^n\varphi^{\star n}(t)$. We also put $v_t^N = \mu N^{-1/2}\sum_{n\geq 0}||A_N^n\mathbf{1}_N||_2\int_0^t s\varphi^{\star n}(t-s)ds$. For any $\eta > 0$, we can find $N_\eta \geq 1$, $t_\eta > 0$ and $0 < c_\eta < C_\eta$ (depending only on $p, \mu, \varphi$ and $\eta$) such that for all $N \geq N_\eta$, on $\Omega_N^2$,*

*(i) for all $t \geq 0$, $v_t^N \leq C_\eta e^{(\alpha_0+\eta)t}$,*

*(ii) for all $t \geq t_\eta$, $v_t^N \geq c_\eta e^{(\alpha_0-\eta)t}$,*

*(iii) for all $t \geq 0$, $\sum_{n\geq 0}\rho_N^n(2N^{-3/8})^{\lfloor n/2\rfloor}\int_0^t \varphi^{\star n}(t-s)ds \leq C_\eta$,*

*(iv) for all $t \geq 0$, $\sum_{n\geq 0}\rho_N^n\int_0^t e^{(\alpha_0+\eta)s/2}\varphi^{\star n}(t-s)ds \leq C_\eta e^{(\alpha_0+\eta)t}$,*

*(v) for all $t \geq 0$, $\sum_{n\geq 0}\rho_N^n\int_0^t s\varphi^{\star n}(t-s)ds \leq C_\eta e^{(\alpha_0+\eta)t}$,*

*(vi) for all $t \geq 0$, $\int_0^t\int_0^t \Gamma^N_{t-r}\Gamma^N_{t-s} e^{(\alpha_0+\eta)(r\wedge s)}drds \leq C_\eta e^{2(\alpha_0+\eta)t}$.*

*Proof.* We fix $\eta > 0$ and work with $N$ large enough and on $\Omega^2_N$, so that we can use Lemmas 35 and 42.

We start with (i). We know from Lemma 35-(iii) that $||A^n_N \mathbf{1}_N||_2 \leq 2\sqrt{N}\rho^n_N$, whence $v^N_t \leq 2\mu\int_0^t s\Gamma^N_{t-s}ds \leq C_\eta \int_0^t s e^{(\alpha_0+\eta)(t-s)}ds = C_\eta e^{(\alpha_0+\eta)t}\int_0^t se^{-(\alpha_0+\eta)s}ds \leq C_\eta e^{(\alpha_0+\eta)t}$.

The LHS of point (iv) is nothing but $\int_0^t e^{(\alpha_0+\eta)s/2}\Gamma^N_{t-s}ds \leq C_\eta\int_0^t e^{(\alpha_0+\eta)s/2}e^{(\alpha_0+\eta)(t-s)}ds = C_\eta e^{(\alpha_0+\eta)t}\int_0^t e^{-(\alpha_0+\eta)s/2}ds \leq C_\eta e^{(\alpha_0+\eta)t}$.

Point (v) follows from point (iv).

The LHS of point (vi) is smaller than $C_\eta\int_0^t\int_0^t e^{(\alpha_0+\eta)(t-r)}e^{(\alpha_0+\eta)(t-s)}e^{(\alpha_0+\eta)(r\wedge s)}drds$, which equals $2C_\eta\int_0^t e^{(\alpha_0+\eta)(t-s)}\int_0^s e^{(\alpha_0+\eta)(t-r)}e^{(\alpha_0+\eta)r}drds = 2C_\eta e^{2(\alpha_0+\eta)t}\int_0^t se^{-(\alpha_0+\eta)s}ds \leq C_\eta e^{2(\alpha_0+\eta)t}$.

Setting $\Lambda = \int_0^\infty \varphi(t)dt$, the LHS of (iii) is bounded by $\sum_{n\geq 0}(\Lambda\rho_N)^n(2N^{-3/8})^{\lfloor n/2\rfloor}$ which is itself bounded by $\sum_{n\geq 0}(2\Lambda p)^n(2N^{-3/8})^{\lfloor n/2\rfloor}$ since $\rho_N \leq 2p$ on $\Omega^2_N$ by Lemma 34. This is uniformly bounded, as soon as $N$ is large enough so that $2\Lambda p(2N^{-3/8})^{1/2} \leq 1/2$.

We finally check (ii). We know from Lemma 35-(iii) that, on $\Omega^2_N$, $||A^n_N\mathbf{1}_N||_2 \geq \sqrt{N}\rho^n_N/2$, whence $v^N_t \geq (\mu/2)\int_0^t s\Gamma^N_{t-s}ds \geq (\mu/2)\int_1^2 s\Gamma^N_{t-s}ds \geq (\mu/2)\int_{t-2}^{t-1}\Gamma^N_s ds$ if $t\geq 2$. By Lemma 42, we thus have $v^N_t \geq (\mu/2)\int_{t-2}^{t-1}(c_\eta e^{(\alpha_0-\eta)s}-1)ds \geq (\mu/2)[c_\eta e^{(\alpha_0-\eta)(t-2)}-1]$. The conclusion easily follows: we can find $t_\eta \geq 2$ and $c_\eta > 0$ such that for all $t\geq t_\eta$, $v^N_t \geq c_\eta e^{(\alpha_0-\eta)t}$. $\square$

5.3. **Preliminary stochastic analysis.** We now prove a few estimates concerning the processes introduced in Notation 9. We recall that $\alpha_0$ was defined in Remark 5 and that $\rho_N$ and $V_N$ were defined in Lemma 34. We start from Lemma 11 to write (with as usual $\varphi^{\star 0}(t-s)ds = \delta_t(ds)$)

$$\mathbb{E}_\theta[\mathbf{Z}^N_t] = \mu\sum_{n\geq 0}\Big[\int_0^t s\varphi^{\star n}(t-s)ds\Big]A^n_N\mathbf{1}_N = v^N_t V_N + \mathbf{I}^N_t, \tag{12}$$

$$\mathbf{U}^N_t = \mathbf{Z}^N_t - \mathbb{E}_\theta[\mathbf{Z}^N_t] = \sum_{n\geq 0}\int_0^t \varphi^{\star n}(t-s)A^n_N\mathbf{M}^N_s ds = \mathbf{M}^N_t + \mathbf{J}^N_t, \tag{13}$$

where

$$v^N_t = \mu\sum_{n\geq 0}\frac{||A^n_N\mathbf{1}_N||_2}{\sqrt{N}}\int_0^t s\varphi^{\star n}(t-s)ds, \tag{14}$$

$$\mathbf{I}^N_t = \mu\sum_{n\geq 0}\Big[\int_0^t s\varphi^{\star n}(t-s)ds\Big]\Big[A^n_N\mathbf{1}_N - \frac{||A^n_N\mathbf{1}_N||_2}{\sqrt{N}}V_N\Big], \tag{15}$$

$$\mathbf{J}^N_t = \sum_{n\geq 1}\int_0^t \varphi^{\star n}(t-s)A^n_N\mathbf{M}^N_s ds. \tag{16}$$

As usual, we denote by $I^{i,N}_t$ and $J^{i,N}_t$ the coordinates of $\mathbf{I}^N_t$ and $\mathbf{J}^N_t$ and by $\bar{I}^N_t$ and $\bar{J}^N_t$ their empirical mean. We start with some upperbounds concerning $\mathbf{Z}^N_t$ and $\mathbf{U}^N_t$.

**Lemma 44.** *Assume A. For all $\eta > 0$, there are $N_\eta \geq 1$ and $C_\eta > 0$ such that for all $N \geq N_\eta$, all $t \geq 0$, on $\Omega^2_N$,* *(i)* $\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N})^2] \le C_\eta e^{2(\alpha_0+\eta)t}$,

*(ii)* $\max_{i=1,\dots,N} \mathbb{E}_\theta[(U_t^{i,N})^2] \le C_\eta (N^{-1} e^{2(\alpha_0+\eta)t} + e^{(\alpha_0+\eta)t})$,

*(iii)* $\mathbb{E}_\theta[(\bar{U}_t^N)^2] \le C_\eta N^{-1} e^{2(\alpha_0+\eta)t}$.

*Proof.* We fix $\eta > 0$ and work with $N$ large enough and on $\Omega_N^2$, so that we can use Lemmas 35 and 43.

*Step 1.* We first verify that $||\mathbb{E}_\theta[\mathbf{Z}_t^N]||_\infty \le C_\eta e^{(\alpha_0+\eta)t}$. Using (12) and that $||A_N^n \mathbf{1}_N||_\infty \le C\rho_N^n$ for all $n \ge 0$ by Lemma 35-(vii), we see that $||\mathbb{E}_\theta[\mathbf{Z}_t^N]||_\infty \le C \sum_{n\ge0} \rho_N^n \int_0^t s\varphi^{\star n}(t-s)ds$, whence the conclusion by Lemma 43-(v).

*Step 2.* We next show that for all $i = 1, \dots, N$, $\mathbb{E}_\theta[(J_t^{i,N})^2] \le C_\eta N^{-1} e^{2(\alpha_0+\eta)t}$. We start from (16), which gives us

$$\mathbb{E}_\theta[(J_t^{i,N})^2] = \sum_{m,n\ge1} \int_0^t \int_0^t \varphi^{\star m}(t-r)\varphi^{\star n}(t-s) \sum_{j,k=1}^N A_N^m(i,j) A_N^n(i,k) \mathbb{E}_\theta[M_r^{j,N} M_s^{k,N}] drds.$$

But we know from Remark 10 that $\mathbb{E}_\theta[M_r^{j,N} M_s^{k,N}] = \mathbf{1}_{\{j=k\}} \mathbb{E}_\theta[Z_{r\wedge s}^{j,N}] \le C_\eta \mathbf{1}_{\{j=k\}} e^{(\alpha_0+\eta)(r\wedge s)}$ by Step 1. Furthermore, $\sum_{j=1}^N A_N^m(i,j) A_N^n(i,j) \le ||A_N^n e_i||_2 ||A_N^m e_i||_2 \le C N^{-1} \rho_N^{m+n}$ by Lemma 35-(vii) (because $m, n \ge 1$). We thus find, recalling that $\Gamma_t^N = \sum_{n\ge0} \varphi^{\star n}(t)$, that

$$\mathbb{E}_\theta[(J_t^{i,N})^2] = C_\eta N^{-1} \int_0^t \int_0^t \Gamma_{t-r}^N \Gamma_{t-s}^N e^{(\alpha_0+\eta)(r\wedge s)} drds.$$

The conclusion follows from Lemma 43-(vi).

*Step 3.* Point (ii) follows from the facts that $U_t^{i,N} = M_t^{i,N} + J_t^{i,N}$, that $\mathbb{E}_\theta[(M_t^{i,N})^2] = \mathbb{E}_\theta[Z_t^{i,N}] \le C_\eta e^{(\alpha_0+\eta)t}$ by Remark 10 and Step 1 and that $\mathbb{E}_\theta[(J_t^{i,N})^2] \le C_\eta N^{-1} e^{2(\alpha_0+\eta)t}$ by Step 2.

*Step 4.* Since $Z_t^{i,N} = \mathbb{E}_\theta[Z_t^{i,N}] + U_t^{i,N}$, we deduce from Steps 1 and 3 that $\mathbb{E}_\theta[(Z_t^{i,N})^2] \le C_\eta(e^{2(\alpha_0+\eta)t} + e^{(\alpha_0+\eta)t} + N^{-1} e^{2(\alpha_0+\eta)t}) \le C_\eta e^{2(\alpha_0+\eta)t}$, whence point (i).

*Step 5.* Finally, we write $\bar{U}_t^N = \bar{M}_t^N + \bar{J}_t^N$. It is clear from Step 2 that $\mathbb{E}_\theta[(\bar{J}_t^N)^2] \le C_\eta N^{-1} e^{2(\alpha_0+\eta)t}$. Remark 10 implies that $\mathbb{E}_\theta[(\bar{M}_t^N)^2] = N^{-2} \sum_{i=1}^N \mathbb{E}_\theta[Z_t^{i,N}] \le C_\eta N^{-1} e^{(\alpha_0+\eta)t}$ by Step 1. Point (iii) is checked. ☐

We next show that the term $\mathbf{I}_t^N$ is very small in the present scales.

**Lemma 45.** *Assume A. For all $\eta > 0$, there are $N_\eta \ge 1$ and $C_\eta > 0$ such that for all $N \ge N_\eta$, all $t \ge 0$, on $\Omega_N^2$, $||\mathbf{I}_t^N||_2 \le C_\eta N^{1/8} t$.*

*Proof.* We fix $\eta > 0$ and work with $N$ large enough and on $\Omega_N^2$, so that we can use Lemmas 35 and 43. Using the Minkowski inequality and then Lemma 35-(iii)-(v), we find

$$\begin{aligned} ||\mathbf{I}_t^N||_2 \le& \mu \sum_{n\ge0} \Big[\int_0^t s\varphi^{\star n}(t-s)ds\Big] \Big|\Big| A_N^n \mathbf{1}_N - \frac{||A_N^n \mathbf{1}_N||_2}{\sqrt{N}} V_N \Big|\Big|_2 \\ \le& 6\mu t \sum_{n\ge0} \Big[\int_0^t \varphi^{\star n}(t-s)ds\Big] N^{1/2} \rho_N^n (2N^{-3/8})^{\lfloor n/2\rfloor+1} \\ \le& 12\mu t N^{1/8} \sum_{n\ge0} \rho_N^n (2N^{-3/8})^{\lfloor n/2\rfloor} \int_0^t \varphi^{\star n}(t-s)ds. \end{aligned}$$

The conclusion follows from Lemma 43-(iii). □

We now study the empirical variance of $\mathbf{J}_t^N$.

**Lemma 46.** *Assume A. For all $\eta > 0$, there are $N_\eta \geq 1$ and $C_\eta > 0$ such that for all $N \geq N_\eta$, all $t \geq 0$, on $\Omega_N^2$, $\mathbb{E}_\theta[||\mathbf{J}_t^N - \bar{J}_t^N \mathbf{1}_N||_2^2] \leq C_\eta[e^{(\alpha_0+\eta)t} + N^{-1}||V_N - \bar{V}_N\mathbf{1}_N||_2^2 e^{2(\alpha_0+\eta)t}]$.*

*Proof.* As usual, we fix $\eta > 0$ and work with $N$ large enough and on $\Omega_N^2$, so that we can use Lemmas 35 and 43. Starting from (16) and using the Minkowski inequality, we find

$$\mathbb{E}_\theta[||\mathbf{J}_t^N - \bar{J}_t^N \mathbf{1}_N||_2^2]^{1/2} \leq \sum_{n\geq 1}\int_0^t \varphi^{\star n}(t-s)\mathbb{E}_\theta[||A_N^n M_s^N - \overline{A_N^n M_s^N}\mathbf{1}_N||_2^2]^{1/2}ds.$$

But using Remark 10 and then Lemma 44-(i), we see that

$$\begin{aligned}\mathbb{E}_\theta[||A_N^n M_s^N - \overline{A_N^n M_s^N}\mathbf{1}_N||_2^2] =&\sum_{i=1}^N \mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N A_N^n(i,j)M_s^{j,N} - \frac{1}{N}\sum_{j,k=1}^N A_N^n(k,j)M_s^{j,N}\Big)^2\Big]\\ =&\sum_{i,j=1}^N \Big(A_N^n(i,j) - \frac{1}{N}\sum_{k=1}^N A_N^n(k,j)\Big)^2\mathbb{E}_\theta[Z_s^{j,N}]\\ \leq& C_\eta e^{(\alpha_0+\eta)s}\sum_{j=1}^N ||A_N^n e_j - \overline{A_N^n e_j}\mathbf{1}_N||_2^2.\end{aligned}$$

Using next that, for all $x, y \in \mathbb{R}^N$, $\big|||x-\bar{x}\mathbf{1}_N||_2 - ||y - \bar{y}\mathbf{1}_N||_2\big| \leq ||x-y||_2$ (with the notation $\bar{x} = N^{-1}\sum_{i=1}^N x_i$ and $\bar{y} = N^{-1}\sum_{i=1}^N y_i$), we write

$$\begin{aligned}||A_N^n e_j - \overline{A_N^n e_j}\mathbf{1}_N||_2 \leq& \Big|\Big|A_N^n e_j - \frac{||A_N^n e_j||_2}{\sqrt{N}}V_N\Big|\Big|_2 + \frac{||A_N^n e_j||_2}{\sqrt{N}}||V_N - \bar{V}_N\mathbf{1}_N||_2\\ \leq& ||A_N^n e_j||_2\Big(12(2N^{-3/8})^{\lfloor n/2\rfloor} + \frac{||V_N - \bar{V}_N\mathbf{1}_N||_2}{\sqrt{N}}\Big)\end{aligned}$$

by Lemma 35-(vi). Since $||A_N^n e_j||_2 \leq C\rho_N^n/\sqrt{N}$ by Lemma 35-(vii) (because $n \geq 1$), we conclude that

$$\mathbb{E}_\theta[||A_N^n M_s^N - \overline{A_N^n M_s^N}\mathbf{1}_N||_2^2]^{1/2} \leq C_\eta e^{(\alpha_0+\eta)s/2}\rho_N^n\Big((2N^{-3/8})^{\lfloor n/2\rfloor} + \frac{||V_N - \bar{V}_N\mathbf{1}_N||_2}{\sqrt{N}}\Big).$$

Consequently,

$$\begin{aligned}\mathbb{E}_\theta[||\mathbf{J}_t^N - \bar{J}_t^N\mathbf{1}_N||_2^2]^{1/2} \leq& C_\eta\sum_{n\geq 1}\rho_N^n\Big((2N^{-3/8})^{\lfloor n/2\rfloor} + \frac{||V_N - \bar{V}_N\mathbf{1}_N||_2}{\sqrt{N}}\Big)\int_0^t \varphi^{\star n}(t-s)e^{(\alpha_0+\eta)s/2}ds\\ \leq& C_\eta e^{(\alpha_0+\eta)t/2}\sum_{n\geq 1}\rho_N^n(2N^{-3/8})^{\lfloor n/2\rfloor}\int_0^t\varphi^{\star n}(t-s)ds\\ &+ C_\eta\frac{||V_N - \bar{V}_N\mathbf{1}_N||_2}{\sqrt{N}}\sum_{n\geq 1}\rho_N^n\int_0^t\varphi^{\star n}(t-s)e^{(\alpha_0+\eta)s/2}ds\\ \leq& C_\eta e^{(\alpha_0+\eta)t/2} + C_\eta\frac{||V_N - \bar{V}_N\mathbf{1}_N||_2}{\sqrt{N}}e^{(\alpha_0+\eta)t}\end{aligned}$$

by Lemma 43-(iii)-(iv). This completes the proof. □

The last lemma of the subsection concerns the martingale $\mathbf{M}_t^N$. In point (ii) below, $(\cdot,\cdot)$ stands for the usual scalar product in $\mathbb{R}^N$.

**Lemma 47.** *Assume A. For all $\eta>0$, there are $N_\eta\geq 1$ and $C_\eta>0$ such that for all $N\geq N_\eta$, all $t\geq 0$, on $\Omega_N^2$,*

*(i)* $\mathbb{E}_\theta[||\mathbf{M}_t^N-\bar M_t^N\mathbf{1}_N||_2^2]\leq C_\eta N e^{(\alpha_0+\eta)t}$,

*(ii)* $\mathbb{E}_\theta[(\mathbf{M}_t^N-\bar M_t^N\mathbf{1}_N,V_N-\bar V_N\mathbf{1}_N)^2]\leq C_\eta||V_N-\bar V_N\mathbf{1}_N||_2^2e^{(\alpha_0+\eta)t}$,

*(iii) setting* $X_t^N=||\mathbf{M}_t^N-\bar M_t^N\mathbf{1}_N||_2^2-N\bar Z_t^N$, *we have* $\mathbb{E}_\theta[|X_t^N|]\leq C_\eta\sqrt{N}e^{(\alpha_0+\eta)t}$.

*Proof.* We fix $\eta>0$ and work with $N$ large enough and on $\Omega_N^2$, so that we can use Lemmas 35 and 43.

To check point (ii), we write $\mathbb{E}_\theta[(\mathbf{M}_t^N-\bar M_t^N\mathbf{1}_N,V_N-\bar V_N\mathbf{1}_N)^2]=\mathbb{E}_\theta[(\mathbf{M}_t^N,V_N-\bar V_N\mathbf{1}_N)^2]=\mathbb{E}_\theta[(\sum_{i=1}^N(V_N(i)-\bar V_N)M_t^{i,N})^2]$. By Remark 10, this equals $\sum_{i=1}^N(V_N(i)-\bar V_N)^2\mathbb{E}_\theta[Z_t^{i,N}]$, which is controled by $C_\eta||V_N-\bar V_N\mathbf{1}_N||_2^2e^{(\alpha_0+\eta)t}$ by Lemma 44-(i).

For point (iii), we first observe that $X_t^N=Y_t^N-N(\bar M_t^N)^2$, where $Y_t^N=||\mathbf{M}_t^N||_2^2-N\bar Z_t^N$. Using as usual Remark 10, we deduce that $\mathbb{E}_\theta[N(\bar M_t^N)^2]=N^{-1}\sum_{i=1}^N\mathbb{E}_\theta[Z_t^{i,N}]\leq C_\eta e^{(\alpha_0+\eta)t}$ by Lemma 44-(i). Next, we see that $||\mathbf{M}_t^N||_2^2=\sum_{i=1}^N(M_t^{i,N})^2=2\sum_{i=1}^N\int_0^tM_{s-}^{i,N}dM_s^{i,N}+\sum_{i=1}^NZ_t^{i,N}$, since for each $i$, $[M^{i,N},M^{i,N}]_t=Z_t^{i,N}$, see Remark 10. Thus $Y_t^N=2\sum_{i=1}^N\int_0^tM_{s-}^{i,N}dM_s^{i,N}$. The martingales $\int_0^tM_{s-}^{i,N}dM_s^{i,N}$ are orthogonal and $[\int_0^\cdot M_{s-}^{i,N}dM_s^{i,N},\int_0^\cdot M_{s-}^{i,N}dM_s^{i,N}]_t=\int_0^t(M_{s-}^{i,N})^2dZ_s^{i,N}\leq Z_t^{i,N}\sup_{[0,t]}(M_s^{i,N})^2$. As a conclusion,

$$\mathbb{E}_\theta[(Y_t^N)^2]=4\sum_{i=1}^N\mathbb{E}_\theta\Big[\int_0^t(M_{s-}^{i,N})^2dZ_s^{i,N}\Big]\leq 4\sum_{i=1}^N\mathbb{E}_\theta\Big[(Z_t^{i,N})^2\Big]^{1/2}\mathbb{E}_\theta\Big[\sup_{[0,t]}(M_s^{i,N})^4\Big]^{1/2}.$$

Using again that $[M^{i,N},M^{i,N}]_t=Z_t^{i,N}$ and the Doob inequality, we see that $\mathbb{E}_\theta[\sup_{[0,t]}(M_s^{i,N})^4]\leq C\mathbb{E}_\theta[(Z_t^{i,N})^2]$. This shows that $\mathbb{E}_\theta[(Y_t^N)^2]\leq C\sum_{i=1}^N\mathbb{E}_\theta[(Z_t^{i,N})^2]\leq C_\eta Ne^{2(\alpha_0+\eta)t}$ by Lemma 44-(i). Hence $\mathbb{E}_\theta[|Y_t^N|]\leq C_\eta\sqrt{N}e^{(\alpha_0+\eta)t}$ and $\mathbb{E}_\theta[|X_t^N|]\leq\mathbb{E}_\theta[|Y_t^N|]+\mathbb{E}[N(\bar M_t^N)^2]\leq C_\eta\sqrt{N}e^{(\alpha_0+\eta)t}$.

Finally, (i) follows from (iii), since $\mathbb{E}_\theta[||\mathbf{M}_t^N-\bar M_t^N\mathbf{1}_N||_2^2]\leq\mathbb{E}_\theta[|X_t^N|]+N\mathbb{E}_\theta[\bar Z_t^N]$ and since $N\mathbb{E}_\theta[\bar Z_t^N]\leq C_\eta Ne^{(\alpha_0+\eta)t}$ by Lemma 44-(i) again. ☐

5.4. **Conclusion.** We now conclude the proof of Theorem 6. We recall that

$$\mathcal{U}_t^N=\Big[\sum_{i=1}^N\Big(\frac{Z_t^{i,N}-\bar Z_t^N}{\bar Z_t^N}\Big)^2-\frac{N}{\bar Z_t^N}\Big]\mathbf{1}_{\{\bar Z_t^N>0\}}=\Big[\frac{||\mathbf{Z}_t^N-\bar Z_t^N\mathbf{1}_N||_2^2-N\bar Z_t^N}{(\bar Z_t^N)^2}\Big]\mathbf{1}_{\{\bar Z_t^N>0\}},$$

that $V_N$ was introduced in Lemma 34 and that

$$\mathcal{U}_\infty^N=\sum_{i=1}^N\Big(\frac{V_N(i)-\bar V_N}{\bar V_N}\Big)^2=\frac{||V_N-\bar V_N\mathbf{1}_N||_2^2}{(\bar V_N)^2}.$$

We first proceed to a suitable decomposition of the error.

**Remark 48.** *Assume that $p\in(0,1]$. We introduce $\mathcal{D}_t^N=|\mathcal{U}_t^N-(1/p-1)|$ and recall that $v_t^N$ was defined in* (14). *There is $N_0$ (depending only on $p$) such that for all $N\geq N_0$, on the event $\Omega_N^2\cap\{\bar Z_t^N\geq v_t^N/4>0\}$,*

$$\mathcal{D}_t^N\leq 16\mathcal{D}_t^{N,1}+128||V_N-\bar V_N\mathbf{1}_N||_2^2\mathcal{D}_t^{N,2}+|\mathcal{U}_\infty^N-(1/p-1)|,$$

*where*

$$\mathcal{D}_t^{N,1}=\frac{1}{(v_t^N)^2}\Big|||\mathbf{Z}_t^N-\bar Z_t^N\mathbf{1}_N||_2^2-N\bar Z_t^N-(v_t^N)^2||V_N-\bar V_N\mathbf{1}_N||_2^2\Big| \quad \text{and} \quad \mathcal{D}_t^{N,2}=\Big|\frac{\bar Z_t^N}{v_t^N}-\bar V_N\Big|.$$

*Proof.* We work with $N$ sufficiently large so that we can apply Lemma 35. We obviously have $\mathcal{D}_t^N\le|\mathcal{U}_t^N-\mathcal{U}_\infty^N|+|\mathcal{U}_\infty^N-(1/p-1)|$. We next write, on the event $\Omega_N^2\cap\{\bar Z_t^N\ge v_t^N/4>0\}$,

$$\begin{aligned}|\mathcal{U}_t^N-\mathcal{U}_\infty^N|\le&\frac{1}{(\bar Z_t^N)^2}\Big|||\mathbf{Z}_t^N-\bar Z_t^N\mathbf{1}_N||_2^2-N\bar Z_t^N-(v_t^N)^2||V_N-\bar V_N\mathbf{1}_N||_2^2\Big|\\&+||V_N-\bar V_N\mathbf{1}_N||_2^2\Big|\Big(\frac{v_t^N}{\bar Z_t^N}\Big)^2-\frac{1}{(\bar V_N)^2}\Big|\\\le&16\mathcal{D}_t^{N,1}+128||V_N-\bar V_N\mathbf{1}_N||_2^2\mathcal{D}_t^{N,2}.\end{aligned}$$

We used that on the present event, $(\bar Z_t^N)^{-2}\le16(v_t^N)^{-2}$, that $\bar V_N\ge1/2$ (see Lemma 35-(ii)), that $(\bar Z_t^N/v_t^N)\ge1/4$ and that, for all $x,y\ge1/4$, $|x^{-2}-y^{-2}|\le128|x-y|$. □

We now treat the term $\mathcal{D}_t^{N,2}$.

**Lemma 49.** *Assume A. For all $\eta>0$, there are $N_\eta\ge1$, $t_\eta\ge0$ and $C_\eta>0$ such that, for all $N\ge N_\eta$, all $t\ge t_\eta$, on $\Omega_N^2$,*

*(i)* $\mathbb{E}_\theta[\mathcal{D}_t^{N,2}]\le C_\eta e^{2\eta t}(N^{-1/2}+e^{-\alpha_0t})$,

*(ii)* $\Pr_\theta(\bar Z_t^N\le v_t^N/4)\le C_\eta e^{2\eta t}(N^{-1/2}+e^{-\alpha_0t})$.

*Proof.* As usual, we fix $\eta>0$ and consider $N\ge N_\eta$ and $t\ge t_\eta$ and we work on $\Omega_N^2$ so that we can apply Lemmas 35 and 43.

Recalling (12)-(13), we write $\mathbf{Z}_t^N=\mathbb{E}_\theta[\mathbf{Z}_t^N]+\mathbf{U}_t^N=v_t^NV_N+\mathbf{I}_t^N+\mathbf{U}_t^N$, whence $\mathcal{D}_t^{N,2}\le(v_t^N)^{-1}(|\bar I_t^N|+|\bar U_t^N|)$. But we infer from Lemma 45 that $|\bar I_t^N|\le N^{-1/2}||\mathbf{I}_t^N||_2\le C_\eta tN^{1/8-1/2}$, which is obviously bounded by $C_\eta e^{\eta t}$. Next, we know from Lemma 44-(iii) that $\mathbb{E}_\theta[|\bar U_t^N|]\le C_\eta N^{-1/2}e^{(\alpha_0+\eta)t}$. We deduce that $\mathbb{E}_\theta[\mathcal{D}_t^{N,2}]\le C_\eta(v_t^N)^{-1}e^{\eta t}[1+N^{-1/2}e^{\alpha_0t}]$. But since $t\ge t_\eta$, we know from Lemma 43-(ii) that $v_t^N\ge c_\eta e^{(\alpha_0-\eta)t}$. This completes the proof of (i).

By Lemma 35-(ii), $\bar V_N\ge1/2$. Thus $\bar Z_t^N\le v_t^N/4$ implies that $\mathcal{D}_t^{N,2}=|\bar Z_t^N/v_t^N-\bar V_N|\ge1/4$. Hence $\Pr_\theta(\bar Z_t^N\le v_t^N/4)\le4\mathbb{E}_\theta[\mathcal{D}_t^{N,2}]$ and (ii) follows from (i). □

**Lemma 50.** *Assume A. For all $\eta>0$, there are $N_\eta\ge1$, $t_\eta\ge0$ and $C_\eta>0$ such that, for all $N\ge N_\eta$, all $t\ge t_\eta$, on $\Omega_N^2$,*

$$\mathbb{E}_\theta[\mathcal{D}_t^{N,1}]\le C_\eta(1+||V_N-\bar V_N\mathbf{1}_N||_2^2)e^{4\eta t}\Big(\frac{1}{\sqrt N}+\frac{\sqrt N}{e^{\alpha_0t}}+\Big(\frac{\sqrt N}{e^{\alpha_0t}}\Big)^{3/2}\Big).$$

*Proof.* We fix $\eta>0$ and consider $N\ge N_\eta$ and $t\ge t_\eta$ and we work on $\Omega_N^2$ so that we can apply Lemmas 35 and 43. Recalling (12)-(13), we write $\mathbf{Z}_t^N=v_t^NV_N+\mathbf{I}_t^N+M_t^N+\mathbf{J}_t^N$ and

$$\begin{aligned}\mathcal{D}_t^{N,1}=\frac{1}{(v_t^N)^2}\Big|&||\mathbf{I}_t^N-\bar I_t^N\mathbf{1}_N+\mathbf{J}_t^N-\bar J_t^N\mathbf{1}_N||_2^2+||\mathbf{M}_t^N-\bar M_t^N\mathbf{1}_N||_2^2-N\bar Z_t^N\\&+2(\mathbf{I}_t^N-\bar I_t^N\mathbf{1}_N+\mathbf{J}_t^N-\bar J_t^N\mathbf{1}_N,v_t^N(V_N-\bar V_N\mathbf{1}_N)+\mathbf{M}_t^N-\bar M_t^N\mathbf{1}_N)\\&+2v_t^N(V_N-\bar V_N\mathbf{1}_N,\mathbf{M}_t^N-\bar M_t^N\mathbf{1}_N)\Big|.\end{aligned}$$

Recalling that $X_t^N = ||\mathbf{M}_t^N - \bar M_t^N \mathbf{1}_N||_2^2 - N\bar Z_t^N$, see Lemma 47, we deduce that

$$\mathcal{D}_t^{N,1} \le \frac{1}{(v_t^N)^2}\Big|2||\mathbf{I}_t^N - \bar I_t^N \mathbf{1}_N||_2^2 + 2||\mathbf{J}_t^N - \bar J_t^N \mathbf{1}_N||_2^2 + |X_t^N| + 2(||\mathbf{I}_t^N - \bar I_t^N \mathbf{1}_N||_2 + ||\mathbf{J}_t^N - \bar J_t^N \mathbf{1}_N||_2)(v_t^N||V_N - \bar V_N \mathbf{1}_N||_2 + ||\mathbf{M}_t^N - \bar M_t^N \mathbf{1}_N||_2) + 2v_t^N|(V_N - \bar V_N \mathbf{1}_N, \mathbf{M}_t^N - \bar M_t^N \mathbf{1}_N)|\Big|.$$

We know from Lemma 43-(i)-(ii) that $v_t^N \ge c_\eta e^{(\alpha_0-\eta)t}$ and $v_t^N \le C_\eta e^{(\alpha_0+\eta)t}$, from Lemma 45 that $||\mathbf{I}_t^N - \bar I_t^N \mathbf{1}_N||_2 \le ||\mathbf{I}_t^N||_2 \le C_\eta N^{1/8} t \le C_\eta N^{1/8} e^{\eta t}$ and from Lemma 46 that $\mathbb{E}_\theta[||\mathbf{J}_t^N - \bar J_t^N \mathbf{1}_N||_2^2] \le C_\eta[e^{(\alpha_0+\eta)t} + N^{-1}||V_N - \bar V_N \mathbf{1}_N||_2^2 e^{2(\alpha_0+\eta)t}]$. And Lemma 47 tells us that $\mathbb{E}_\theta[|X_t^N|] \le C_\eta \sqrt{N} e^{(\alpha_0+\eta)t}$, $\mathbb{E}_\theta[||\mathbf{M}_t^N - \bar M_t^N \mathbf{1}_N||_2^2] \le C_\eta N e^{(\alpha_0+\eta)t}$ and $\mathbb{E}_\theta[|(\mathbf{M}_t^N - \bar M_t^N \mathbf{1}_N, V_N - \bar V_N \mathbf{1}_N)|] \le C_\eta ||V_N - \bar V_N \mathbf{1}_N||_2 e^{(\alpha_0+\eta)t/2}$. Using the Cauchy-Schwarz inequality, we find

$$\mathbb{E}_\theta[\mathcal{D}_t^{N,1}] \le C_\eta(1+||V_N - \bar V_N \mathbf{1}_N||_2^2)e^{-2(\alpha_0-\eta)t}\Big(N^{1/4}e^{2\eta t} + [e^{(\alpha_0+\eta)t} + N^{-1}e^{2(\alpha_0+\eta)t}] + N^{1/2}e^{(\alpha_0+\eta)t} + [N^{1/8}e^{\eta t} + e^{(\alpha_0+\eta)t/2} + N^{-1/2}e^{(\alpha_0+\eta)t}][e^{(\alpha_0+\eta)t} + N^{1/2}e^{(\alpha_0+\eta)t/2}] + e^{3(\alpha_0+\eta)t/2}\Big).$$

We easily deduce that

$$\mathbb{E}_\theta[\mathcal{D}_t^{N,1}] \le C_\eta(1+||V_N - \bar V_N \mathbf{1}_N||_2^2)e^{4\eta t}\Big(N^{-1/2} + e^{-\alpha_0 t/2} + N^{1/2}e^{-\alpha_0 t} + N^{5/8}e^{-3\alpha_0 t/2}\Big).$$

To conclude, it suffices to notice that $e^{-\alpha_0 t/2} \le N^{-1/2} + N^{1/2}e^{-\alpha_0 t}$ and that $N^{5/8}e^{-3\alpha_0 t/2} \le N^{3/4}e^{-3\alpha_0 t/2} = (N^{1/2}e^{-\alpha_0 t})^{3/2}$. □

We now have all the weapons to give the

*Proof of Theorem 6.* We assume $A$ and fix $\eta > 0$.

*Step 1.* Starting from Remark 48 and using Lemmas 49 and 50, we deduce that there is $N_\eta \ge 1$, $t_\eta \ge 0$ and $C_\eta > 0$ such that for all $N \ge N_\eta$, all $t \ge t_\eta$,

$$\mathbf{1}_{\Omega_N^2}\mathbb{E}_\theta\Big[\mathbf{1}_{\{\bar Z_t^N \ge v_t^N/4 > 0\}}\Big|\mathcal{U}_t^N - \Big(\frac{1}{p}-1\Big)\Big|\Big] \le \mathbf{1}_{\Omega_N^2}\Big|\mathcal{U}_\infty^N - \Big(\frac{1}{p}-1\Big)\Big| + C_\eta \mathbf{1}_{\Omega_N^2}(1+||V_N - \bar V_N \mathbf{1}_N||_2^2)e^{4\eta t}\Big(\frac{1}{\sqrt N} + \frac{\sqrt N}{e^{\alpha_0 t}} + \Big(\frac{\sqrt N}{e^{\alpha_0 t}}\Big)^{3/2}\Big),$$

which implies, by Proposition 40, that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^2 \cap \{\bar Z_t^N \ge v_t^N/4 > 0\}}\Big|\mathcal{U}_t^N - \Big(\frac{1}{p}-1\Big)\Big|\Big] \le C_\eta e^{4\eta t}\Big(\frac{1}{\sqrt N} + \frac{\sqrt N}{e^{\alpha_0 t}} + \Big(\frac{\sqrt N}{e^{\alpha_0 t}}\Big)^{3/2}\Big)$$

and thus, for all $\varepsilon \in (0,1)$,

$$\Pr\Big(\Omega_N^2, \bar Z_t^N \ge v_t^N/4 > 0, \Big|\mathcal{U}_t^N - \Big(\frac{1}{p}-1\Big)\Big| \ge \varepsilon\Big) \le \frac{C_\eta}{\varepsilon}e^{4\eta t}\Big(\frac{1}{\sqrt N} + \frac{\sqrt N}{e^{\alpha_0 t}} + \Big(\frac{\sqrt N}{e^{\alpha_0 t}}\Big)^{3/2}\Big) \le \frac{C_\eta}{\varepsilon}e^{4\eta t}\Big(\frac{1}{\sqrt N} + \frac{\sqrt N}{e^{\alpha_0 t}}\Big).$$

For the last inequality, we used that either $N^{1/2}e^{-\alpha_0 t} \ge 1$ and then the inequality is trivial or $N^{1/2}e^{-\alpha_0 t} \le 1$ and then $(N^{1/2}e^{-\alpha_0 t})^{3/2} \le N^{1/2}e^{-\alpha_0 t}$. But we know from Lemma (43)-(ii) that

$v_t^N > 0$ on $\Omega_N^2$ (because $t \geq t_\eta$) and from Lemmas 33 and 49-(ii) that

$$\Pr((\Omega_N^2)^c \text{ or } \bar{Z}_t^N < v_t^N/4) \leq Ce^{-cN^{1/4}} + C_\eta e^{2\eta t}\Big(\frac{1}{\sqrt{N}} + e^{-\alpha_0 t}\Big) \leq C_\eta e^{2\eta t}\Big(\frac{1}{\sqrt{N}} + \frac{\sqrt{N}}{e^{\alpha_0 t}}\Big),$$

whence finally,

$$\Pr\left(\Big|\mathcal{U}_t^N - \Big(\frac{1}{p} - 1\Big)\Big| \geq \varepsilon\right) \leq \frac{C_\eta}{\varepsilon} e^{4\eta t}\Big(\frac{1}{\sqrt{N}} + \frac{\sqrt{N}}{e^{\alpha_0 t}}\Big).$$

We have proved this inequality only for $N \geq N_\eta$ and $t \geq t_\eta$, but it obviously extends, enlarging $C_\eta$ is necessary, to any values of $N \geq 1$ and $t \geq 0$.

*Step 2.* We next recall that $\mathcal{P}_t^N = \Phi(\mathcal{U}_t^N)$, where $\Phi(u) = (1+u)^{-1}\mathbf{1}_{\{u \geq 0\}}$ and observe that $p = \Phi(1/p - 1)$. The function $\Phi$ is Lipschitz continuous on $[0, \infty)$ with Lipschitz constant 1. Thus for any $\varepsilon \in (0,1)$, $|\mathcal{P}_t^N - p| > \varepsilon$ implies that either $|\mathcal{U}_t^N - (1/p-1)| > \varepsilon$ or $\mathcal{U}_t^N < 0$, so that in any case, $|\mathcal{U}_t^N - (1/p-1)| > \min\{\varepsilon, (1/p-1)\}$. We thus conclude from Step 1 that for any $N \geq 1$, any $t \geq 0$, any $\varepsilon \in (0,1)$,

$$\Pr(|\mathcal{P}_t^N - p| \geq \varepsilon) \leq \frac{C_\eta}{\min\{\varepsilon, (1/p-1)\}} e^{4\eta t}\Big(\frac{1}{\sqrt{N}} + \frac{\sqrt{N}}{e^{\alpha_0 t}}\Big) \leq \frac{C_\eta}{\varepsilon} e^{4\eta t}\Big(\frac{1}{\sqrt{N}} + \frac{\sqrt{N}}{e^{\alpha_0 t}}\Big).$$

The proof is complete. □

Finally, we handle the

*Proof of Remark 5.* We assume $A$ and fix $\eta > 0$. We know from Lemma 49-(i) that for all $N \geq N_\eta$, $t \geq t_\eta$, $\mathbf{1}_{\Omega_N^2}\mathbb{E}_\theta[|(\bar{Z}_t^N/v_t^N) - \bar{V}_N|] \leq C_\eta e^{2\eta t}(N^{-1/2} + e^{-\alpha_0 t})$, from Lemma 35-(ii) that $\bar{V}_N \in [1/2, 2]$ (on $\Omega_N^2$) and from Lemma 33 that $\Pr(\Omega_N^2) \geq 1 - Ce^{-cN^{1/4}}$. We also know from Lemma 43-(i)-(ii) that on $\Omega_N^2$, there are $0 < a_\eta < b_\eta$ such that $v_t^N \in [a_\eta e^{(\alpha_0 - \eta)t}, b_\eta e^{(\alpha_0+\eta)t}]$. We easily deduce that, still for $N \geq N_\eta$ and $t \geq t_\eta$,

$$\Pr(\bar{Z}_t^N \notin [(a_\eta/2)e^{(\alpha_0-\eta)t}, 2b_\eta e^{(\alpha_0+\eta)t}]) \leq Ce^{-cN^{1/4}} + C_\eta e^{2\eta t}(N^{-1/2} + e^{-\alpha_0 t}).$$

We conclude that for any $\eta \in (0, \alpha_0/2)$, $\lim_{t\to\infty}\lim_{N\to\infty}\Pr(\bar{Z}_t^N \in [(a_\eta/2)e^{(\alpha_0-\eta)t}, 2b_\eta e^{(\alpha_0+\eta)t}]) = 1$. This of course implies that for any $\eta > 0$, $\lim_{t\to\infty}\lim_{N\to\infty}\Pr(\bar{Z}_t^N \in [e^{(\alpha_0-\eta)t}, e^{(\alpha_0+\eta)t}]) = 1$. □

## 6. Detecting subcriticality and supercriticality

*Proof of Proposition 7.* We first assume $H(1)$. We then know from Lemma 16-(i) with $r = 1$ that, on $\Omega_N^1$, $\mathbb{E}_\theta[\bar{Z}_t^N] \leq Ct < e^{(\log t)^2}/2$ for all $t$ large enough, say for all $t \geq t_0$. We also know from Lemma 18 that, still on $\Omega_N^1$, $\mathbb{E}_\theta[|\bar{Z}_t^N - \mathbb{E}_\theta[\bar{Z}_t^N]|] = \mathbb{E}_\theta[|\bar{U}_t^N|] \leq C(t/N)^{1/2}$ and from Lemma 13 that $\Pr[(\Omega_N^1)^c] \leq Ce^{-cN}$. We easily deduce that

$$\begin{aligned}\Pr(\log(\bar{Z}_t^N) \geq (\log t)^2) \leq& \Pr((\Omega_N^1)^c) + \Pr(\Omega_N^1, \mathbb{E}_\theta[\bar{Z}_t^N] \geq e^{(\log t)^2}/2 \text{ or } |\bar{Z}_t^N - \mathbb{E}_\theta[\bar{Z}_t^N]| \geq e^{(\log t)^2}/2)\\ \leq& Ce^{-cN} + C(t/N)^{1/2}e^{-(\log t)^2}\end{aligned}$$

for all $t \geq t_0$. Enlarging $C$ if necessary, we conclude that for all $t \geq 1$, $\Pr(\log(\bar{Z}_t^N) \geq (\log t)^2) \leq Ce^{-cN} + Ct^{1/2}e^{-(\log t)^2}$.

We next assume $A$ and we fix $\eta \in (0, \alpha_0)$. We know from Lemma 49-(ii) that for all $N \geq N_\eta$ and $t \geq t_\eta$, on $\Omega_N^2$, $\Pr_\theta(\bar{Z}_t^N \leq v_t^N/4) \leq C_\eta e^{2\eta t}(N^{-1/2} + e^{-\alpha_0 t})$, from Lemma 43-(i)-(ii) that, still on $\Omega_N^2$, $v_t^N \ge c_\eta e^{(\alpha_0-\eta)t} \ge 4e^{(\log t)^2}$ (enlarging the value of $t_\eta$ if necessary). Finally, Lemma 33 tells us that $\Pr((\Omega_N^2)^c) \le Ce^{-cN^{1/4}}$. We thus see that

$$\begin{aligned}\Pr(\log(\bar Z_t^N) \le (\log t)^2) \le& \Pr((\Omega_N^2)^c) + \Pr(\Omega_N^2, \bar Z_t^N \le v_t^N/4)\\ \le& Ce^{-cN^{1/4}} + C_\eta e^{2\eta t}(N^{-1/2} + e^{-\alpha_0 t})\\ \le& C_\eta e^{2\eta t}(N^{-1/2}+e^{-\alpha_0 t}).\end{aligned}$$

All this shows that for all $\eta \in (0, \alpha_0)$, we can find $C_\eta$ and $t\eta$ such that for all $t \ge t_\eta$ and all $N \ge N_\eta$, $\Pr(\log(\bar Z_t^N) \le (\log t)^2) \le C_\eta e^{2\eta t}(N^{-1/2} + e^{-\alpha_0 t})$. We easily conclude that for all $\eta > 0$, there is $C_\eta$ such that for all $N \ge 1$ and all $t \ge 1$, $\Pr(\log(\bar Z_t^N) \le (\log t)^2) \le C_\eta e^{2\eta t}(N^{-1/2} + e^{-\alpha_0 t})$ as desired. □

## 7. Numerics

We say that we are in the *independent case* when the family $(\theta_{ij})_{1\le i,j\le N}$ is i.i.d. and Bernoulli($p$)-distributed, as in the whole paper. We say we are in the *symmetric case* when the family $(\theta_{ij})_{1\le i\le j\le N}$ is i.i.d. and Bernoulli($p$)-distributed and when $\theta_{ji} = \theta_{ij}$ for all $1 \le i < j \le N$. We will see that this does not change much the numerical results (with the very same estimators). Also, we assume that we observe only $(Z_s^{i,N})_{s\in[0,T],i=1,\dots,K}$ for some (large) $K$ smaller than $N$. The theoretical results of this paper only apply when $K = N$. We adapt the estimators as follows. We introduce $\bar Z_t^{N,K} = K^{-1}\sum_{i=1}^K Z_t^{i,N}$ and

$$\mathcal{E}_t^{N,K} = \frac{\bar Z_{2t}^{N,K} - \bar Z_t^{N,K}}{t}, \qquad \mathcal{V}_t^{N,K} = \frac{N}{K}\sum_{i=1}^K \Big(\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \mathcal{E}_t^{N,K}\Big)^2 - \frac{N}{t}\mathcal{E}_t^{N,K},$$

$$\mathcal{W}_{\Delta,t}^{N,K} = 2\mathcal{Z}_{2\Delta,t}^{N,K} - \mathcal{Z}_{\Delta,t}^{N,K}, \quad \text{where} \quad \mathcal{Z}_{\Delta,t}^{N,K} = \frac{N}{t}\sum_{k=t/\Delta+1}^{2t/\Delta}\Big(\bar Z_{k\Delta}^{N,K} - \bar Z_{(k-1)\Delta}^{N,K} - \Delta\mathcal{E}_t^{N,K}\Big)^2,$$

as well as

$$\mathcal{P}_{\Delta,T}^{sub,N,K} = \Phi_3\Big(\mathcal{E}_{T/2}^{N,K}, \mathcal{V}_{T/2}^{N,K}, \Big|\mathcal{W}_{\Delta,T/2}^{N,K} - \frac{N-K}{K}\mathcal{E}_{T/2}^{N,K}\Big|\Big),$$

with $\Phi$ defined in Corollary 4. We added the absolute value around the last argument of $\Phi_3$ for practical reasons: by this way, $\mathcal{P}_{\Delta,T}^{sub,N,K}$ is always well-defined (and seems closer to the reality than $\Psi_3$ which is 0 when $w \le 0$). This does not change the theory (since $\mathcal{W}_{\Delta,T/2}^{N,K} - \frac{N-K}{K}\mathcal{E}_{T/2}^{N,K}$ is asymptoticaly positive, at least when $N = K$). We also put

$$\mathcal{U}_T^{N,K} = \Big[\frac{N}{K}\sum_{i=1}^K\Big(\frac{Z_T^{i,N} - \bar Z_T^{N,K}}{\bar Z_T^{N,K}}\Big)^2 - \frac{N}{\bar Z_T^{N,K}}\Big]\mathbf{1}_{\{\bar Z_T^{N,K}>0\}} \quad \text{and} \quad \mathcal{P}_T^{sup,N,K} = \frac{1}{\mathcal{U}_T^{N,K}+1}\mathbf{1}_{\{\mathcal{U}_T^{N,K}\ge 0\}}.$$

We set $\Delta_t = t/(2\lfloor t^{9/13}\rfloor)$, which corresponds to the (quite arbitrary) choice $q = 12$, and

$$\hat p_T^{N,K} = \mathcal{P}_{\Delta_T,T}^{sub,N,K}\mathbf{1}_{\{\log(\bar Z_T^{N,K})<(\log T)^2\}} + \mathcal{P}_T^{sup,N,K}\mathbf{1}_{\{\log(\bar Z_T^{N,K})>(\log T)^2\}}.$$

7.1. **Choice of the estimators.** Let us explain briefly how we have modified the estimators when observing only $(Z_s^{i,N})_{s\in[0,T],i=1,\dots,K}$. We adopt the notation of Section 2, in particular $A_N(i,j) = N^{-1}\theta_{ij}$, and we follow the considerations therein.

In the subcritical case, we recall that $Q_N = (I - \Lambda A_N)^{-1}$ and that $\ell_N(i) = \sum_{j=1}^N Q_N(i,j)$. Following closely the argumentation of Subsection 2.1, we expect that, for $t$ (and $\Delta$) large and in a suitable regime, we should have $\mathcal{E}_t^{N,K} \simeq \mu\bar\ell_N^K$ (where $\bar\ell_N^K = K^{-1}\sum_{i=1}^K \ell_N(i)$), $\mathcal{V}_t^{N,K} \simeq$

$\mu^2(N/K)\sum_{i=1}^K(\ell_N(i)-\bar\ell_N^K)^2$, and $\mathcal{W}_{\Delta,t}^{N,K}\simeq\mu(N/K^2)\sum_{j=1}^N(\sum_{i=1}^K Q_N(i,j))^2\ell_N(j)$. Recalling now that $\ell_N(i)\simeq 1+\Lambda(1-\Lambda p)^{-1}L_N(i)$ and that $NL_N$ is a vector composed of i.i.d. Binomial$(N,p)$ random variables, we expect that $\mathcal{E}_t^{N,K}\simeq\mu\mathbb{E}[\ell_N(1)]\simeq\mu/(1-\Lambda p)$ and $\mathcal{V}_t^{N,K}\simeq\mu^2N\mathbb{V}\mathrm{ar}\,(\ell_N(1))\simeq\mu^2\Lambda^2p(1-p)/(1-\Lambda p)^2$ for $N$, $K$ and $t$ large. For the last estimator, one first has to get convinced, following again the arguments of Subsection 2.1, that $\sum_{i=1}^K Q_N(i,j)\simeq 1+(K/N)\Lambda p/(1-\Lambda p)$ if $j\in\{1,\dots,K\}$ while $\sum_{i=1}^K Q_N(i,j)\simeq(K/N)(\Lambda p/(1-\Lambda p))$ if $j\in\{K+1,\dots,N\}$. Since we still have $\ell_N(j)\simeq 1+\Lambda p/(1-\Lambda p)=1/(1-\Lambda p)$, we find that

$$\mathcal{W}_{\Delta,t}^{N,K}\simeq\frac{\mu N}{K^2(1-\Lambda p)}\Big(K\Big[1+\frac{K\Lambda p}{N(1-\Lambda p)}\Big]^2+(N-K)\Big[\frac{K\Lambda p}{N(1-\Lambda p)}\Big]^2\Big)=\frac{\mu}{(1-\Lambda p)^3}+\frac{(N-K)\mu}{K(1-\Lambda p)}.$$

Recalling that $\mathcal{E}_t^{N,K}\simeq\mu/(1-\Lambda p)$, we conclude that $\mathcal{W}_{\Delta,t}^{N,K}-(N-K)\mathcal{E}_t^{N,K}/K\simeq\mu/(1-\Lambda p)^3$. For $N$, $K$, $t$ and $\Delta$ large, we thus should have

$$\Phi_3\Big(\mathcal{E}_t^{N,K},\mathcal{V}_t^{N,K},\Big|\mathcal{W}_{\Delta,t}^{N,K}-\frac{N-K}{K}\mathcal{E}_t^{N,K}\Big|\Big)\simeq\Phi_3\Big(\frac{\mu}{1-\Lambda p},\frac{\mu^2\Lambda^2p(1-p)}{(1-\Lambda p)^2},\frac{\mu}{(1-\Lambda p)^3}\Big)=p.$$

We introduce the conjectured limit of $\mathcal{P}_{\Delta_t,t}^{sub,N,K}$ as $t\to\infty$:

$$\mathcal{P}_{\infty,\infty}^{sub,N,K}=\Phi_3\Big(\mu\bar\ell_N^K,\frac{\mu^2N}{K}\sum_{i=1}^K(\ell_N(i)-\bar\ell_N^K)^2,\Big|\frac{\mu N}{K^2}\sum_{j=1}^N(\sum_{i=1}^K Q_N(i,j))^2\ell_N(j)-\frac{N-K}{K}\mu\bar\ell_N^K\Big|\Big).$$

In the supercritical case, we follow Subsection 2.2 and deduce that for $t$ large, we should have $\mathcal{U}_t^{N,K}\simeq(N/K)(\bar V_N^K)^{-2}\sum_{i=1}^K(V_N(i)-\bar V_N^K)^2$, where $V_N$ is the Perron-Frobenius eigenvector of $A_N$ and where $\bar V_N^K=K^{-1}\sum_{i=1}^K V_N(i)$. Recalling now that $V_N$ is almost colinear to $L_N$ and that $NL_N$ is a vector composed of i.i.d. Binomial$(N,p)$ random variables, we conclude that indeed, it should hold that $\mathcal{U}_t^{N,K}\simeq N(\mathbb{E}[V_N(1)])^{-2}\mathbb{V}\mathrm{ar}\,(V_N(1))\simeq 1/p-1$ for $N$, $K$ and $t$ large, whence $\mathcal{P}_t^{sup,N,K}\simeq p$. Here we introduce the conjectured limit of $\mathcal{P}_t^{sup,N,K}$ as $t\to\infty$:

$$\mathcal{P}_\infty^{sup,N,K}=\Big(1+\frac{N}{K(\bar V_N^K)^2}\sum_{i=1}^K(V_N(i)-\bar V_N^K)^2\Big)^{-1}.$$

7.2. **Numerical results.** From now on, we assume that $\varphi(t)=a\exp(-bt)$ for some $a,b>0$, which satisfies all our assumptions and is easy to simulate. We also always assume that $a=2$ and $b=1$, whence $\Lambda=2$. We did not find interesting different behaviors when using other values.

On all the pictures below, we plot the time evolution of the three quartiles, using 1000 simulations, of $\hat p_t^{N,K}-p$, as a function of time $t\in[0,T]$. We always choose $T$ in such a way that $\bar Z_T^N\simeq 3000$, so that on the right of all the pictures below, we always have more or less the same quantity of data (for a given value of $K$). The curves are not smooth because we use only 1000 simulations, but this already takes a lot of time.

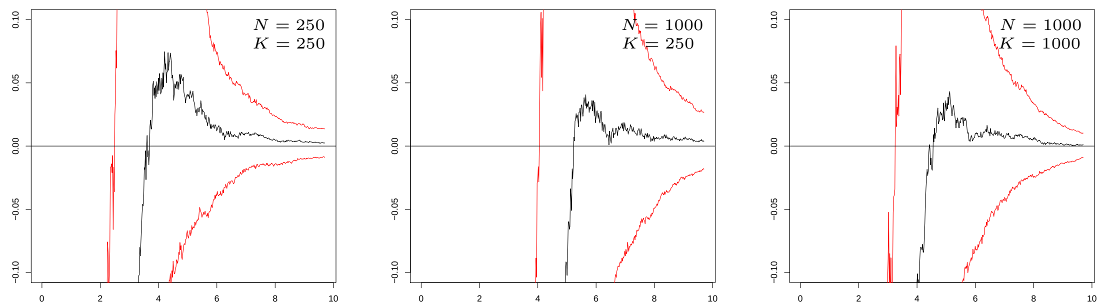
For a given simulation, we say that the choice is *good* when $\hat p_t^{N,K}=\mathcal{P}_{\Delta_t,t}^{sub,N,K}$ and $\Lambda p<1$ or $\hat p_t^{N,K}=\mathcal{P}_t^{sup,N,K}$ and $\Lambda p>1$. When the choice is almost always good (that is, for a large proportion of the simulations), we also indicate below the picture the three quartiles of $\hat p_\infty^{N,K}-p$, where $\hat p_\infty^{N,K}$ is the conjectured limit as $t\to\infty$ of $\hat p_t^{N,K}$, given by $\hat p_\infty^{N,K}=\mathcal{P}_{\infty,\infty}^{sub,N,K}$ when $\Lambda p<1$ and $\hat p_\infty^{N,K}=\mathcal{P}_\infty^{sup,N,K}$ when $\Lambda p>1$.

We start with the independent case. As the pictures below show, the estimation of $p$ is more precise in the fairly supercritical case. Also, around the critical case, $\hat p_t^{N,K}$ is far from always making the good choice, but this does not, however, produce too bad results. On the contrary, the estimation of $p$ when $p$ is very small does not work very well.

Observe that the results with $N = K = 1000$ are most often not better than those with $N = K = 250$. This is not so surprising for a given value of $T$, since our rate of convergence resembles $T^{-1}N^{1/2} + N^{-1/2}$ in the subcritical case and something similar in the supercritical case.

Finally, in all the trials below, it seems that $|\hat{p}_{\infty}^{N,K} - p|$ is much smaller than $|\hat{p}_t^{N,K} - p|$.

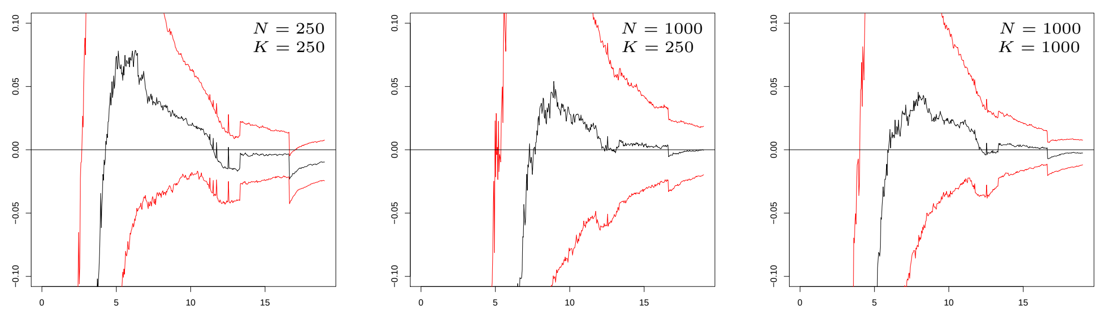
Independent case, $p = 0.85$, $\mu = 1$ (fairly supercritical). The choice is always good for $t \in [1, 9.7]$.




$-0.0073, 0.00097, 0.0091$ $-0.0067, 0.0011, 0.0080$ $-0.0038, 0.000086, 0.0041$

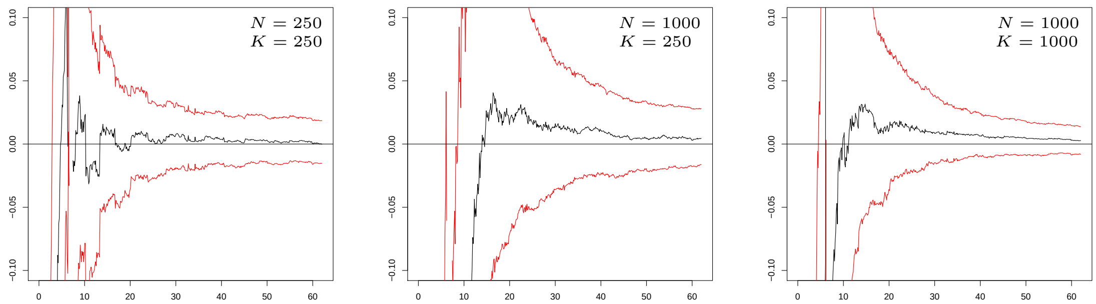
These pictures illustrate that this situation (fairly supercritical) is quite favorable.

Independent case, $p = 0.65$, $\mu = 1$ (supercritical). The choice is always bad for $t \in [14, 19]$.



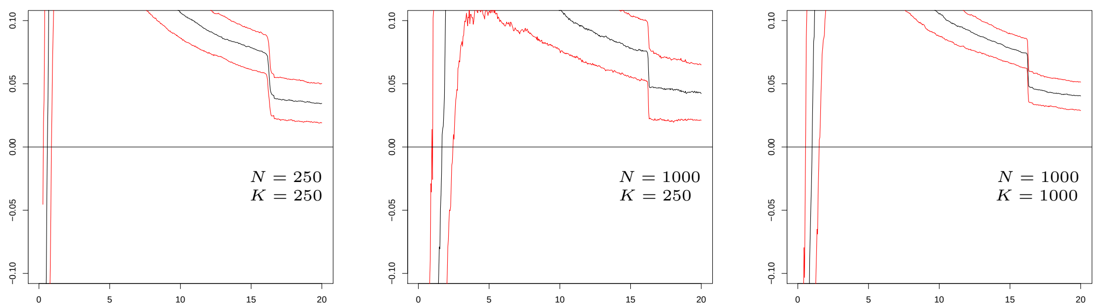

However, the error (using the wrong estimator) does not seem so large. The jumps correspond to jumps of $t \mapsto \Delta_t$. This is particularly visible on these pictures because the time intervall is short. Let us mention that the choice becomes good around $t = 22$.

Independent case, $p = 0.51$, $\mu = 1$ (slightly supercritical). The choice is always bad for $t \in [9, 62]$.




However, the error (using the wrong estimator) does not seem so large.

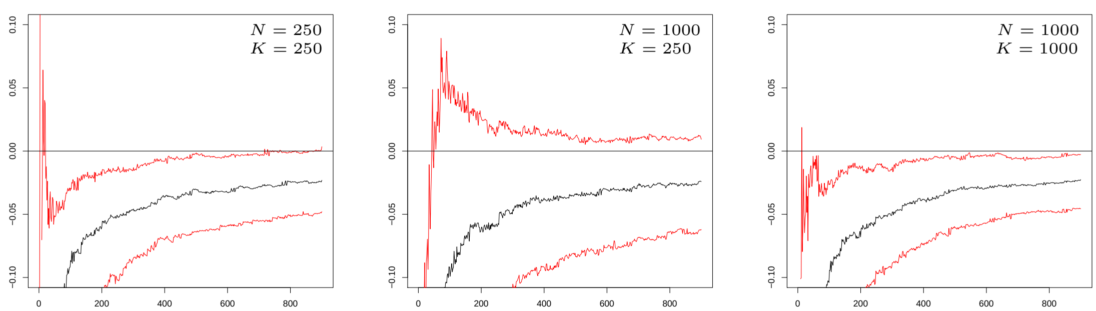
Independent case, $p = 0.48$, $\mu = 20$ (slightly subcritical). The choice is always bad for $t \in [1, 15]$ and always good for $t \in [17, 20]$.




$-0.014, 0.0010, 0.016$ $\quad$ $-0.015, 0.0017, 0.016$ $\quad$ $-0.0070, 0.00078, 0.0076$

We clearly see the change of choice around $t = 16$.

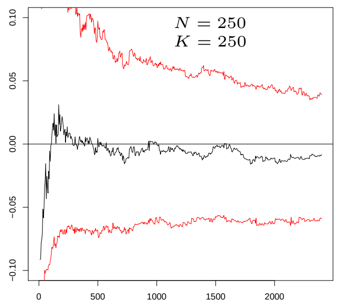
Independent case, $p = 0.35$, $\mu = 1$ (fairly subcritical). The choice is always good for $t \in (0, 900]$.



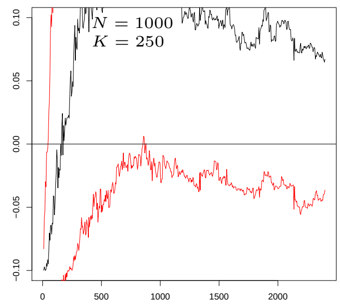

$-0.012, 0.0012, 0.014$ $\quad$ $-0.012, 0.00072, 0.015$ $\quad$ $-0.0065, 0.00021, 0.0070$

These results show that the bias is rather large, of the same order as the standard deviation.

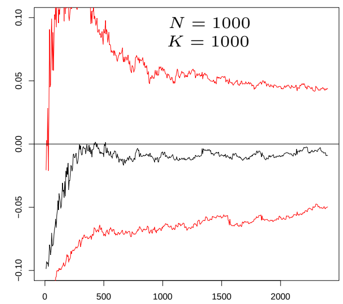
Independent case, $p = 0.1$, $\mu = 1$ (fairly subcritical). The choice is always good.




$-0.0038, 0.0013, 0.0067$



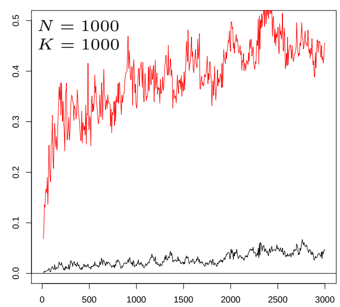

$-0.0045, 0.00057, 0.0061$




$-0.0021, 0.00045, 0.0032$

These pictures illustrate that this situation ($p$ small) is not quite favorable.

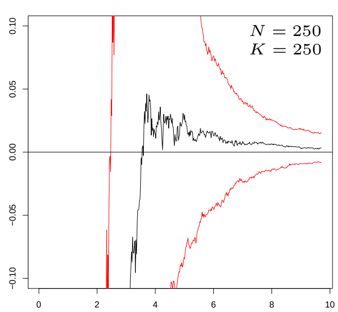
Independent case, $p = 0$, $\mu = 1$ (subcritical). The choice is always good.



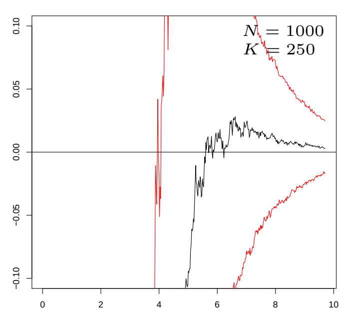

This is catastrophic.

In the symmetric case, we obtain very similar numerical results.

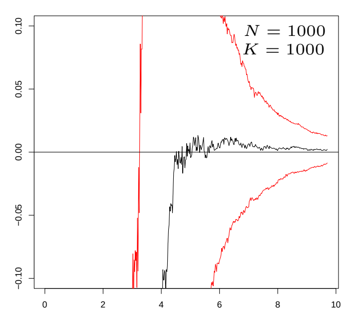
Symmetric case, $p = 0.85$, $\mu = 1$ (fairly supercritical). The choice is always good for $t \in [1, 9.7]$.




$-0.0057, 0.0021, 0.0091$

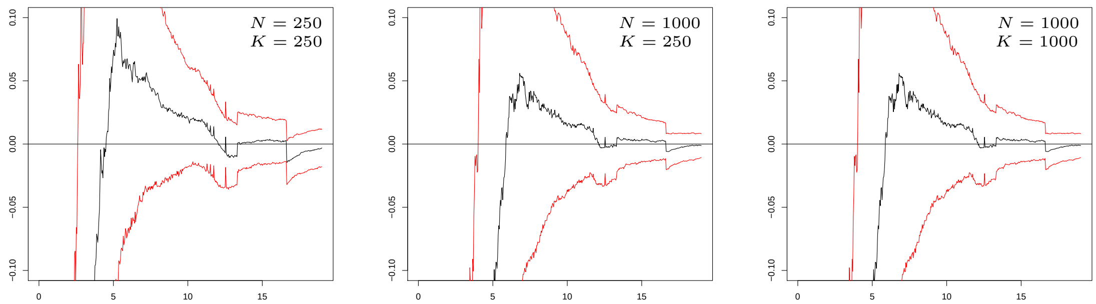



$-0.0062, 0.0012, 0.0084$

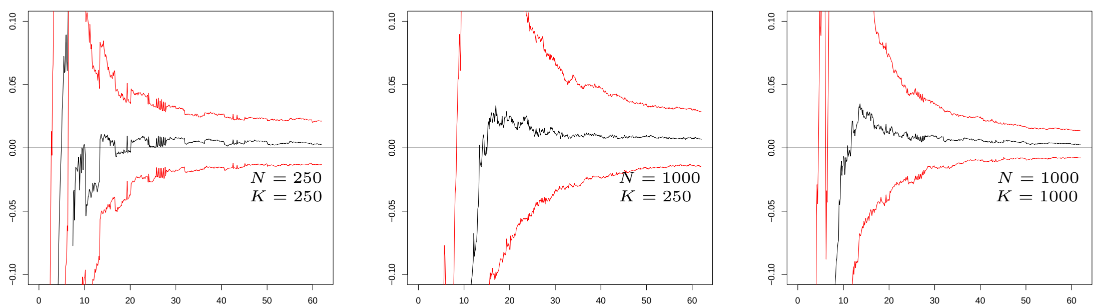



$-0.0029, 0.00079, 0.0044$

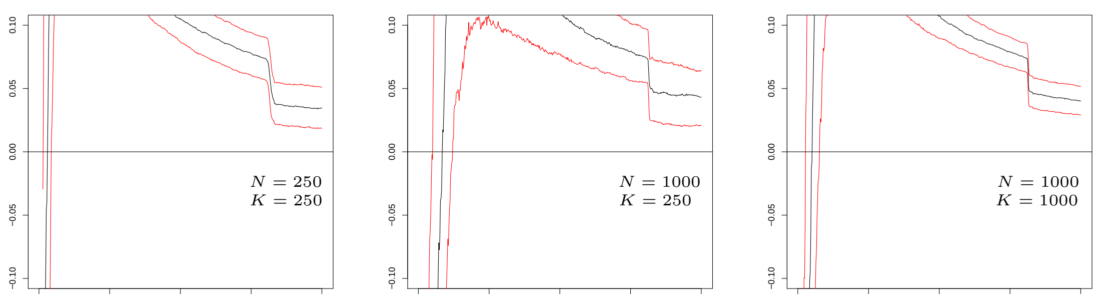
Symmetric case, $p = 0.65$, $\mu = 1$ (supercritical). The choice is always bad for $t \in [14, 19]$.




Symmetric case, $p = 0.51$, $\mu = 1$ (slightly supercritical). The choice is always bad for $t \in [9, 62]$.

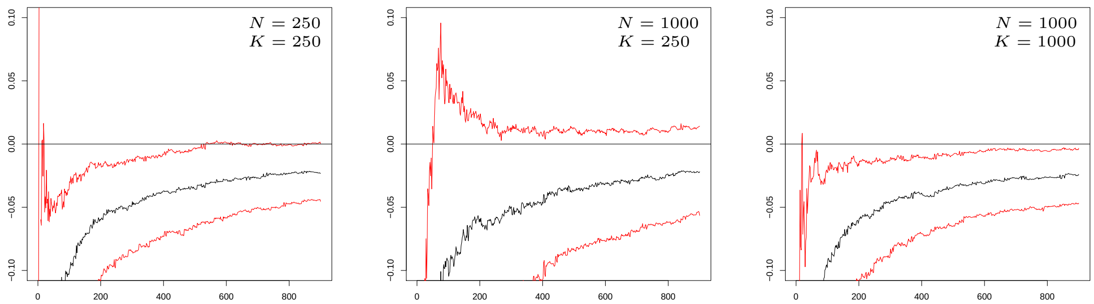



Symmetric case, $p = 0.48$, $\mu = 20$ (slightly subcritical but large $\mu$). The choice is always bad for $t \in [1, 15]$ and always good for $t \in [17, 20]$.



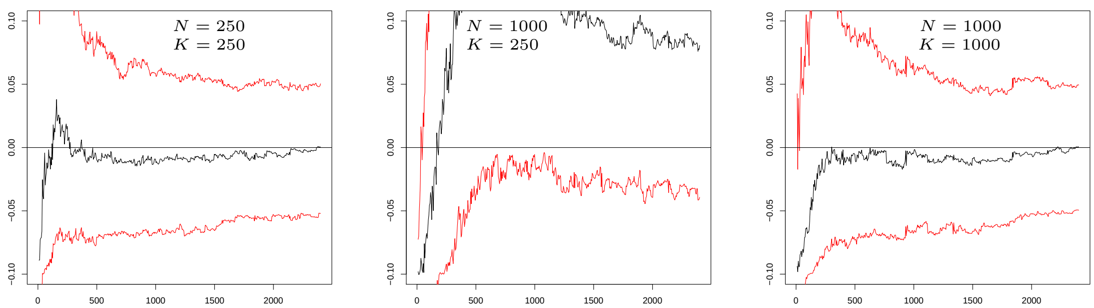

$-0.014, 0.00095, 0.018$ $\quad$ $-0.013, 0.0020, 0.017$ $\quad$ $-0.0074, 0.00053, 0.0078$

Symmetric case, $p = 0.35$, $\mu = 1$ (fairly subcritical). The choice is always good for $t \in (0, 900]$.




$-0.010, 0.0021, 0.0165$ $\quad$ $-0.012, 0.0021, 0.018$ $\quad$ $-0.0067, 0.0011, 0.0075$

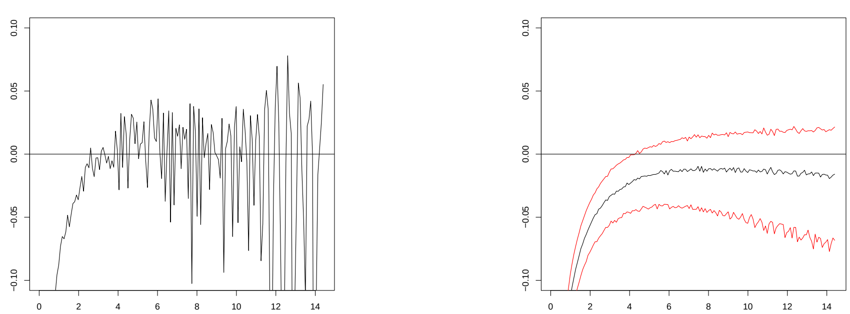
Symmetric case, $p = 0.1$, $\mu = 1$ (fairly subcritical). The choice is always good.




$-0.0030, 0.0029, 0.0087$ $\quad$ $-0.0043, 0.00091, 0.0029$ $\quad$ $-0.0022, 0.00056, 0.0029$

Finally, we discuss the practical choice of $\Delta$.

Independent case, $\mu = 1$, $p = 0.35$ (fairly subcritical), with $T = 900$ and $N = K = 1000$. On the left, we have plotted $\mathcal{P}^{sub,N,K}_{\Delta,T} - p$ as a function of $\Delta \in [1, 15]$ obtained with one simulation. On the right, we have plotted the quartiles of the same quantity using 1000 simulations.



Our (arbitrary) choice $\Delta_T = T/(2\lfloor T^{9/13} \rfloor) \simeq 4.1$ seems rather suitable: we see on the right picture that the "optimal" $\Delta$ lies between 4 and 6. This is mainly due to chance and probably depends strongly on the parameters of the model. We see on the left picture that given one set of data, $\mathcal{P}^{sub,N,K}_{\Delta,T}$ varies a lot.

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