

ASYMPTOTIC OF GRAZING COLLISIONS AND PARTICLE APPROXIMATION FOR THE KAC EQUATION WITHOUT CUTOFF.

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ABSTRACT. The subject of this article is the Kac equation without cutoff. We first show that in the asymptotic of grazing collisions, the Kac equation can be approximated by a Fokker-Planck equation. The convergence is uniform in time and we give an explicit rate of convergence. Next, we replace the small collisions by a small diffusion term in order to approximate the solution of the Kac equation and study the resulting error. We finally build a system of stochastic particles undergoing collisions and diffusion, that we can easily simulate, which approximates the solution of the Kac equation without cutoff. We give some estimates on the rate of convergence.

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1. INTRODUCTION

1.1. **The model.** The spatially homogeneous Boltzmann equation (see Cercignani [5], Villani [35]) describes the density $f_t(v)$ of particles in a gas, which move with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The Kac equation is a one-dimensional *caricature* of the Boltzmann equation. It writes

$$(1.1) \quad \frac{\partial f_t}{\partial t}(v) = \int_{v_* \in \mathbb{R}} \int_{\theta=-\pi}^{\pi} \left(f_t(v') f_t(v'_*) - f_t(v) f_t(v_*) \right) \beta(\theta) d\theta dv_*,$$

where $t \geq 0$, $v \in \mathbb{R}$ and where the post-collisional velocities are given by

$$(1.2) \quad v' = v \cos \theta - v_* \sin \theta, \quad v'_* = v \sin \theta + v_* \cos \theta.$$

The function $\beta : [-\pi, \pi] - \{0\} \rightarrow \mathbb{R}_+$ is an even function called *cross section*. Each pair of particles with velocities v and v_* collides to give particles with velocities v' and v'_* with a rate proportional to $\beta(\theta)$. See Kac [20, 21] and Desvillettes [9] for more precisions. If we have $\int_0^\pi \beta(\theta)d\theta = \infty$, then there is an infinite number of collisions for each particle during any time interval. The case where we assume $\int_0^\pi \beta(\theta)d\theta < \infty$ (case with cutoff) has been much studied. We will focus here on the real physical situation where we only assume $\int_0^\pi \theta^2 \beta(\theta)d\theta < \infty$ (case without cutoff). By analogy with the 3d-Boltzmann equation, we will include the case where, for some $0 < \nu < 2$,

$$(1.3) \quad \beta(\theta) \stackrel{\theta=0}{\approx} |\theta|^{-1-\nu}.$$

We will use in this article Wasserstein distances. Let us recall that for $p \geq 1$, if f and g are two probability measures on \mathbb{R} with a moment of order p ,

$$W_p(f, g) = \inf \left\{ \mathbb{E}(|U - V|^p)^{1/p}, U \sim f, V \sim g \right\},$$

where the infimum is taken over all random variables U with law f and V with law g . See e.g. Villani [36] for many details on the subject. In particular, it is known that the infimum is reached : one can build $U \sim f$ and $V \sim g$ such that $W_p^p(f, g) = \mathbb{E}(|U - V|^p)$.

1.2. Asymptotic of grazing collisions. Assume that there are more and more collisions, but that these collisions generate smaller and smaller deviations. For example, consider $\beta_\epsilon(\theta) = \frac{1}{\epsilon^3} \beta\left(\frac{\pi\theta}{\epsilon}\right) \mathbb{1}_{|\theta| < \epsilon}$. Then, we have $\int_0^\pi \theta^2 \beta_\epsilon(\theta)d\theta = \text{const}$ and $\int_0^\pi \theta^4 \beta_\epsilon(\theta)d\theta \rightarrow 0$. It is known that in this case, the solutions of Boltzmann's equation converge to the solution of the Fokker-Planck-Landau equation. To be more precise, Degond and Lucquin-Desreux [6] and Desvillettes [7] have shown the convergence of the operators (not of the solutions) and Villani [34] has shown some compactness results and the convergence of subsequences. The uniqueness results of [14] show the true convergence (under some more restrictive assumptions).

If we denote by $(g_t^\epsilon)_{t \geq 0}$ the solution of equation (1.1) with cross section β_ϵ and initial condition $g_0^\epsilon(v) = g_0(v)$ and if we assume $\int_{\mathbb{R}} v^4 g_0(v)dv < \infty$ we will show that $\sup_{t \in [0, \infty)} W_2(g_t^\epsilon, g_t) \leq C\epsilon$, with $(g_t)_{t \geq 0}$ starting from g_0 and solving

$$(1.4) \quad \frac{\partial}{\partial t} g_t(v) = \frac{\mathcal{E}}{2} \frac{\partial^2}{\partial v^2} g_t(v) + \frac{1}{2} \frac{\partial}{\partial v} (v g_t(v)),$$

where $\mathcal{E} := \int_{\mathbb{R}} v^2 g_0(v)dv$. This limit equation is *nonlinear*, but the nonlinearity appears only through $\int_{\mathbb{R}} v^2 g_t(v)dv$, which is constant in time.

A similar result has already been proved by Toscani [31] with a stronger distance but the rate of convergence is not very explicit. We believe that the present rate of convergence is optimal.

1.3. Replacing grazing collisions by a small diffusion term. We come back to the Kac equation (1.1) with fixed cross section β . Numerically, we must truncate small collisions, since they are in infinite number. There are two possibilities.

One may truncate roughly small collisions by replacing β by $\tilde{\beta}_\epsilon(\theta) = \beta(\theta) \mathbb{1}_{|\theta| > \epsilon}$. We denote by $(\tilde{f}_t^\epsilon)_{t \geq 0}$ the solution of (1.1) with this $\tilde{\beta}_\epsilon$.

One may replace small collisions by a small diffusion term in the spirit of grazing collisions. We denote by $(f_t^\epsilon)_{t \geq 0}$ the solution to

$$(1.5) \quad \begin{aligned} \frac{\partial}{\partial t} f_t^\epsilon(v) = & b_\epsilon \frac{\partial}{\partial v} \left[v f_t^\epsilon(v) \right] + \mathcal{E} b_\epsilon \frac{\partial^2}{\partial v^2} f_t^\epsilon(v) \\ & + \int_{v_* \in \mathbb{R}} \int_{|\theta| \geq \epsilon} \left(f_t^\epsilon(v') f_t^\epsilon(v'_*) - f_t^\epsilon(v) f_t^\epsilon(v_*) \right) \beta(\theta) d\theta dv_*, \end{aligned}$$

where

$$(1.6) \quad b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta \quad \text{and} \quad \mathcal{E} = \int_{\mathbb{R}} v^2 f_0^\epsilon(v) dv.$$

We will show that $\sup_{t \in [0, T]} W_2(f_t, f_t^\epsilon) \leq C\epsilon(1 + \sqrt{T})$ if $\int_{\mathbb{R}} v^4 f_0(dv) < \infty$. Observe that when neglecting roughly grazing collisions, we get $\sup_{t \in [0, T]} W_2(\tilde{f}_t^\epsilon, f_t) \leq C_T \epsilon^{1-\nu/2}$ (see Desvillettes-Graham-Mélard [12]) if β is as in (1.3). We can yet notice that there is no dependance on ν in our result. This is due to the fact that the more ν is close to 2, the more we neglect small collisions, but the more small collisions are well-approximated by the diffusion term. The proof is inspired by [13].

1.4. A finite system of stochastic particles. Let β be a given cross section and f_0 an initial datum with $\int_{\mathbb{R}} v^4 f_0(dv) < \infty$. We consider a solution $(f_t)_{t \geq 0}$ of (1.1).

For $\epsilon > 0$ fixed, we are going to build a system of n stochastic particles that we can simulate with a cost of order $Tn \int_{|\theta| > \epsilon} \beta(\theta) d\theta$ on $[0, T]$, which is at worst of order $T\epsilon^{-2}n$. If we denote by $\mu_t^{n, \epsilon}$ the empirical measure associated to this system of particles and by μ_t^n the empirical measure associated with a system of n i.i.d. particles with law f_t , we will show that

$$\sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n, \epsilon}) \right] \leq C(1 + T)^3 \left(\epsilon^2 + \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right] \right).$$

Our system of particles is thus as efficient as the system of particles with *true i.i.d. particles* with law f_t which are not simulable because of the nonlinearity. If we assume that f_0 has infinitely many moments, we will get

$$\sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n, \epsilon}) \right] \leq C(1 + T)^3 \left(\epsilon^2 + \frac{1}{n^{(1/2)^-}} \right).$$

This system of particles uses the ideas of the previous section : we replace small collisions by a small diffusion term, which gives an error of order ϵ .

In Desvillettes-Graham-Mélard [12], they just cutoff small collisions and they get, roughly, something like $\sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \tilde{\mu}_t^{n, \epsilon}) \right] \leq C_T \left(\epsilon^{2-\nu} + \frac{\epsilon^{C_T \Lambda_\epsilon}}{n} \right)$, with $\Lambda_\epsilon = \int_{|\theta| > \epsilon} \beta(\theta) d\theta \approx \epsilon^{-\nu}$ if β is as in (1.3). If we compare this result with our result, we can observe the following.

- In the first term, we get an error of order ϵ^2 instead of $\epsilon^{2-\nu}$. It is due to the fact that we replace small collisions by a small diffusion term.
- In the second term, we get a bound which does not depend on ϵ . It is because we use a Wasserstein distance which is well-adapted for this study. In Desvillettes-Graham-Mélard [12], they give the final result with a Wasserstein distance, but to get this result they use a variation distance.
- The cost of simulation for the two systems of particles is similar.

1.5. Comments. We managed to obtain some bounds uniform in time for the asymptotic of grazing collisions. For our two other main results, we tried to limit the time dependance. We thus avoid getting bounds with exponential terms.

The bound we get for $\mathbb{E}[W_2^2(f_t, \mu_t^n)]$ is not very satisfactory. *A priori*, it is of order $n^{-(1/2)-}$ (if the initial condition has infinitely many moments, see Lemma A.4) which gives a bound for $\mathbb{E}[W_2(f_t, \mu_t^n)]$ of order $n^{-(1/4)-}$. We expected to get a bound of order $n^{-1/2}$ as in the central limit theorem, but we cannot get it. See Peyre [26] for example to get more details. It seems to be the only defect of W_2 for this study.

Assuming that $\int_0^\pi \theta \beta(\theta) d\theta < \infty$ (e.g. if we assume (1.3) with $\nu \in (0, 1)$), we get a bound for $\mathbb{E}[W_1(f_t, \mu_t^{n, \epsilon})]$ which is of order $\epsilon + n^{-1/2}$ but with an exponential dependance in time. If $\int_0^\pi \theta^\gamma \beta(\theta) d\theta < \infty$ (e.g. if $\nu < \gamma$) for some $\gamma \in (1, 2)$, we also study $\mathbb{E}[W_\gamma^\gamma(f_t, \mu_t^{n, \epsilon})]$.

In a future work, we will apply the same kind of methods to the homogeneous Boltzmann equation. We hope to get some results which will probably be much less optimal.

1.6. Bibliographic comments. In this article, we use probabilistic methods. The first who used probabilistic methods to study the Kac (or Boltzmann) equation is McKean [22, 23]. He was investigating the convergence to equilibrium and he proposed some probabilistic representation of Wild's sums, using some tools now known as the McKean graphs. This idea has been used and refined in many papers, let us quote a few of them by Carlen-Carvalho-Gabetta [2, 3, 4] and more recently by Dolera-Gabetta-Regazzini [10, 11]. The exact rate of convergence to equilibrium for the Kac equation is now very precisely known.

Another pioneer of probabilistic methods for kinetic equations is Tanaka. In [29], he discovered the noticeable property that the Wasserstein distance with quadratic cost is non-increasing along solutions to the Kac equation. This idea was extended in [30] to the Boltzmann equation for Maxwell molecules. His study was based on the use of some nonlinear stochastic processes related to the Kac and Boltzmann equations. The paper of Graham-Méléard [16] (about the regularization properties of the Kac equation via Malliavin calculus) is based on the work of Tanaka, as well as [15, 14] dealing with the uniqueness of solutions to the Boltzmann equation. Our work is also strongly inspired by Tanaka's papers.

The present paper is also closely linked with the original problem of Kac [20, 21], concerning the propagation of chaos for the Kac equation. The idea is to make a step toward the rigorous justification of the Boltzmann equation, by showing that some simplified particle systems converge to a simplified Boltzmann equation. On this topic, some important results have been obtained by Sznitman [28] (Boltzmann equation for hard spheres), Desvillettes-Graham-Méléard [12] (Kac equation without cutoff) and more recently by Peyre [26] (large deviations for the Boltzmann equation for Maxwell molecules) and by Mischler-Mouhot [24] (quantitative propagation of chaos for the Boltzmann equation for hard spheres and Maxwell molecules). Let us mention that we provide here a very simple proof of the propagation of chaos for the Kac equation (in a slightly modified context) with very satisfying quantitative estimates, see Section 6 until Proposition 6.2.

To end with probabilistic references, let us mention that our results strongly rely on a recent paper of Rio [27] giving some precise rate of convergence for the standard central limit theorem in Wasserstein distance.

To conclude, we quote the article of Pareschi-Toscani-Villani [25]. This paper is devoted to a numerical method, based on the use of Fourier transforms, for the homogeneous Boltzmann equation. It also uses an approximation of the grazing collisions by a small Landau-type diffusive term. Of course, our numerical method can be immediately extended to the true homogeneous Boltzmann equation. From a numerical point of view, we are not able to decide which method is better. In [25], periodic solutions are considered, which is not physical. One would need to study the error due to such an approximation. Furthermore, the theoretical results in [25] only concern the rate of convergence of the operators (when dealing with quite regular functions), not of the solutions. So the main advantage of our paper is that we really provide an (almost optimal) estimate of the numerical error, taking into account all the approximation parameters (number of particles and cutoff parameter). Unfortunately, we only study a simplified model. We believe that our estimates also hold true for the true homogeneous Boltzmann equation, but we are far from being able to prove it.

1.7. Plan of the paper. In the next section, we will state more precisely our three main results. In Section 3, we will give a probabilistic interpretation of the three equations. Sections 4, 5 and 6 are devoted to the proofs of our main results. Some numerical illustrations will be given in Section 7. At the end of the paper, we will give an appendix with some results about the Wasserstein distance between a compensated Poisson integral and a centered Gaussian law with same variance, the rate of convergence of an empirical measure using Wasserstein distances, the moments of the solution to (1.1) and the well-posedness for a certain kind of P.D.E.s.

2. RESULTS

2.1. Weak solutions. Let β be a cross section satisfying

$$(2.1) \quad \int_{-\pi}^{\pi} \theta^2 \beta(\theta) d\theta < \infty.$$

For $k \geq 0$, we denote by $\mathcal{P}_k(\mathbb{R})$ the set of probability measures on \mathbb{R} admitting a moment of order k and by $C_b^2(\mathbb{R})$ the space of real bounded functions which are in $C^2(\mathbb{R})$ with first and second derivatives bounded. We say that a family of probability measures $(f_t)_{t \geq 0}$ is in $L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ if $\sup_{[0, T]} \int_{\mathbb{R}} v^2 f_t(dv) < \infty$ for all T . If $\varphi \in C_b^2(\mathbb{R})$ and $(v, v_*) \in \mathbb{R}^2$, we set

$$(2.2) \quad K_\beta^\varphi(v, v_*) = \int_{-\pi}^{\pi} \left[\varphi(v \cos \theta - v_* \sin \theta) - \varphi(v) - (v(\cos \theta - 1) - v_* \sin \theta) \varphi'(v) \right] \beta(\theta) d\theta - bv \varphi'(v),$$

with

$$(2.3) \quad b = \int_{-\pi}^{\pi} (1 - \cos \theta) \beta(\theta) d\theta.$$

If $\int_0^\pi \theta \beta(\theta) d\theta < \infty$, then one easily checks, using that β is even, that

$$(2.4) \quad K_\beta^\varphi(v, v_*) = \int_{-\pi}^{\pi} \left[\varphi(v \cos \theta - v_* \sin \theta) - \varphi(v) \right] \beta(\theta) d\theta$$

We now define precisely the notion of solutions that we will use.

Definition 2.1. Consider a cross section β satisfying (2.1).

(1) We say that $(f_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ solves (1.1) if for any φ in $C_b^2(\mathbb{R})$, any $t \geq 0$,

$$(2.5) \quad \int_{\mathbb{R}} \varphi(v) f_t(dv) = \int_{\mathbb{R}} \varphi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} K_{\beta}^{\varphi}(v, v_*) f_s(dv) f_s(dv_*) ds.$$

(2) We say that $(g_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ solves (1.4) if for any φ in $C_b^2(\mathbb{R})$, any $t \geq 0$,

$$(2.6) \quad \int_{\mathbb{R}} \varphi(v) g_t(dv) = \int_{\mathbb{R}} \varphi(v) g_0(dv) + \frac{1}{2} \mathcal{E} \int_0^t \int_{\mathbb{R}} \varphi''(v) g_s(dv) ds \\ - \frac{1}{2} \int_0^t \int_{\mathbb{R}} v \varphi'(v) g_s(dv) ds,$$

where $\mathcal{E} := \int v^2 g_0(dv)$.

(3) For $\epsilon \in (0, 1)$ fixed, we say that $(f_t^\epsilon)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ solves (1.5) if for any φ in $C_b^2(\mathbb{R})$, any $t \geq 0$,

$$(2.7) \quad \int_{\mathbb{R}} \varphi(v) f_t^\epsilon(dv) = \int_{\mathbb{R}} \varphi(v) f_0^\epsilon(dv) - b_\epsilon \int_0^t \int_{\mathbb{R}} v \varphi'(v) f_s^\epsilon(dv) ds \\ + \mathcal{E} b_\epsilon \int_0^t \int_{\mathbb{R}} \varphi''(v) f_s^\epsilon(dv) ds + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} K_{\beta_\epsilon}^{\varphi}(v, v_*) f_s^\epsilon(dv) f_s^\epsilon(dv_*) ds,$$

where

$$(2.8) \quad \beta_\epsilon(\theta) = \beta(\theta) \mathbb{1}_{|\theta| > \epsilon}, \quad b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta \quad \text{and} \quad \mathcal{E} = \int_{\mathbb{R}} v^2 f_0^\epsilon(dv).$$

Observe that all the terms in the above equations are well-defined. For example in (2.5), the last term is well-defined because for $\varphi \in C_b^2(\mathbb{R})$, $|K_{\beta}^{\varphi}(v, v_*)| \leq C \int_0^\pi \theta^2 \beta(\theta) d\theta (|v|^2 + |v_*|^2) \|\varphi''\|_\infty + b|v| \|\varphi'\|_\infty$.

Proposition 2.2. *Let f_0 , g_0 and f_0^ϵ be in $\mathcal{P}_2(\mathbb{R})$ and let β satisfy (2.1). There is existence and uniqueness of solutions $(f_t)_{t \geq 0}$, $(g_t)_{t \geq 0}$ and $(f_t^\epsilon)_{t \geq 0}$ to equations (1.1), (1.4) and (1.5) starting from f_0 , g_0 and f_0^ϵ respectively, in the sense of Definition 2.1. Furthermore, we have energy conservation: for any $t \geq 0$*

$$(2.9) \quad \int_{\mathbb{R}} v^2 f_t(dv) = \int_{\mathbb{R}} v^2 f_0(dv), \quad \int_{\mathbb{R}} v^2 g_t(dv) = \int_{\mathbb{R}} v^2 g_0(dv)$$

and

$$(2.10) \quad \int_{\mathbb{R}} v^2 f_t^\epsilon(dv) = \int_{\mathbb{R}} v^2 f_0^\epsilon(dv).$$

For the proof of the previous result, one can see Toscani-Villani [32] for (1.1). For (1.4), use Proposition A.6 with $a = \frac{1}{2} \mathcal{E}$, $b = -\frac{1}{2}$ and $q = r = 0$. For (1.5), use Proposition A.6 with $a = \mathcal{E} b_\epsilon$, $b = -b_\epsilon$, $q = 0$ and $r(t, v, v_*, dh)$ defined by $r(t, v, v_*, A) = \int_{-\pi}^\pi \mathbb{1}_{|\theta| > \epsilon} \mathbb{1}_A(v(\cos \theta - 1) - v_* \sin \theta) \beta(\theta) d\theta$ for all Borel subset A of \mathbb{R} , which indeed satisfies $\sup_{t, v, v_*} r(t, v, v_*, \mathbb{R}) = \int_{-\pi}^\pi \mathbb{1}_{|\theta| > \epsilon} \beta(\theta) d\theta < \infty$ and $\sup_{t \geq 0} \int_{\mathbb{R}} (h^2 + 2vh) r(t, v, v_*, dh) = \int_{|\theta| > \epsilon} \sin^2 \theta \beta(\theta) d\theta (v^2 + v_*^2) = C(v^2 + v_*^2)$. To get energy conservation, it suffices to apply (2.5), (2.6) and (2.7) with $\varphi(v) = v^2$.

2.2. Asymptotic of grazing collisions. Our first main result is the following.

Theorem 2.3. *Let $g_0 \in \mathcal{P}_4(\mathbb{R})$ and let $(\beta_\epsilon)_{\epsilon \in (0,1)}$ be a family of cross sections verifying $\int_{-\pi}^{\pi} \theta^2 \beta_\epsilon(\theta) d\theta = 1$ and $\int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta \xrightarrow{\epsilon \rightarrow 0} 0$. For $\epsilon \in (0, 1)$, let $(g_t^\epsilon)_{t \geq 0}$ be the solution of (1.1) with g_0 for initial datum and β_ϵ for cross section. If $(g_t)_{t \geq 0}$ is the solution of (1.4) with the same g_0 for initial datum, then for all $\epsilon \in (0, 1)$,*

$$\sup_{t \in [0, \infty)} W_2^2(g_t^\epsilon, g_t) \leq C \frac{\int_{\mathbb{R}} v^4 g_0(dv)}{\mathcal{E}} \int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta,$$

where C is a universal constant.

This convergence result was already known (see for example Toscani [31]), but we get here an explicit and probably optimal rate of convergence, which, to our knowledge, had never been done so far.

Remark 2.4. *If we consider a cross section β with $\int_{-\pi}^{\pi} \theta^2 \beta(\theta) d\theta = 1$ and if for any $\epsilon \in (0, 1)$, we set $\beta_\epsilon(\theta) = \frac{\pi^3}{\epsilon^3} \beta\left(\frac{\pi\theta}{\epsilon}\right) \mathbb{1}_{|\theta| < \epsilon}$, then $\int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta \leq \epsilon^2$.*

2.3. Error when we replace the small collisions by a small diffusion term.

Let us explain briefly why (2.7) approximates (2.5): consider a cross section β satisfying (2.1) and $\varphi \in C_b^2(\mathbb{R})$. Write, using that β is even,

$$\begin{aligned} K_{\beta}^{\varphi}(v, v_*) &= \int_{-\epsilon}^{\epsilon} \left[\varphi(v \cos \theta - v_* \sin \theta) - \varphi(v) - (v \cos \theta - 1) - v_* \sin \theta \varphi'(v) \right] \beta(\theta) d\theta \\ &\quad + K_{\beta_\epsilon}^{\varphi}(v, v_*) - b_\epsilon v \varphi'(v) \\ &\approx \int_{-\epsilon}^{\epsilon} \left[v \cos \theta - 1 - v_* \sin \theta \right]^2 \beta(\theta) d\theta \frac{\varphi''(v)}{2} + K_{\beta_\epsilon}^{\varphi}(v, v_*) - b_\epsilon v \varphi'(v) \\ &\approx \frac{\varphi''(v)}{2} v_*^2 \int_{-\epsilon}^{\epsilon} \sin^2 \theta \beta(\theta) d\theta + \frac{\varphi''(v)}{2} v^2 \int_{-\epsilon}^{\epsilon} (\cos \theta - 1)^2 \beta(\theta) d\theta \\ &\quad + K_{\beta_\epsilon}^{\varphi}(v, v_*) - b_\epsilon v \varphi'(v) \\ &\approx \varphi''(v) v_*^2 b_\epsilon + K_{\beta_\epsilon}^{\varphi}(v, v_*) - b_\epsilon v \varphi'(v). \end{aligned}$$

We decided to neglect the second term in the fourth line of this approximate equality, since it is much smaller than the other terms, because $\int_{-\epsilon}^{\epsilon} (\cos \theta - 1)^2 \beta(\theta) d\theta \leq \int_{-\epsilon}^{\epsilon} \theta^4 \beta(\theta) d\theta \leq \epsilon^2 \int_{-\epsilon}^{\epsilon} \theta^2 \beta(\theta) d\theta \approx \epsilon^2 \int_{-\epsilon}^{\epsilon} \sin^2 \theta \beta(\theta) d\theta$. In order to obtain an equation preserving the kinetic energy, we replaced $\frac{1}{2} \int_{-\epsilon}^{\epsilon} \sin^2 \theta \beta(\theta) d\theta$ by b_ϵ (both are approximately equal to $\frac{1}{2} \int_{-\epsilon}^{\epsilon} \theta^2 \beta(\theta) d\theta$). Our second main result is the following.

Theorem 2.5. *Let $f_0 \in \mathcal{P}_4(\mathbb{R})$ and let β be a cross section satisfying (2.1). For $\epsilon \in (0, 1)$, we consider $(f_t)_{t \geq 0}$ and $(f_t^\epsilon)_{t \geq 0}$ solutions of (1.1) and (1.5) respectively, both starting from f_0 . Then for any $T > 0$, any $\epsilon \in (0, 1)$, we have*

$$\sup_{t \in [0, T]} W_2^2(f_t, f_t^\epsilon) \leq C \epsilon^2 \min \left(1 + T, \frac{1}{\int_{|\theta| < \epsilon} \theta^2 \beta(\theta) d\theta} \right),$$

where C depends only on $\int_{\mathbb{R}} v^2 f_0(dv)$, $\int_{\mathbb{R}} v^4 f_0(dv)$ and on $\int_0^\pi \theta^2 \beta(\theta) d\theta$.

We can observe that we are not so far to get a bound uniform in time for ϵ^2 (we do not have exponential bounds).

Remark 2.6. *If β is as in (1.3), we get a bound in $C \min(\epsilon^2(1+T), \epsilon^\nu)$.*

2.4. System of particles. Let f_0 be a probability measure on \mathbb{R} and let β be a cross section satisfying (2.1). We fix an integer n and we consider:

- a family of i.i.d. random variables $(V_0^i)_{i \in \{1, \dots, n\}}$ with law f_0 ,
- a family of i.i.d. Poisson measures $(N^{i,n}(dsd\theta dj))_{i \in \{1, \dots, n\}}$ on $[0, \infty) \times [-\pi, \pi] \times \{1, \dots, n\}$ with intensity measure $ds\beta(\theta)d\theta \frac{1}{n} \sum_{k=1}^n \delta_k(dj)$,
- a family of i.i.d. Brownian motions $(B_t^i)_{t \geq 0, i \in \{1, \dots, n\}}$.

For $\epsilon \in (0, 1)$, we consider $(V_t^{i,n,\epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ solution of the following system of SDEs: for $i = 1, \dots, n$, for all $t \geq 0$,

(2.11)

$$\begin{aligned} V_t^{i,n,\epsilon} = & V_0^i + \int_0^t \int_{|\theta| > \epsilon} \int_{j \in \{1, \dots, n\}} \left[(\cos \theta - 1) V_{s-}^{i,n,\epsilon} - \sin \theta V_{s-}^{j,n,\epsilon} \right] N^{i,n}(dsd\theta dj) \\ & - b_\epsilon \int_0^t V_s^{i,n,\epsilon} ds + \sqrt{2\mathcal{E}b_\epsilon} B_t^i, \end{aligned}$$

where $\mathcal{E} = \int_{\mathbb{R}} v^2 f_0(dv)$ and $b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta$.

The quantity $V_t^{i,n,\epsilon}$ has to be thought as the velocity of the i -th particle at time t . The behavior of $(V_t^{i,n,\epsilon})_{t \geq 0}$ is the following: after an exponential time τ with parameter $\Lambda_\epsilon = \int_{|\theta| \geq \epsilon} \beta(\theta) d\theta$, it collides with another particle labelled j chosen at random and then we set $V_\tau^{i,n,\epsilon} = (\cos \Theta) V_{\tau-}^{i,n,\epsilon} - (\sin \Theta) V_{\tau-}^{j,n,\epsilon}$, where Θ is $\Lambda_\epsilon^{-1} \beta(\theta) \mathbb{1}_{|\theta| \geq \epsilon} d\theta$ -distributed. Between two jumps, $V_t^{i,n,\epsilon}$ behaves like an Ornstein-Uhlenbeck process

$$V_t^{i,n,\epsilon} = V_s^{i,n,\epsilon} - b_\epsilon \int_s^t V_u^{i,n,\epsilon} du + \sqrt{2\mathcal{E}b_\epsilon} (B_t^i - B_s^i).$$

We can solve explicitly this last SDE and we get

$$V_t^{i,n,\epsilon} = V_s^{i,n,\epsilon} e^{-b_\epsilon(t-s)} + \sqrt{2\mathcal{E}b_\epsilon} e^{-b_\epsilon(t-s)} \int_s^t e^{b_\epsilon u} dB_u^i.$$

Hence the strong existence and uniqueness of a solution $(V_t^{i,n,\epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ to (2.11) is straightforward.

We can observe that to simulate our system of particles on $[0, T]$, we need to simulate in mean $nT \int_{|\theta| \geq \epsilon} \beta(\theta) d\theta$ jumps. We thus have a cost of simulation of order $nT \int_{|\theta| \geq \epsilon} \beta(\theta) d\theta$. The fact that we can explicitly solve the previous SDE is fundamental in order to have such a cost of simulation.

Theorem 2.7. *Let $f_0 \in \mathcal{P}_4(\mathbb{R})$ and let β be a cross section satisfying (2.1). We consider $(f_t)_{t \geq 0}$ solution to the Kac equation (1.1) starting from f_0 . For $n \in \mathbb{N}^*$ and $\epsilon \in (0, 1)$, we consider the solution $(V_t^{i,n,\epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ to (2.11). We set $\mu_t^{n,\epsilon} = \frac{1}{n} \sum_{i=1}^n \delta_{V_t^{i,n,\epsilon}}$. Then for any $T > 0$, any $n \geq 2$ and any $\epsilon \in (0, 1)$, we have*

$$\sup_{t \in [0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n,\epsilon}) \right] \leq C(1+T)^3 \left(\epsilon^2 + \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right] \right),$$

where C depends only on $\int_{\mathbb{R}} v^2 f_0(dv)$, $\int_{\mathbb{R}} v^4 f_0(dv)$ and on $\int_0^\pi \theta^2 \beta(\theta) d\theta$, and where for all $t \geq 0$, $\mu_t^n = \frac{1}{n} \sum_{i=1}^n \delta_{V_t^i}$, where $(V_t^i)_{i \in \{1, \dots, n\}}$ is a family of i.i.d. particles with law f_t .

Applying Lemma A.4 of the appendix we will deduce the following consequence:

Corollary 2.8. *Under the same assumptions and notation as in Theorem 2.7, if f_0 has a moment of order $p \geq 4$ with p even, then for all $T > 0$, all $n \geq 2$ and all $\epsilon \in (0, 1)$,*

$$\sup_{t \in [0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n, \epsilon}) \right] \leq C(1 + T)^3 \left(\epsilon^2 + \frac{1}{n^{\frac{p-2}{2p-2}}} \right),$$

where C depends only on p , f_0 and $\int_0^\pi \theta^2 \beta(\theta) d\theta$.

We end this section with a result using another Wasserstein distance.

Proposition 2.9. *Under the same assumptions and notation as in Theorem 2.7, if the cross section β satisfies the stronger assumption $\int_0^\pi \theta \beta(\theta) d\theta < \infty$, then for all $T > 0$, all $n \geq 2$ and all $\epsilon \in (0, 1)$,*

$$\sup_{t \in [0, T]} \mathbb{E} \left[W_1(f_t, \mu_t^{n, \epsilon}) \right] \leq C_T \left(\epsilon + \frac{1}{\sqrt{n}} \right),$$

where C_T depends only on T , $\int_{\mathbb{R}} v^4 f_0(dv)$ and on $\int_0^\pi \theta \beta(\theta) d\theta$.

We thus have a better dependence in n , but we get exponential bounds in time.

3. PROBABILISTIC INTERPRETATION OF THE EQUATIONS

This section is strongly inspired by Tanaka [30] and Desvillettes-Graham-Méléard [12]. Until the end of the article, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ will designate a Polish filtered probability space satisfying the usual conditions. Such a space is Borel isomorphic to the Lebesgue space $([0, 1], \mathcal{B}([0, 1]), d\alpha)$ which we will use as an auxiliary space. To be as clear as possible, we will use the notation \mathbb{E} for the expectation and \mathcal{L} for the law of a random variable or process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and we will use the notation \mathbb{E}_α and \mathcal{L}_α for the expectation and law of random variables or processes on $([0, 1], \mathcal{B}([0, 1]), d\alpha)$. The processes on $([0, 1], \mathcal{B}([0, 1]), d\alpha)$ will be called α -processes.

We say that a \mathbb{R} -valued process $(V_t)_{t \geq 0}$ is a L^2 -process if it is càdlàg, adapted and if $\mathbb{E}(\sup_{[0, T]} V_t^2) < \infty$ for all $T \geq 0$.

Now, we introduce a nonlinear stochastic differential equation linked with (1.1).

Proposition 3.1. *Let β be a cross section satisfying (2.1). Let $f_0 \in \mathcal{P}_2(\mathbb{R})$ and let $(f_t)_{t \geq 0}$ be the solution to (1.1) starting from f_0 . Consider any α -process $(W_t)_{t \geq 0}$ such that $\mathcal{L}_\alpha(W_t) = f_t$ for all $t \geq 0$. Let also N be a $(\mathcal{F}_t)_{t \geq 0}$ -Poisson measure on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $d s d \alpha \beta(\theta) d \theta$, and V_0 a \mathcal{F}_0 -measurable random variable with law f_0 . Then there exists a unique L^2 -process $(V_t)_{t \geq 0}$ such that for all $t \geq 0$,*

(3.1)

$$V_t = V_0 + \int_0^t \int_0^1 \int_{-\pi}^\pi \left[(\cos \theta - 1) V_{s-} - \sin \theta W_{s-}(\alpha) \right] \tilde{N}(d s d \alpha d \theta) - b \int_0^t V_s d s,$$

with b given by (2.3). Furthermore, $\mathcal{L}(V_t) = f_t$ for all $t \geq 0$.

Proof. While stated in a slightly different way, this result is almost contained in Desvillettes-Graham-Méléard [12, Theorem 3.4]. See the proof of Proposition 3.3 below for similar arguments. \square

Let us now write down a probabilistic interpretation of (1.4).

Proposition 3.2. *Let $g_0 \in \mathcal{P}_2(\mathbb{R})$ and set $\mathcal{E} = \int_{\mathbb{R}} v^2 g_0(dv)$. Consider a \mathcal{F}_0 -measurable random variable Y_0 with law g_0 and a $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion $(B_t)_{t \geq 0}$. Then there exists a unique L^2 -process $(Y_t)_{t \geq 0}$ such that for all $t \geq 0$,*

$$(3.2) \quad Y_t = Y_0 - \frac{1}{2} \int_0^t Y_s ds + \sqrt{\mathcal{E}} B_t.$$

Furthermore, $\mathcal{L}(Y_t) = g_t$ for all $t \geq 0$, where $(g_t)_{t \geq 0}$ is the unique solution to (1.4).

Proof. The existence and uniqueness of Y is classical since (3.2) is a S.D.E. with Lipschitz coefficients. By Itô's formula, we have for any $\varphi \in C_b^2(\mathbb{R})$

$$\varphi(Y_t) = \varphi(Y_0) + \int_0^t \varphi'(Y_s) \left(-\frac{1}{2} Y_s ds + \sqrt{\mathcal{E}} dB_s\right) + \frac{\mathcal{E}}{2} \int_0^t \varphi''(Y_s) ds.$$

Taking expectations and setting $\mu_t = \mathcal{L}(Y_t)$, we get for any $\varphi \in C_b^2(\mathbb{R})$

$$\int_{\mathbb{R}} \varphi(v) \mu_t(dv) = \int_{\mathbb{R}} \varphi(v) g_0(dv) - \frac{1}{2} \int_0^t \int_{\mathbb{R}} v \varphi'(v) \mu_s(dv) ds + \frac{\mathcal{E}}{2} \int_0^t \int_{\mathbb{R}} \varphi''(v) \mu_s(dv).$$

Thus $(\mu_t)_{t \geq 0}$ solves (1.4) in the sense of Definition 2.1. We get $(\mu_t)_{t \geq 0} = (g_t)_{t \geq 0}$ by uniqueness (see Proposition 2.2). \square

It remains to give a probabilistic interpretation of (1.5).

Proposition 3.3. *Let $\epsilon \in (0, 1)$ be fixed. Consider a cross-section β satisfying (2.1), a probability measure $f_0^\epsilon \in \mathcal{P}_2(\mathbb{R})$, and the corresponding unique solution $(f_t^\epsilon)_{t \geq 0}$ to (1.5). Consider any α -process $(W_t^\epsilon)_{t \geq 0}$ such that for all $t \geq 0$, $\mathcal{L}_\alpha(W_t^\epsilon) = f_t^\epsilon$. Let V_0^ϵ be a \mathcal{F}_0 -measurable random variable with law f_0^ϵ , let N be a $(\mathcal{F}_t)_{t \geq 0}$ -Poisson measure on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $ds d\alpha \beta(\theta) d\theta$ and let $(B_t)_{t \geq 0}$ be a $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion independent of N . Then there exists a unique L^2 -process $(V_t^\epsilon)_{t \geq 0}$ such that for all $t \geq 0$,*

$$(3.3) \quad \begin{aligned} V_t^\epsilon = & V_0^\epsilon + \int_0^t \int_{|\theta| \geq \epsilon} \int_0^1 \left[(\cos \theta - 1) V_{s-}^\epsilon - \sin \theta W_{s-}^\epsilon(\alpha) \right] N(d\theta d\alpha ds) \\ & - b_\epsilon \int_0^t V_s^\epsilon ds + \sqrt{2\mathcal{E}b_\epsilon} B_t, \end{aligned}$$

with b_ϵ defined in (2.8). Furthermore, $\mathcal{L}(V_t^\epsilon) = f_t^\epsilon$ for all $t \geq 0$.

Proof. See Ikeda-Watanabe [17, Theorem 9.1] for existence and uniqueness of $(V_t^\epsilon)_{t \geq 0}$: (3.3) is a classical jumping S.D.E. with Lipschitz coefficients. Let $\varphi \in C_b^2(\mathbb{R})$. By Itô's formula for jump processes (see e.g. Ikeda-Watanabe [17, Theorem 5.1]), we have

$$\begin{aligned} \varphi(V_t^\epsilon) = & \varphi(V_0^\epsilon) + \int_0^t \varphi'(V_s^\epsilon) \left(-b_\epsilon V_s^\epsilon ds + \sqrt{2\mathcal{E}b_\epsilon} dB_s\right) + \mathcal{E}b_\epsilon \int_0^t \varphi''(V_s^\epsilon) ds \\ & + \int_0^t \int_{|\theta| \geq \epsilon} \int_0^1 \left[\varphi(\cos \theta V_{s-}^\epsilon - \sin \theta W_{s-}^\epsilon(\alpha)) - \varphi(V_{s-}^\epsilon) \right] N(d\theta d\alpha ds). \end{aligned}$$

Taking expectations and setting $\mu_t^\epsilon = \mathcal{L}(V_t^\epsilon)$, we get for any $\varphi \in C_b^2(\mathbb{R})$, using that $\mathcal{L}_\alpha(W_t^\epsilon) = f_t^\epsilon$,

$$\begin{aligned} \int_{\mathbb{R}} \varphi(v) \mu_t^\epsilon(dv) &= \int_{\mathbb{R}} \varphi(v) f_0^\epsilon(dv) - b_\epsilon \int_0^t \int_{\mathbb{R}} v \varphi'(v) \mu_s^\epsilon(dv) ds \\ &\quad + \mathcal{E} b_\epsilon \int_0^t \int_{\mathbb{R}} \varphi''(v) \mu_s^\epsilon(dv) ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{|\theta| \geq \epsilon} \left[\varphi(v \cos \theta - v_* \sin \theta) - \varphi(v) \right] \beta(\theta) d\theta \mu_s^\epsilon(dv) f_s^\epsilon(dv_*) ds. \end{aligned}$$

But $(f_t^\epsilon)_{t \geq 0}$ solves the same equation since it solves (1.5) in the sense of Definition 2.1. Since $(f_s^\epsilon)_{s \geq 0}$ is given, this equation is linear and we have uniqueness of the solution. Indeed, we use Proposition A.6 with $a = \mathcal{E} b_\epsilon$, $b = -b_\epsilon$, $r = 0$ and $q(t, v, A) = \int_{-\pi}^\pi \int_{\mathbb{R}} \mathbb{1}_{|\theta| > \epsilon} \mathbb{1}_A(v \cos \theta - v_* \sin \theta) f_t^\epsilon(dv_*) \beta(\theta) d\theta$ for all Borel subset $A \subset \mathbb{R}$, which satisfies $\sup_{t, v} q(t, v, \mathbb{R}) = \int_{|\theta| > \epsilon} \beta(\theta) d\theta < \infty$ and $\sup_{t \geq 0} \int_{\mathbb{R}} (h^2 + 2vh) q(t, v, dh) = \int_{|\theta| > \epsilon} \sin^2 \theta \beta(\theta) d\theta \left(\int_{\mathbb{R}} v_*^2 f_0(dv_*) + v^2 \right) \leq C(1 + v^2)$. Finally, we get $(\mu_t^\epsilon)_{t \geq 0} = (f_t^\epsilon)_{t \geq 0}$. \square

4. THE GRAZING COLLISIONS LIMIT

We consider a family of cross sections $(\beta_\epsilon)_{\epsilon \in (0, 1)}$ with $\int_{-\pi}^\pi \theta^2 \beta_\epsilon(\theta) d\theta = 1$ and $\int_{-\pi}^\pi \theta^4 \beta_\epsilon(\theta) d\theta \xrightarrow{\epsilon \rightarrow 0} 0$. Let $g_0 \in \mathcal{P}_4(\mathbb{R})$. For any $\epsilon \in (0, 1)$, we consider $(g_t^\epsilon)_{t \geq 0}$ the unique solution of (1.1) with cross section β_ϵ starting from g_0 . We also consider $(g_t)_{t \geq 0}$ the unique solution of (1.4) starting from g_0 . For $\epsilon \in (0, 1)$, we consider a \mathcal{F}_0 -measurable random variable V_0 with law g_0 , and a $(\mathcal{F}_t)_{t \geq 0}$ -Poisson measure N^ϵ on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $ds d\alpha \beta_\epsilon(\theta) d\theta$. We also consider an α -process $(W_t^\epsilon)_{t \geq 0}$ such that $\mathcal{L}_\alpha(W_t^\epsilon) = g_t^\epsilon$ for all $t \geq 0$. Let $(B_t)_{t \geq 0}$ be a $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion. We consider $(V_t^\epsilon)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ solutions of the following S.D.E.s

$$V_t^\epsilon = V_0 + \int_0^t \int_0^1 \int_{-\pi}^\pi \left[(\cos \theta - 1) V_{s-}^\epsilon - \sin \theta W_{s-}^\epsilon(\alpha) \right] \tilde{N}^\epsilon(ds d\alpha d\theta) - b_\epsilon \int_0^t V_s^\epsilon ds,$$

where $b_\epsilon = \int_{-\pi}^\pi (1 - \cos \theta) \beta_\epsilon(\theta) d\theta$ and

$$(4.1) \quad Y_t = V_0 - \frac{1}{2} \int_0^t Y_s ds + \sqrt{\mathcal{E}} B_t.$$

Theorem 2.3 is a corollary of the following statement.

Theorem 4.1. *For any $t \geq 0$ and any $\epsilon \in (0, 1)$, we can couple the Poisson measure N^ϵ and the Brownian motion B in such a way that*

$$\begin{aligned} \mathbb{E}[(V_t^\epsilon - Y_t)^2] &\leq 4 \left[\frac{\mathcal{E} \int_{-\pi}^\pi (1 - \cos \theta)^2 \beta_\epsilon(\theta) d\theta}{2b_\epsilon} + C \frac{(\mathbb{E}(V_0^4) + 3\mathcal{E}^2) \int_{-\pi}^\pi \sin^4 \theta \beta_\epsilon(\theta) d\theta}{\mathcal{E} \gamma_\epsilon} \right. \\ &\quad \left. + \mathcal{E} \left(|\ln(2b_\epsilon)|^2 + \left(\frac{\gamma_\epsilon}{2b_\epsilon} + 1 \right) |2b_\epsilon - 1| + 2|\gamma_\epsilon - 1| \right) \right], \end{aligned}$$

where $b_\epsilon = \int_{-\pi}^\pi (1 - \cos \theta) \beta_\epsilon(\theta) d\theta$, $\gamma_\epsilon = \int_{-\pi}^\pi \sin^2 \theta \beta_\epsilon(\theta) d\theta$, $\mathcal{E} = \mathbb{E}[V_0^2]$ and C is a universal constant.

Let us insist on the fact that the coupling between N^ϵ and B depends on t . Assuming for a moment that this result holds true, we can prove Theorem 2.3.

Proof of Theorem 2.3. First recalling that $\mathcal{L}(V_t^\epsilon) = g_t^\epsilon$ by Proposition 3.1 and $\mathcal{L}(Y_t) = g_t$ by Proposition 3.2, we have $W_2^2(g_t^\epsilon, g_t) \leq \mathbb{E}[(V_t^\epsilon - Y_t)^2]$. If $\int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta > 1$, we have $\mathbb{E}[(V_t^\epsilon - Y_t)^2] \leq 2\mathbb{E}[(V_t^\epsilon)^2] + 2\mathbb{E}[Y_t^2] = 4\mathcal{E} \leq 4\mathcal{E} \int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta$. We now suppose that $\int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta < 1$. Using the Taylor-Lagrange inequality, we have

$$\begin{aligned} |1 - \cos \theta| &\leq \theta^2/2, & |2(1 - \cos \theta) - \theta^2| &\leq \theta^4/12 \\ |\sin \theta| &\leq |\theta| & \text{and } |\sin^2 \theta - \theta^2| &\leq \theta^4/3. \end{aligned}$$

Using these inequalities, we get

$$\int_{-\pi}^{\pi} (1 - \cos \theta)^2 \beta_\epsilon(\theta) d\theta \leq \int_{-\pi}^{\pi} (\theta^4/4) \beta_\epsilon(\theta) d\theta, \quad \int_{-\pi}^{\pi} \sin^4 \theta \beta_\epsilon(\theta) d\theta \leq \int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta,$$

and, recalling that $\int_{-\pi}^{\pi} \theta^2 \beta_\epsilon(\theta) d\theta = 1$,

$$|2b_\epsilon - 1| \leq \int_{-\pi}^{\pi} (\theta^4/12) \beta_\epsilon(\theta) d\theta, \quad |\gamma_\epsilon - 1| \leq \int_{-\pi}^{\pi} (\theta^4/3) \beta_\epsilon(\theta) d\theta.$$

Since $\int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta < 1$ by assumption, we have $2b_\epsilon \geq 11/12$, $2/3 \leq \gamma_\epsilon \leq 4/3$ and $2b_\epsilon - 1 \in [-1/12, 1/12]$ which allows us to write $|\ln(2b_\epsilon)|^2 = |\ln(1 + (2b_\epsilon - 1))|^2 \leq 4|2b_\epsilon - 1|^2$. We thus get $\mathbb{E}[(V_t^\epsilon - Y_t)^2] \leq C(\mathcal{E} + \frac{\mathbb{E}(V_0^4)}{\mathcal{E}}) \int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta$, which concludes the proof, since $\mathcal{E}^2 = \mathbb{E}[V_0^2]^2 \leq \mathbb{E}[V_0^4]$ by the Cauchy-Schwarz inequality. \square

It remains to prove Theorem 4.1. Let us start with the following lemma.

Lemma 4.2. *For $\epsilon \in (0, 1)$, let Y^ϵ be the unique solution of*

$$(4.2) \quad Y_t^\epsilon = V_0 - b_\epsilon \int_0^t Y_s^\epsilon ds - \int_0^t \int_0^1 \int_{-\pi}^{\pi} \sin \theta W_{s-}^\epsilon(\alpha) \tilde{N}^\epsilon(ds d\alpha d\theta)$$

(since W^ϵ is a given α -process, this is a classical S.D.E. with Lipschitz coefficients). Then for all $t \geq 0$,

$$\mathbb{E}\left((V_t^\epsilon - Y_t^\epsilon)^2\right) \leq \frac{\mathcal{E} \int_{-\pi}^{\pi} (1 - \cos \theta)^2 \beta_\epsilon(\theta) d\theta}{2b_\epsilon}.$$

Proof. Observing that

$$V_t^\epsilon - Y_t^\epsilon = \int_0^t \int_0^1 \int_{-\pi}^{\pi} (\cos \theta - 1) V_{s-}^\epsilon \tilde{N}^\epsilon(ds d\alpha d\theta) - b_\epsilon \int_0^t (V_s^\epsilon - Y_s^\epsilon) ds,$$

we get by Itô's formula

$$\begin{aligned}
(V_t^\epsilon - Y_t^\epsilon)^2 &= \int_0^t \int_0^1 \int_{-\pi}^\pi \left[\left(V_{s-}^\epsilon - Y_{s-}^\epsilon + (\cos \theta - 1) V_{s-}^\epsilon \right)^2 \right. \\
&\quad \left. - (V_{s-}^\epsilon - Y_{s-}^\epsilon)^2 \right] \tilde{N}^\epsilon(ds d\alpha d\theta) \\
&\quad + \int_0^t \int_0^1 \int_{-\pi}^\pi \left[\left(V_s^\epsilon - Y_s^\epsilon + (\cos \theta - 1) V_s^\epsilon \right)^2 - (V_s^\epsilon - Y_s^\epsilon)^2 \right. \\
&\quad \left. - 2(V_s^\epsilon - Y_s^\epsilon)(\cos \theta - 1) V_s^\epsilon \right] ds d\alpha \beta_\epsilon(\theta) d\theta \\
&\quad - 2b_\epsilon \int_0^t (V_s^\epsilon - Y_s^\epsilon)^2 ds \\
&= M_t^\epsilon + \int_0^t \int_{-\pi}^\pi (\cos \theta - 1)^2 (V_s^\epsilon)^2 ds \beta_\epsilon(\theta) d\theta - 2b_\epsilon \int_0^t (V_s^\epsilon - Y_s^\epsilon)^2 ds,
\end{aligned}$$

where M_t^ϵ is a martingale with mean 0. So using that $\mathbb{E}[(V_t^\epsilon)^2] = \int_{\mathbb{R}} v^2 g_t^\epsilon(dv) = \int_{\mathbb{R}} v^2 g_0(dv) = \mathcal{E}$ for all $t \geq 0$ by (2.9), we have

$$\begin{aligned}
\mathbb{E}[(V_t^\epsilon - Y_t^\epsilon)^2] &= \int_0^t \int_{-\pi}^\pi (1 - \cos \theta)^2 \mathbb{E}[(V_s^\epsilon)^2] \beta_\epsilon(\theta) d\theta ds - 2b_\epsilon \int_0^t \mathbb{E}[(V_s^\epsilon - Y_s^\epsilon)^2] ds \\
&= \mathcal{E} t \int_{-\pi}^\pi (1 - \cos \theta)^2 \beta_\epsilon(\theta) d\theta - 2b_\epsilon \int_0^t \mathbb{E}[(V_s^\epsilon - Y_s^\epsilon)^2] ds.
\end{aligned}$$

Differentiating this equality with respect to t , we find an O.D.E. that can be solved explicitly. This gives

$$\mathbb{E}[(V_t^\epsilon - Y_t^\epsilon)^2] = \frac{\mathcal{E} \int_{-\pi}^\pi (1 - \cos \theta)^2 \beta_\epsilon(\theta) d\theta}{2b_\epsilon} (1 - e^{-2b_\epsilon t}).$$

The conclusion follows. \square

In the following lemma, using Corollary A.2, we will find a suitable coupling between our Poisson measure N^ϵ and our Brownian motion B .

Lemma 4.3. *Let \tilde{Y}^ϵ be the unique solution of*

$$(4.3) \quad \tilde{Y}_t^\epsilon = V_0 - b_\epsilon \int_0^t \tilde{Y}_s^\epsilon ds + \sqrt{\mathcal{E} \gamma_\epsilon} B_t.$$

We consider the process Y^ϵ defined in Lemma 4.2. For any $\epsilon \in (0, 1)$ and for each $t \geq 0$, we can couple the Poisson measure N^ϵ and the Brownian motion B in such a way that

$$\mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t^\epsilon)^2] \leq C \frac{(\mathbb{E}(V_0^4) + 3\mathcal{E}^2) \int_{-\pi}^\pi \sin^4 \theta \beta_\epsilon(\theta) d\theta}{\mathcal{E} \gamma_\epsilon},$$

where C is a universal constant and $\gamma_\epsilon = \int_{-\pi}^\pi \sin^2 \theta \beta_\epsilon(\theta) d\theta$.

Observe that for each t we need a suitable coupling. We are not able to find a coupling working simultaneously for all values of t .

Proof. Applying Itô's formula, we get $\tilde{Y}_t^\epsilon e^{b_\epsilon t} = V_0 + \sqrt{\mathcal{E} \gamma_\epsilon} \int_0^t e^{b_\epsilon s} dB_s$ and $Y_t^\epsilon e^{b_\epsilon t} = V_0 - \int_0^t \int_0^1 \int_{-\pi}^\pi e^{b_\epsilon s} \sin \theta W_{s-}^\epsilon(\alpha) \tilde{N}^\epsilon(ds d\alpha d\theta)$. We observe that the random variable $\sqrt{\mathcal{E} \gamma_\epsilon} \int_0^t e^{b_\epsilon s} dB_s$ follows a centered normal law with variance $\mathcal{E} \gamma_\epsilon \int_0^t e^{2b_\epsilon s}$

which is equal to $\int_0^t \int_0^1 \int_{-\pi}^\pi e^{2b_\epsilon s} \sin^2 \theta (W_s^\epsilon(\alpha))^2 \beta_\epsilon(\theta) d\theta d\alpha ds$ because $\mathcal{L}_\alpha(W_s^\epsilon) = g_s^\epsilon$ and due to (2.9). So using Corollary A.2, we get

$$\begin{aligned} W_2^2(\mathcal{L}(\tilde{Y}_t^\epsilon e^{b_\epsilon t}), \mathcal{L}(Y_t^\epsilon e^{b_\epsilon t})) &\leq C_0 \frac{\int_0^t \int_0^1 \int_{-\pi}^\pi e^{4b_\epsilon s} \sin^4 \theta (W_s^\epsilon(\alpha))^4 \beta_\epsilon(\theta) ds d\alpha d\theta}{\mathcal{E} \gamma_\epsilon \int_0^t e^{2b_\epsilon s} ds} \\ &= C_0 \frac{\int_0^t e^{4b_\epsilon s} \mathbb{E}_\alpha((W_s^\epsilon)^4) ds \int_{-\pi}^\pi \sin^4 \theta \beta_\epsilon(\theta) d\theta}{\mathcal{E} \gamma_\epsilon \int_0^t e^{2b_\epsilon s} ds}. \end{aligned}$$

Using Lemma A.5, since $\mathcal{L}(W_s^\epsilon) = g_s^\epsilon$ and since g^ϵ solves (1.1) (with the cross section β_ϵ), we deduce $\mathbb{E}_\alpha[(W_s^\epsilon)^2] \leq \int_{\mathbb{R}} v^4 g_0(dv) + 3\mathcal{E}^2 = \mathbb{E}[V_0^4] + 3\mathcal{E}^2$ for all $s \geq 0$. Hence, using that $\frac{\int_0^t e^{4b_\epsilon s} ds}{\int_0^t e^{2b_\epsilon s} ds} \leq e^{2b_\epsilon t}$, we have

$$W_2^2(\mathcal{L}(\tilde{Y}_t^\epsilon e^{b_\epsilon t}), \mathcal{L}(Y_t^\epsilon e^{b_\epsilon t})) \leq C_0 \frac{(\mathbb{E}(V_0^4) + 3\mathcal{E}^2) \int_{-\pi}^\pi \sin^4 \theta \beta_\epsilon(\theta) d\theta e^{2b_\epsilon t}}{\mathcal{E} \gamma_\epsilon}.$$

Consequently,

$$W_2^2(\mathcal{L}(\tilde{Y}_t^\epsilon), \mathcal{L}(Y_t^\epsilon)) \leq C_0 \frac{(\mathbb{E}(V_0^4) + 3\mathcal{E}^2) \int_{-\pi}^\pi \sin^4 \theta \beta_\epsilon(\theta) d\theta}{\mathcal{E} \gamma_\epsilon}.$$

To conclude, it suffices to take N^ϵ and B in such a way that $\mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t^\epsilon)^2] = W_2^2(\mathcal{L}(\tilde{Y}_t^\epsilon), \mathcal{L}(Y_t^\epsilon))$. \square

Let us now give the last lemma needed to prove Theorem 4.1.

Lemma 4.4. *Consider the unique solutions Y and \tilde{Y}^ϵ to (4.1) and (4.3) respectively, driven by the same Brownian motion B . Then for all $t \geq 0$ fixed and for all $\epsilon \in (0, 1)$,*

$$\mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t)^2] \leq \mathcal{E} \left(|\ln(2b_\epsilon)|^2 + \left(\frac{\gamma_\epsilon}{2b_\epsilon} + 1 \right) |2b_\epsilon - 1| + 2|\gamma_\epsilon - 1| \right).$$

Proof. We have $\tilde{Y}_t^\epsilon = V_0 e^{-b_\epsilon t} + \sqrt{\mathcal{E} \gamma_\epsilon} e^{-b_\epsilon t} \int_0^t e^{b_\epsilon s} dB_s$ and $Y_t = V_0 e^{-t/2} + \sqrt{\mathcal{E}} e^{-t/2} \int_0^t e^{s/2} dB_s$ as in the proof of Lemma 4.3. Since B and V_0 are independent, we have

$$\begin{aligned} \mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t)^2] &= \mathbb{E}(V_0^2) (e^{-b_\epsilon t} - e^{-t/2})^2 \\ &\quad + \mathbb{E} \left[\left(\int_0^t (\sqrt{\mathcal{E} \gamma_\epsilon} e^{-b_\epsilon t} e^{b_\epsilon s} - \sqrt{\mathcal{E}} e^{-t/2} e^{s/2}) dB_s \right)^2 \right] \\ &= \mathcal{E} (e^{-b_\epsilon t} - e^{-t/2})^2 + \mathcal{E} \int_0^t (\sqrt{\gamma_\epsilon} e^{-b_\epsilon(t-s)} - e^{-(t-s)/2})^2 ds \\ &= \mathcal{E} (e^{-b_\epsilon t} - e^{-t/2})^2 + \mathcal{E} \int_0^t (\sqrt{\gamma_\epsilon} e^{-b_\epsilon s} - e^{-s/2})^2 ds. \end{aligned}$$

We set $h(t) = (e^{-b_\epsilon t} - e^{-t/2})^2$. The function h reaches its maximum at $t_0 = \ln(2b_\epsilon)/(b_\epsilon - 1/2)$. Moreover, $|h(t_0)| = |e^{-b_\epsilon t_0} - e^{-t_0/2}|^2 \leq |b_\epsilon t_0 - t_0/2|^2 = |t_0|^2 |b_\epsilon - 1/2|^2 = |\ln(2b_\epsilon)|^2$. Next,

$$\begin{aligned}
\int_0^t (\sqrt{\gamma_\epsilon} e^{-b_\epsilon s} - e^{-s/2})^2 ds &\leq \int_0^\infty (\sqrt{\gamma_\epsilon} e^{-b_\epsilon s} - e^{-s/2})^2 ds \\
&= \frac{\gamma_\epsilon}{2b_\epsilon} + 1 - \frac{2\sqrt{\gamma_\epsilon}}{b_\epsilon + 1/2} \\
&= \frac{\gamma_\epsilon(b_\epsilon + 1/2) + 2b_\epsilon(b_\epsilon + 1/2) - 4\sqrt{\gamma_\epsilon}b_\epsilon}{2b_\epsilon(b_\epsilon + 1/2)} \\
&\leq \frac{1}{b_\epsilon} \left[\gamma_\epsilon(1/2 - b_\epsilon) + 2b_\epsilon(\gamma_\epsilon + b_\epsilon + 1/2 - 2\sqrt{\gamma_\epsilon}) \right] \\
&= \frac{\gamma_\epsilon}{2b_\epsilon} (1 - 2b_\epsilon) + 2 \left[(\sqrt{\gamma_\epsilon} - 1)^2 + (b_\epsilon - 1/2) \right] \\
&\leq \left(\frac{\gamma_\epsilon}{2b_\epsilon} + 1 \right) |2b_\epsilon - 1| + 2|\gamma_\epsilon - 1|,
\end{aligned}$$

the last inequality coming from $(\sqrt{x} - \sqrt{y})^2 \leq |x - y|$. The lemma is proved. \square

We can now conclude this section.

Proof of Theorem 4.1. For $\epsilon \in (0, 1)$ and $t \geq 0$ fixed, we take the Poisson measure N^ϵ and the Brownian motion B as in Lemma 4.3 and we consider the processes V^ϵ , Y , Y^ϵ and \tilde{Y}^ϵ build with this N^ϵ and this B . Then, writing

$$\mathbb{E}[(V_t^\epsilon - Y_t)^2] \leq 4 \left[\mathbb{E}[(V_t^\epsilon - Y_t^\epsilon)^2] + \mathbb{E}[(Y_t^\epsilon - \tilde{Y}_t^\epsilon)^2] + \mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t)^2] \right],$$

and using Lemmas 4.2, 4.3 and 4.4, we immediately conclude. \square

5. CUTOFF APPROXIMATION WITH DIFFUSION

The whole section is dedicated to the proof of Theorem 2.5. Let thus $f_0 \in \mathcal{P}_4(\mathbb{R})$ and let β be a cross section satisfying (2.1). We fix $\epsilon \in (0, 1)$, and we consider the solutions $(f_t)_{t \geq 0}$ and $(f_t^\epsilon)_{t \geq 0}$ to (1.1) and (1.5) respectively, both starting from f_0 .

We will proceed as follows. We fix some $t_0 \geq 0$ for the whole proof. We will build some solutions $(V_t)_{t \geq 0}$ and $(V_t^\epsilon)_{t \geq 0}$ to (3.1) and (3.3), both starting from some initial value V_0 with law f_0 , coupled in such a way that $\mathbb{E}[(V_{t_0} - V_{t_0}^\epsilon)^2]$ is as small as possible.

We divide the proof into five steps. In the first step, we introduce the (suitably coupled) processes $(V_t)_{t \geq 0}$, $(V_t^\epsilon)_{t \geq 0}$ as well as an intermediate process $(\tilde{V}_t^\epsilon)_{t \geq 0}$. In Step 2, we upperbound $\mathbb{E}[(V_{t_0}^\epsilon - \tilde{V}_{t_0}^\epsilon)^2]$. Step 3 is dedicated to the study of $\mathbb{E}[(\tilde{V}_{t_0}^\epsilon - V_{t_0})^2]$. In Step 4, we show that $\mathbb{E}[(V_{t_0}^\epsilon - \tilde{V}_{t_0}^\epsilon)(\tilde{V}_{t_0}^\epsilon - V_{t_0})] = 0$. We conclude in Step 5.

In the whole section, we will use the notation

$$(5.1) \quad b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta, \quad c_\epsilon = 2b_\epsilon + \int_{|\theta| \geq \epsilon} \sin^2 \theta \beta(\theta) d\theta, \\ d_\epsilon = \int_{|\theta| \geq \epsilon} \sin^2 \theta \beta(\theta) d\theta \quad \text{and} \quad \gamma_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta)^2 \beta(\theta) d\theta.$$

Step 1: the coupling.

- Let $(\Omega_i, \mathcal{F}^i, (\mathcal{F}_t^i)_{t \geq 0}, \mathbb{P}_i)$, $i = 1, 2$, be two Polish filtered probability spaces satisfying the usual conditions and consider the following filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}^1 \otimes \mathcal{F}^2, (\mathcal{F}_t^1 \otimes \mathcal{F}_t^2)_{t \geq 0}, \mathbb{P}_1 \otimes \mathbb{P}_2)$. We denote by \mathbb{E} the expectation under \mathbb{P} and by \mathbb{E}_i the expectation under \mathbb{P}_i .

- On $(\Omega_1, \mathcal{F}^1, (\mathcal{F}_t^1)_{t \geq 0}, \mathbb{P}_1)$, we consider a f_0 -distributed random variable V_0 \mathcal{F}_0^1 -measurable, as well as a $(\mathcal{F}_t^1)_{t \geq 0}$ -Poisson measure $N_{|\theta| \geq \epsilon}$ on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $ds d\alpha \beta(\theta) \mathbb{1}_{|\theta| \geq \epsilon} d\theta$. We set

$$(5.2) \quad X_t = \int_0^t \int_0^1 \int_{-\pi}^\pi (\cos \theta - 1) N_{|\theta| \geq \epsilon}(ds d\alpha d\theta) - b_\epsilon t,$$

We consider the Doléans-Dade exponential of X , see Jacod-Shiryaev [19, Theorem 4.61], defined by

$$(5.3) \quad Z_t = 1 + \int_0^t Z_{s-} dX_s.$$

There holds

$$(5.4) \quad Z_t = e^{X_t} \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s} = e^{-b_\epsilon t} \prod_{i \geq 1} \cos \theta_i \mathbb{1}_{T_i \leq t},$$

where $(T_i, \theta_i, \alpha_i)_{i \geq 1}$ are the marks of the Poisson measure $N_{|\theta| \geq \epsilon}$. Observe that a.s., $Z_t \neq 0 \forall t \geq 0$, because $\beta(\theta) d\theta$ does not give weight to $\left\{ -\frac{\pi}{2}, \frac{\pi}{2} \right\}$.

Of course, the processes $(X_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ depend on ϵ but we do not write this dependence in order to lighten notations.

- For each $t \geq 0$, we consider some α -random variables W_t and W_t^ϵ with respective laws f_t and f_t^ϵ verifying

$$(5.5) \quad W_2^2(f_t, f_t^\epsilon) = \mathbb{E}_\alpha[(W_t - W_t^\epsilon)^2].$$

- Recall that $t_0 \geq 0$ is fixed. Fix also $\omega_1 \in \Omega_1$. On $(\Omega_2, \mathcal{F}^2, (\mathcal{F}_t^2)_{t \geq 0}, \mathbb{P}_2)$, we consider a $(\mathcal{F}_t^2)_{t \geq 0}$ -Poisson measure $N_{|\theta| < \epsilon}^{\omega_1}$ on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $ds d\alpha \beta(\theta) \mathbb{1}_{|\theta| < \epsilon} d\theta$ and a Brownian motion $(B_t^{\omega_1})_{t \geq 0}$ (we do not write the dependence in t_0 and ϵ) such that:

$$(5.6) \quad W_2^2(\mu_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1}) = \mathbb{E}_2 \left[\left(\int_0^{t_0} \int_0^1 \int_{-\pi}^\pi (Z_{t_0} Z_{s-}^{-1})(\omega_1) \sin(-\theta) W_{s-}(\alpha) \tilde{N}_{|\theta| < \epsilon}^{\omega_1}(\omega_2, ds d\alpha d\theta) \right. \right. \\ \left. \left. - \int_0^{t_0} \sqrt{2\mathcal{E}b_\epsilon}(Z_{t_0} Z_{s-}^{-1})(\omega_1) dB_s^{\omega_1}(\omega_2) \right)^2 \right],$$

where

$$(5.7) \quad \mu_{t_0}^{\omega_1} = \mathcal{L}_2 \left(\int_0^{t_0} \int_0^1 \int_{-\pi}^\pi (Z_{t_0} Z_{s-}^{-1})(\omega_1) \sin(-\theta) W_{s-}(\alpha) \tilde{N}_{|\theta| < \epsilon}^{\omega_1}(\omega_2, ds d\alpha d\theta) \right),$$

$$(5.8) \quad \nu_{t_0}^{\omega_1} = \mathcal{L}_2 \left(\int_0^{t_0} \sqrt{2\mathcal{E}b_\epsilon} (Z_{t_0} Z_s^{-1})(\omega_1) dB_s^{\omega_1}(\omega_2) \right).$$

Here again we do not write the dependence in ϵ of $\mu_{t_0}^{\omega_1}$ and $\nu_{t_0}^{\omega_1}$.

- For $(\omega_1, \omega_2) \in \Omega$, we can now set $N(\omega_1, \omega_2) = N_{|\theta| \geq \epsilon}(\omega_1) + N_{|\theta| < \epsilon}^{\omega_1}(\omega_2)$ and $(B_t(\omega_1, \omega_2))_{t \geq 0} = (B_t^{\omega_1}(\omega_2))_{t \geq 0}$. Clearly, as random objects on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, the process $(B_t)_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion and N is a $(\mathcal{F}_t)_{t \geq 0}$ -Poisson measure on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $ds d\alpha \beta(\theta) d\theta$.

- Setting $\mathcal{E} := \mathbb{E}[V_0^2]$, for $0 < \epsilon < 1$, we consider the processes $(V_t)_{t \geq 0}$, $(V_t^\epsilon)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ solutions to (3.1) and (3.3) with B, N, W, W^ϵ defined previously, both starting from V_0 . We also introduce the process $(\tilde{V}_t^\epsilon)_{t \geq 0}$ solution of the following S.D.E.:

$$(5.9) \quad \begin{aligned} \tilde{V}_t^\epsilon = & V_0 + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} [(\cos \theta - 1) \tilde{V}_{s-}^\epsilon - \sin \theta W_{s-}(\alpha)] N(ds d\alpha d\theta) \\ & - \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \sin \theta W_{s-}(\alpha) \tilde{N}(ds d\alpha d\theta) - b_\epsilon \int_0^t \tilde{V}_s^\epsilon ds. \end{aligned}$$

By Proposition 3.1 and Proposition 3.3, $\mathcal{L}(V_t)$ and $\mathcal{L}(V_t^\epsilon)$ are nothing but f_t and f_t^ϵ respectively. We set $\Delta_t^\epsilon = V_t - \tilde{V}_t^\epsilon$, $\tilde{\Delta}_t^\epsilon = \tilde{V}_t^\epsilon - V_t^\epsilon$ and $\delta_t^\epsilon(\alpha) = W_s(\alpha) - W_s^\epsilon(\alpha)$.

Step 2: the aim is here to prove that

$$(5.10) \quad \mathbb{E} \left((\tilde{\Delta}_{t_0}^\epsilon)^2 \right) \leq d_\epsilon e^{-c_\epsilon t_0} \int_0^{t_0} e^{c_\epsilon s} \mathbb{E}_\alpha(\delta_s^2) ds + C\epsilon^2,$$

where C depends only on \mathcal{E} and $\mathbb{E}[V_0^4]$ and where c_ϵ and d_ϵ are defined in (5.1). Making the difference between (5.9) and (3.3), we get

$$(5.11) \quad \begin{aligned} \tilde{\Delta}_t^\epsilon = & \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} [(\cos \theta - 1) \tilde{\Delta}_{s-}^\epsilon - \sin \theta \delta_{s-}^\epsilon(\alpha)] N(ds d\alpha d\theta) - b_\epsilon \int_0^t \tilde{\Delta}_s^\epsilon ds \\ & - \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \sin \theta W_{s-}(\alpha) \tilde{N}(ds d\alpha d\theta) - \sqrt{2\mathcal{E}b_\epsilon} B_t \\ = & H_t + \int_0^t \tilde{\Delta}_{s-}^\epsilon dX_s, \end{aligned}$$

with $(X_t)_{t \geq 0}$ defined in (5.2) and with

$$\begin{aligned} H_t = & - \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \sin \theta \delta_{s-}^\epsilon(\alpha) N(ds d\alpha d\theta) \\ & - \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \sin \theta W_{s-}(\alpha) \tilde{N}(ds d\alpha d\theta) - \sqrt{2\mathcal{E}b_\epsilon} B_t. \end{aligned}$$

We do not write the dependence in ϵ for H . According to Jacod [18], $\tilde{\Delta}_t^\epsilon = (L_t + D_t)Z_t$, where Z_t was defined in (5.4) and where

$$(5.12) \quad D_t = - \int_0^t \int_0^1 \int_{|\theta| < \epsilon} Z_{s-}^{-1} \sin \theta W_{s-}(\alpha) \tilde{N}(ds d\alpha d\theta) - \sqrt{2\mathcal{E}b_\epsilon} \int_0^t Z_s^{-1} dB_s$$

and

$$(5.13) \quad L_t = - \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} Z_{s-}^{-1} \frac{\sin \theta}{\cos \theta} \delta_{s-}^\epsilon(\alpha) N(ds d\alpha d\theta).$$

To verify this, it suffices to apply Itô's formula and observe that the process $((L_t + D_t)Z_t)_{t \geq 0}$ satisfies the same S.D.E. than $(\tilde{\Delta}_t^\epsilon)_{t \geq 0}$, i.e $(L_t + D_t)Z_t = H_t + \int_0^t (L_{s-} + D_{s-})Z_{s-} dX_s$. This S.D.E. has Lipschitz coefficients and thus has a unique solution. The processes $(D_t)_{t \geq 0}$ and $(L_t)_{t \geq 0}$ depend on ϵ but we do not write this dependence.

Hence

$$(5.14) \quad \mathbb{E}[(\tilde{\Delta}_t^\epsilon)^2] = \mathbb{E}[L_t^2 Z_t^2] + \mathbb{E}[D_t^2 Z_t^2] + 2\mathbb{E}[L_t D_t Z_t^2].$$

- First,

$$(5.15) \quad \mathbb{E}[L_t D_t Z_t^2] = \mathbb{E}_1[\mathbb{E}_2(L_t D_t Z_t^2)] = \mathbb{E}_1[L_t Z_t^2 \mathbb{E}_2(D_t)] = 0,$$

because for $\omega = (\omega_1, \omega_2)$, we have $(L_t D_t Z_t^2)(\omega) = (L_t Z_t^2)(\omega_1) D_t(\omega_1, \omega_2)$ and because for ω_1 fixed, $\mathbb{E}_2[D_t(\omega_1, \omega_2)] = 0$. Indeed, recall that $(L_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ depend only on ω_1 and that for ω_1 fixed, $N_{|\theta| < \epsilon}(\omega_1, \omega_2)$ is a Poisson measure while $(B_t(\omega_1, \omega_2))_{t \geq 0}$ is a Brownian motion on $(\Omega_2, \mathcal{F}^2, \mathcal{F}_t^2, \mathbb{P}_2)$, so that $(D_t(\omega_1, \omega_2))_{t \geq 0}$ is a centered martingale (for ω_1 fixed).

- By Itô's formula, we have

$$\begin{aligned} Z_t^2 L_t^2 &= -2b_\epsilon \int_0^t Z_s^2 L_s^2 ds \\ &\quad + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \left[\left(Z_{s-} + (\cos \theta - 1) Z_{s-} \right)^2 \left(L_{s-} - \frac{\sin \theta}{\cos \theta} Z_{s-}^{-1} \delta_{s-}^\epsilon(\alpha) \right)^2 \right. \\ &\quad \quad \left. - Z_{s-}^2 L_{s-}^2 \right] N(ds d\alpha d\theta) \\ &= -2b_\epsilon \int_0^t Z_s^2 L_s^2 ds \\ &\quad + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \left[(\cos^2 \theta - 1) Z_{s-}^2 L_{s-}^2 + \sin^2 \theta (\delta_{s-}^\epsilon(\alpha))^2 \right. \\ &\quad \quad \left. - 2 \cos \theta \sin \theta Z_{s-} L_{s-} \delta_{s-}^\epsilon(\alpha) \right] N(ds d\alpha d\theta). \end{aligned}$$

Taking expectations and recalling (5.1), we get (use that $\cos \theta \sin \theta \beta(\theta) d\theta$ is odd)

$$\mathbb{E}(Z_t^2 L_t^2) = -c_\epsilon \int_0^t \mathbb{E}(Z_s^2 L_s^2) ds + d_\epsilon \int_0^t \mathbb{E}_\alpha[(\delta_s^\epsilon)^2] ds.$$

Solving this differential equation, we find

$$(5.16) \quad \mathbb{E}(Z_t^2 L_t^2) = d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} \mathbb{E}_\alpha[(\delta_s^\epsilon)^2] ds.$$

- It remains to compute $\mathbb{E}(Z_{t_0}^2 D_{t_0}^2)$. Recalling (5.12), we directly obtain

$$\begin{aligned} \mathbb{E}(Z_{t_0}^2 D_{t_0}^2) &= \mathbb{E}\left[\left(Z_{t_0} \int_0^{t_0} \int_0^1 \int_{|\theta|<\epsilon} Z_s^{-1} \sin(-\theta) W_{s-}(\alpha) \tilde{N}(ds d\alpha d\theta) \right. \right. \\ &\quad \left. \left. - Z_{t_0} \int_0^{t_0} Z_s^{-1} \sqrt{2\mathcal{E}b_\epsilon} dB_s\right)^2\right] \\ &= \mathbb{E}_1 \left[\mathbb{E}_2 \left[\left(\int_0^{t_0} \int_0^1 \int_{-\pi}^\pi (Z_{t_0} Z_s^{-1})(\omega_1) \sin(-\theta) W_{s-}(\alpha) \tilde{N}_{|\theta|<\epsilon}^{\omega_1}(\omega_2, ds d\alpha d\theta) \right. \right. \right. \\ &\quad \left. \left. - \int_0^{t_0} (Z_{t_0} Z_s^{-1})(\omega_1) \sqrt{2\mathcal{E}b_\epsilon} dB_s^{\omega_1}(\omega_2) \right)^2 \right] \right]. \end{aligned}$$

We thus obtain $\mathbb{E}(Z_{t_0}^2 D_{t_0}^2) = \mathbb{E}_1(W_2^2(\mu_{t_0}, \nu_{t_0}))$, recall (5.6). We consider

$$(5.17) \quad \eta_{t_0}^{\omega_1} := \mathcal{L}_2 \left(\sigma_\epsilon \int_0^{t_0} (Z_{t_0} Z_s^{-1})(\omega_1) dB_s^{\omega_1}(\omega_2) \right),$$

with

$$\sigma_\epsilon = \sqrt{\mathcal{E} \int_{|\theta|<\epsilon} \sin^2 \theta \beta(\theta) d\theta}.$$

Using the triangular inequality, we have

$$W_2^2(\mu_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1}) \leq 2(W_2^2(\mu_{t_0}^{\omega_1}, \eta_{t_0}^{\omega_1}) + W_2^2(\eta_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1})).$$

By Corollary A.2 and since $\mathbb{E}_\alpha(W_s^2) = \mathcal{E}$ for all $s \geq 0$ by the energy conservation, we have (recall that ω_1 is fixed)

$$W_2^2(\mu_{t_0}^{\omega_1}, \eta_{t_0}^{\omega_1}) \leq C_0 \frac{\int_0^{t_0} \int_{|\theta|<\epsilon} (Z_{t_0} Z_s^{-1})^4(\omega_1) \sin^4 \theta \mathbb{E}_\alpha(W_s^4) \beta(\theta) d\theta ds}{\int_0^{t_0} \int_{|\theta|<\epsilon} (Z_{t_0} Z_s^{-1})^2(\omega_1) \sin^2 \theta \mathcal{E} \beta(\theta) d\theta ds}.$$

But $\mathbb{E}_\alpha(W_s^4) \leq \mathbb{E}(V_0^4) + 3\mathcal{E}^2$ by Lemma A.5. Furthermore, recalling (5.4), we have $|Z_{t_0} Z_s^{-1}| = |e^{-b_\epsilon(t_0-s)} \prod_{s \leq T_i \leq t} \cos \theta_i| \leq 1$. Since finally $\sin^2 \theta \leq \theta \leq \epsilon^2$ on $[-\epsilon, \epsilon]$, we easily deduce that for all ω_1 fixed,

$$W_2^2(\mu_{t_0}^{\omega_1}, \eta_{t_0}^{\omega_1}) \leq \frac{C_0(\mathbb{E}(V_0^4) + 3\mathcal{E}^2)}{\mathcal{E}} \epsilon^2.$$

Finally, it obviously holds, recall (5.8) and (5.17), that for all ω_1 fixed,

$$\begin{aligned} W_2^2(\eta_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1}) &\leq \int_0^{t_0} \left(\sqrt{2\mathcal{E}b_\epsilon} - \sqrt{\mathcal{E} \int_{|\theta|<\epsilon} \sin^2 \theta \beta(\theta) d\theta} \right)^2 (Z_s^{-1} Z_{t_0})^2(\omega_1) ds \\ &\leq \mathcal{E} \left| 2b_\epsilon - \int_{|\theta|<\epsilon} \sin^2 \theta \beta(\theta) d\theta \right| \int_0^{t_0} (Z_s^{-1} Z_{t_0})^2(\omega_1) ds. \end{aligned}$$

We used that $(\sqrt{x} - \sqrt{y})^2 \leq |x - y|$. Recalling that $|Z_s^{-1} Z_{t_0}| \leq e^{-b_\epsilon(t_0-s)}$, we easily get $\int_0^{t_0} (Z_s^{-1} Z_{t_0})^2(\omega_1) ds \leq \frac{1}{2b_\epsilon}$. Furthermore, $\left| 2b_\epsilon - \int_{|\theta|<\epsilon} \sin^2 \theta \beta(\theta) d\theta \right| = \left| \int_{|\theta|<\epsilon} (2(1 - \cos \theta) - \sin^2 \theta) \beta(\theta) d\theta \right| \leq \int_{|\theta|<\epsilon} \theta^4 \beta(\theta) d\theta$. Finally, it is easily checked that, since $\epsilon \in (0, 1)$, $b_\epsilon = \int_{|\theta|<\epsilon} (1 - \cos \theta) \beta(\theta) d\theta \geq \frac{1}{4} \int_{|\theta|<\epsilon} \theta^2 \beta(\theta) d\theta$. Hence it

holds that for all ω_1 fixed,

$$W_2^2(\eta_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1}) \leq 2\mathcal{E} \frac{\int_{|\theta|<\epsilon} \theta^4 \beta(\theta) d\theta}{\int_{|\theta|<\epsilon} \theta^2 \beta(\theta) d\theta} \leq 2\mathcal{E}\epsilon^2.$$

We conclude that $W_2^2(\mu_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1}) \leq C\epsilon^2$ (where C depends on \mathcal{E} and $\mathbb{E}[V_0^4]$), whence

$$(5.18) \quad \mathbb{E}[Z_{t_0}^2 D_{t_0}^2] \leq C\epsilon^2.$$

Gathering (5.14), (5.15), (5.16) and (5.18), we deduce (5.10).

Step 3: in this step, we check that

$$(5.19) \quad \mathbb{E}[(\Delta_t^\epsilon)^2] \leq \frac{\mathcal{E}}{4}\epsilon^2$$

for all $t \geq 0$. We first observe that (3.1) can be rewritten as

$$\begin{aligned} V_t = & V_0 + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} [(\cos \theta - 1)V_{s-} - \sin \theta W_{s-}(\alpha)] N(ds d\alpha d\theta) \\ & + \int_0^t \int_0^1 \int_{|\theta| < \epsilon} [(\cos \theta - 1)V_{s-} - \sin \theta W_{s-}(\alpha)] \tilde{N}(ds d\alpha d\theta) - b_\epsilon \int_0^t V_s ds. \end{aligned}$$

Hence, making the difference with (5.9), we find

$$(5.20) \quad \begin{aligned} \Delta_t^\epsilon = & \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} (\cos \theta - 1) \Delta_{s-}^\epsilon N(ds d\alpha d\theta) \\ & + \int_0^t \int_0^1 \int_{|\theta| < \epsilon} (\cos \theta - 1) V_{s-} \tilde{N}(ds d\alpha d\theta) - b_\epsilon \int_0^t \Delta_s^\epsilon ds. \end{aligned}$$

Applying Itô's formula, we get

$$\begin{aligned} (\Delta_t^\epsilon)^2 = & -2b_\epsilon \int_0^t (\Delta_s^\epsilon)^2 ds + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} (\cos^2 \theta - 1) (\Delta_{s-}^\epsilon)^2 N(ds d\alpha d\theta) \\ & + \int_0^t \int_0^1 \int_{|\theta| < \epsilon} [(\Delta_{s-}^\epsilon + (\cos \theta - 1)V_{s-})^2 - (\Delta_{s-}^\epsilon)^2] \tilde{N}(ds d\alpha d\theta) \\ & + \int_0^t \int_0^1 \int_{|\theta| < \epsilon} [(\Delta_s^\epsilon + (\cos \theta - 1)V_s)^2 - (\Delta_s^\epsilon)^2 \\ & \quad - 2\Delta_s^\epsilon (\cos \theta - 1)V_s] \beta(\theta) ds d\alpha d\theta \\ = & -\left(2b_\epsilon \int_0^t (\Delta_s^\epsilon)^2 ds + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \sin^2 \theta (\Delta_{s-}^\epsilon)^2 N(ds d\alpha d\theta)\right) + M_t \\ & + \int_0^t \int_{|\theta| < \epsilon} V_s^2 (1 - \cos \theta)^2 \beta(\theta) d\theta ds, \end{aligned}$$

where $(M_t)_{t \geq 0}$ is a centered martingale. Taking expectations, this yields, recalling (5.1) and that $\mathbb{E}[V_s^2] = \mathcal{E}$ for all $s \geq 0$ by the energy conservation,

$$\mathbb{E}[(\Delta_t^\epsilon)^2] = -c_\epsilon \int_0^t \mathbb{E}[(\Delta_s^\epsilon)^2] ds + \gamma_\epsilon \mathcal{E} t.$$

Thus

$$\mathbb{E}[(\Delta_t^\epsilon)^2] = \frac{\gamma_\epsilon \mathcal{E}}{c_\epsilon} (1 - e^{-c_\epsilon t}) \leq \frac{\gamma_\epsilon \mathcal{E}}{c_\epsilon}.$$

But $\gamma_\epsilon \leq b_\epsilon \frac{\epsilon^2}{2}$ (because for $|\theta| < \epsilon$, $(1 - \cos \theta) \leq \theta^2/2 < \epsilon^2/2$) and $c_\epsilon \geq 2b_\epsilon$. We deduce that $\gamma_\epsilon/c_\epsilon \leq \epsilon^2/4$ and finally get (5.19).

Step 4: we now check that $\mathbb{E}[\Delta_t^\epsilon \tilde{\Delta}_t^\epsilon] = 0$ for all $t \geq 0$. Applying Itô's formula, using (5.20) and (5.11), we have

$$\begin{aligned} \Delta_t^\epsilon \tilde{\Delta}_t^\epsilon &= -\sqrt{2\mathcal{E}b_\epsilon} \int_0^t \Delta_s^\epsilon dB_s - 2b_\epsilon \int_0^t \Delta_s^\epsilon \tilde{\Delta}_s^\epsilon ds \\ &\quad + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \left[(\cos \theta \Delta_{s-}^\epsilon) (\cos \theta \tilde{\Delta}_{s-}^\epsilon - \sin \theta \delta_{s-}^\epsilon(\alpha)) \right. \\ &\quad \quad \left. - \Delta_{s-}^\epsilon \tilde{\Delta}_{s-}^\epsilon \right] N(ds d\alpha d\theta) \\ &\quad + \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \left[\left(\Delta_{s-}^\epsilon + (\cos \theta - 1)V_{s-} \right) \left(\tilde{\Delta}_{s-}^\epsilon - \sin \theta W_{s-}(\alpha) \right) \right. \\ &\quad \quad \left. - \Delta_{s-}^\epsilon \tilde{\Delta}_{s-}^\epsilon \right] \tilde{N}(ds d\alpha d\theta) \\ &\quad + \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \left[\left(\Delta_s^\epsilon + (\cos \theta - 1)V_s \right) \left(\tilde{\Delta}_s^\epsilon - \sin \theta W_s(\alpha) \right) \right. \\ &\quad \quad \left. - \Delta_s^\epsilon \tilde{\Delta}_s^\epsilon - \tilde{\Delta}_s^\epsilon (\cos \theta - 1)V_s + \Delta_s^\epsilon \sin \theta W_s(\alpha) \right] \beta(\theta) ds d\alpha d\theta. \end{aligned}$$

Taking expectation and using that β is even, we get

$$\mathbb{E}[\Delta_t^\epsilon \tilde{\Delta}_t^\epsilon] = -2b_\epsilon \int_0^t \mathbb{E}[\Delta_s^\epsilon \tilde{\Delta}_s^\epsilon] ds + \int_0^t \int_{|\theta| \geq \epsilon} (\cos^2 \theta - 1) \mathbb{E}[\Delta_s^\epsilon \tilde{\Delta}_s^\epsilon] \beta(\theta) d\theta ds.$$

So the function $t \mapsto \mathbb{E}[\Delta_t^\epsilon \tilde{\Delta}_t^\epsilon]$ solves the O.D.E. $y' = -c_\epsilon y$, see (5.1). Since $y(0) = 0$, we easily conclude.

Step 5: conclusion. Using Steps 2, 3 and 4, we find that

$$\begin{aligned} \mathbb{E}[(V_{t_0} - V_{t_0}^\epsilon)^2] &= \mathbb{E}[(\Delta_{t_0}^\epsilon)^2] + \mathbb{E}[(\tilde{\Delta}_{t_0}^\epsilon)^2] + 2\mathbb{E}[\Delta_{t_0}^\epsilon \tilde{\Delta}_{t_0}^\epsilon] \\ &\leq d_\epsilon e^{-c_\epsilon t_0} \int_0^{t_0} e^{c_\epsilon s} \mathbb{E}_\alpha[(\delta_s^\epsilon)^2] ds + K\epsilon^2, \end{aligned}$$

where K depends only on \mathcal{E} and $\mathbb{E}[V_0^4]$. We set $u(t) = W_2^2(f_t, f_t^\epsilon) = \mathbb{E}_\alpha[(\delta_s^\epsilon)^2]$ by (5.5). Since $\mathcal{L}(V_{t_0}) = f_{t_0}$ and $\mathcal{L}(V_{t_0}^\epsilon) = f_{t_0}^\epsilon$, we have $u(t_0) \leq \mathbb{E}[(V_{t_0} - V_{t_0}^\epsilon)^2]$. Since $t_0 \geq 0$ is arbitrary, we get, for all $t \geq 0$,

$$u(t) \leq d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} u(s) ds + K\epsilon^2 =: v(t).$$

Consequently,

$$\begin{aligned} v'(t) &= -c_\epsilon (v(t) - K\epsilon^2) + d_\epsilon u(t) \\ &\leq -c_\epsilon (v(t) - K\epsilon^2) + d_\epsilon v(t) \\ &\leq (d_\epsilon - c_\epsilon) v(t) + c_\epsilon K\epsilon^2. \end{aligned}$$

We first observe that $d_\epsilon \leq c_\epsilon$, so that $v(t) \leq v(0) + Kc_\epsilon \epsilon^2 t \leq K\epsilon^2(1 + c_\epsilon)(1 + t) \leq C\epsilon^2(1 + t)$, because $c_\epsilon \leq \int_{-\pi}^\pi \theta^2 \beta(\theta) d\theta$, see (5.1).

We can also obtain a uniform in time bound. Recall that $v'(t) \leq (d_\epsilon - c_\epsilon)v(t) + c_\epsilon K\epsilon^2 = -2b_\epsilon v(t) + c_\epsilon K\epsilon^2$. We observe in fact that $v'(t) \leq 0$ as soon as $v(t) \geq \frac{c_\epsilon K\epsilon^2}{2b_\epsilon}$. Since $v(0) = K\epsilon^2 \leq \frac{c_\epsilon K\epsilon^2}{2b_\epsilon}$, we classically deduce that $v(t) \leq \frac{c_\epsilon K\epsilon^2}{2b_\epsilon} \leq C\frac{\epsilon^2}{b_\epsilon}$ for all $t \geq 0$.

So we have $W_2^2(f_t, f_t^\epsilon) = u(t) \leq v(t) \leq C \min\left(\epsilon^2(1+t), \frac{\epsilon^2}{b_\epsilon}\right)$ for all $t \geq 0$. To complete the proof of Theorem 2.5, it suffices to observe that $4b_\epsilon \geq \int_{|\theta| < \epsilon} \theta^2 \beta(\theta) d\theta$ for any $\epsilon \in (0, 1)$. \square

6. CONVERGENCE OF THE PARTICLE SYSTEM

In this section, we prove the results about the approximation of the solution of the Kac equation by a system of particles. Let thus $f_0 \in \mathcal{P}_4(\mathbb{R})$ and let β be a cross section satisfying (2.1). We fix $\epsilon \in (0, 1)$, and we consider the solutions $(f_t)_{t \geq 0}$ and $(f_t^\epsilon)_{t \geq 0}$ to (1.1) and (1.5) respectively, both starting from f_0 .

In the first part, we will rewrite the system of particles (2.11) in a suitable way and in the second part, we will introduce a system of i.i.d. particles with law $(f_t^\epsilon)_{t \geq 0}$. Using these systems of particles, we will be able to prove Theorem 2.7 and its corollary. We will end this section with the proof of Proposition 2.9 and with an extension about the Wasserstein distance W_γ for $\gamma \in (1, 2)$.

We recall a useful result (see e.g. Villani [36, Remark 2.19 (iii)]).

Proposition 6.1. *If μ and ν are two probability measures on \mathbb{R} , for $\gamma \geq 1$, we have $W_\gamma^\gamma(\mu, \nu) = \int_0^1 (F_\mu^{-1}(\alpha) - F_\nu^{-1}(\alpha))^\gamma d\alpha$ where $F_\mu(x) = \mu((-\infty, x])$ and $F_\nu(x) = \nu((-\infty, x])$.*

6.1. Another way to write system (2.11). We fix an integer n and we consider:

- a family of i.i.d. random variables $(V_0^i)_{i \in \{1, \dots, n\}}$ with law f_0 ,
- a family of i.i.d. Poisson measures $(N^i)_{i \in \{1, \dots, n\}}$ on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $dt d\alpha \beta(\theta) d\theta$,
- a family of i.i.d. Brownian motions $(B_t^i)_{t \geq 0, i \in \{1, \dots, n\}}$.

For $\epsilon \in (0, 1)$, we consider $(V_t^{i, n, \epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ solution of the following system of S.D.E.s: for $i = 1, \dots, n$, for all $t \geq 0$,

$$\begin{aligned} V_t^{i, n, \epsilon} = & V_0^i + \int_0^t \int_0^1 \int_{|\theta| > \epsilon} \left[(\cos \theta - 1) V_{s-}^{i, n, \epsilon} - \sin \theta \left(F_{s-}^{n, \epsilon} \right)^{-1}(\alpha) \right] N^i(d\theta d\alpha ds) \\ & - b_\epsilon \int_0^t V_s^{i, n, \epsilon} ds + \sqrt{2\mathcal{E}b_\epsilon} B_t^i, \end{aligned}$$

where $F_t^{n, \epsilon} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{V_t^{i, n, \epsilon} \leq x}$, $\mathcal{E} = \int_{\mathbb{R}} v^2 f_0(dv)$ and $b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta$.

This particle system is identical (in law) to the one introduced in (2.11). Indeed, it suffices to note that given $(V_{s-}^{i, n, \epsilon})_{i \in \{1, \dots, n\}}$, the law of $(F_{s-}^{n, \epsilon})^{-1}(\alpha)$ (with α uniformly distributed on $[0, 1]$) is the same as that of $V_{s-}^{j, n, \epsilon}$ (with j uniformly distributed in $\{1, \dots, n\}$).

6.2. A system of i.i.d. particles. For $i \in \{1, \dots, n\}$ and $\epsilon > 0$, we consider the process $(\bar{V}_t^{i,\epsilon})_{t \geq 0}$ solution of the following S.D.E. (with the same random objects V_0^i , N^i and B^i as previously),

$$\begin{aligned} \bar{V}_t^{i,\epsilon} = & V_0^i + \int_0^t \int_0^1 \int_{|\theta| > \epsilon} \left[(\cos \theta - 1) \bar{V}_{s-}^{i,\epsilon} - \sin \theta \left(F_{s-}^\epsilon \right)^{-1}(\alpha) \right] N^i(dsd\alpha d\theta) \\ & - b_\epsilon \int_0^t \bar{V}_s^{i,\epsilon} ds + \sqrt{2\mathcal{E}b_\epsilon} B_t^i, \end{aligned}$$

where $F_t^\epsilon(x) = \int_{-\infty}^x f_t^\epsilon(dv)$.

For each $s \geq 0$, it holds that $\mathcal{L}_\alpha((F_s^\epsilon)^{-1}) = f_s^\epsilon$. Hence we can apply Proposition 3.3 with $W_s^\epsilon = (F_s^\epsilon)^{-1}$ and deduce that for each $i \in \{1, \dots, n\}$, each $t \geq 0$, $\mathcal{L}(\bar{V}_t^{i,\epsilon}) = f_t^\epsilon$. Furthermore, the processes $(\bar{V}_t^i)_{t \geq 0}$ are obviously i.i.d. (for $i = 1, \dots, n$).

6.3. Proof of Theorem 2.7. We start with the following result.

Proposition 6.2. *We set $\mu_t^{n,\epsilon} = \frac{1}{n} \sum_1^n \delta_{V_t^{i,n,\epsilon}}$ and $\bar{\mu}_t^{n,\epsilon} = \frac{1}{n} \sum_1^n \delta_{\bar{V}_t^{i,\epsilon}}$. Then for any $T > 0$,*

$$\sup_{[0,T]} \mathbb{E} \left[W_2^2(f_t^\epsilon, \mu_t^{n,\epsilon}) \right] \leq C(1+T)^2 \sup_{[0,T]} \mathbb{E} \left[W_2^2(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon}) \right],$$

where C depends only on $\int_0^\pi \theta^2 \beta(\theta) d\theta$.

Proof. To lighten notation, we set $V_t^i = V_t^{i,n,\epsilon}$ and $\bar{V}_t^i = \bar{V}_t^{i,\epsilon}$ for the whole proof. By the triangular inequality, we have $W_2(f_t^\epsilon, \mu_t^{n,\epsilon}) \leq W_2(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon}) + W_2(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon})$. Hence, by squaring and taking expectations

$$(6.1) \quad \begin{aligned} \mathbb{E} \left[W_2^2(f_t^\epsilon, \mu_t^{n,\epsilon}) \right] \leq & \mathbb{E} \left[W_2^2(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon}) \right] + \mathbb{E} \left[W_2^2(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon}) \right] \\ & + 2\mathbb{E} \left[W_2(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon}) W_2(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon}) \right]. \end{aligned}$$

Using the fact that $W_2^2\left(\frac{1}{n} \sum_1^n \delta_{x_i}, \frac{1}{n} \sum_1^n \delta_{y_i}\right) \leq \frac{1}{n} \sum_1^n |x_i - y_i|^2$, we have

$$\mathbb{E} \left[W_2^2(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon}) \right] \leq \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n |\bar{V}_t^i - V_t^i|^2 \right) = \mathbb{E} \left(|\bar{V}_t^1 - V_t^1|^2 \right).$$

We set $\Delta_t = \bar{V}_t^1 - V_t^1$. It holds that

$$\Delta_t = \int_0^t \int_0^1 \int_{|\theta| > \epsilon} \left[(\cos \theta - 1) \Delta_{s-} - \sin \theta \delta_{s-}(\alpha) \right] N^1(dsd\alpha d\theta) - b_\epsilon \int_0^t \Delta_s ds,$$

where $\delta_t(\alpha) = (F_t^{n,\epsilon})^{-1}(\alpha) - (F_t^\epsilon)^{-1}(\alpha)$. Applying Itô's formula, we get

$$\begin{aligned} \Delta_t^2 = & \int_0^t \int_0^1 \int_{|\theta| > \epsilon} \left[(\Delta_s + (\cos \theta - 1) \Delta_s - \sin \theta \delta_s(\alpha))^2 - \Delta_s^2 \right] N^1(dsd\alpha d\theta) \\ & - 2b_\epsilon \int_0^t \Delta_s^2 ds. \end{aligned}$$

Taking expectations and using Proposition 6.1, we get, with c_ϵ and d_ϵ defined in (5.1),

$$\begin{aligned} v(t) &:= \mathbb{E}(\Delta_t^2) = -c_\epsilon \int_0^t \mathbb{E}(\Delta_s^2) ds + d_\epsilon \int_0^t \mathbb{E} \left(\int_0^1 \delta_s^2(\alpha) d\alpha \right) ds \\ &= -c_\epsilon \int_0^t v(s) ds + d_\epsilon \int_0^t \mathbb{E}[W_2^2(f_s^\epsilon, \mu_s^{n,\epsilon})] ds \\ &= d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} \mathbb{E}[W_2^2(f_s^\epsilon, \mu_s^{n,\epsilon})] ds, \end{aligned}$$

the last equality being obtained by solving the differential equation satisfied by v . If we set $u(t) := \mathbb{E}[W_2^2(f_t^\epsilon, \mu_t^{n,\epsilon})]$, $s_n := \sup_{[0,T]} \mathbb{E}[W_2^2(f_s^\epsilon, \bar{\mu}_s^{n,\epsilon})]$ and if we return to (6.1), we thus find, for all $t \in [0, T]$,

$$\begin{aligned} u(t) &\leq s_n + v(t) + 2\mathbb{E}\left[W_2(f_t^\epsilon, \bar{\mu}_t^n)W_2(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon})\right] \\ &\leq s_n + v(t) + 2\sqrt{s_n}\sqrt{v(t)}, \end{aligned}$$

by the Cauchy-Schwarz inequality. We thus have

$$\begin{aligned} v(t) &= d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} u(s) ds \\ &\leq d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} \left(s_n + v(s) + 2\sqrt{s_n}\sqrt{v(s)} \right) ds \\ &\leq s_n + d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} \left(v(s) + 2\sqrt{s_n}\sqrt{v(s)} \right) ds =: w(t). \end{aligned}$$

We used that $d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} ds = \frac{d_\epsilon}{c_\epsilon} (1 - e^{-c_\epsilon t}) \leq 1$, since $d_\epsilon \leq c_\epsilon$, recall (5.1). Differentiating w , we get

$$\begin{aligned} w'(t) &= -c_\epsilon (w(t) - s_n) + d_\epsilon (v(t) + 2\sqrt{s_n}\sqrt{v(t)}) \\ &\leq c_\epsilon s_n + w(t)(-c_\epsilon + d_\epsilon) + 2d_\epsilon \sqrt{s_n}\sqrt{w(t)} \\ &\leq c_\epsilon s_n + 2d_\epsilon \sqrt{s_n}\sqrt{w(t)} \\ &\leq a s_n + 2a \sqrt{s_n}\sqrt{w(t)}, \end{aligned}$$

where $a = \int_{-\pi}^\pi \theta^2 \beta(\theta) d\theta$.

Putting $x(t) = w(t)/s_n$, we deduce that $x(0) = 1$ and $x'(t) \leq a(1 + 2\sqrt{x(t)}) \leq 2a(1 + \sqrt{x(t)}) \leq 4a\sqrt{1+x(t)}$, whence $\sqrt{1+x(t)} - \sqrt{1+x(0)} \leq 2at$, which gives $x(t) \leq C(1+t)^2$, and so $w(t) \leq C(1+t)^2 s_n$.

To summarize, we have $v(t) \leq w(t) \leq C s_n (1+t)^2$ and, for all $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}\left[W_2^2(f_t^\epsilon, \mu_t^{n,\epsilon})\right] &= u(t) \leq s_n + v(t) + 2\sqrt{s_n}\sqrt{v(t)} \\ &\leq s_n + C s_n (1+t)^2 + 2\sqrt{s_n}\sqrt{C}\sqrt{s_n}(1+t) \\ &\leq C s_n (1+t)^2 \\ &= C \sup_{[0,T]} \mathbb{E}\left[W_2^2(f_s^\epsilon, \bar{\mu}_s^{n,\epsilon})\right] (1+t)^2. \end{aligned}$$

This concludes the proof. \square

Theorem 2.7 follows almost immediately.

Proof of Theorem 2.7. For each $t \geq 0$, we consider an i.i.d. sequence $(\hat{V}_t^i)_{i \in \{1, \dots, n\}}$ with law f_t such that for each i , $\mathbb{E}[(\hat{V}_t^i - \bar{V}_t^{i, \epsilon})^2] = W_2^2(f_t, f_t^\epsilon)$. Then we set $\mu_t^n = \frac{1}{n} \sum_{i=1}^n \delta_{\hat{V}_t^i}$. Using the triangular inequality, Theorem 2.5 and Proposition 6.2, we have

$$(6.2) \quad \begin{aligned} \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n, \epsilon}) \right] &\leq 2 \sup_{[0, T]} W_2^2(f_t, f_t^\epsilon) + 2 \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t^\epsilon, \mu_t^{n, \epsilon}) \right] \\ &\leq C(1+T)\epsilon^2 + C(1+T)^2 \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t^\epsilon, \bar{\mu}_t^{n, \epsilon}) \right]. \end{aligned}$$

We use again the triangular inequality to obtain

$$\mathbb{E} \left[W_2^2(f_t^\epsilon, \bar{\mu}_t^{n, \epsilon}) \right] \leq 4 \left(W_2^2(f_t^\epsilon, f_t) + \mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right] + \mathbb{E} \left[W_2^2(\mu_t^n, \bar{\mu}_t^{n, \epsilon}) \right] \right).$$

But $\mathbb{E} \left[W_2^2(\mu_t^n, \bar{\mu}_t^{n, \epsilon}) \right] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left(|\hat{V}_t^i - \bar{V}_t^{i, \epsilon}|^2 \right) = W_2^2(f_t, f_t^\epsilon)$. So using Theorem 2.5, we get

$$(6.3) \quad \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t^\epsilon, \bar{\mu}_t^{n, \epsilon}) \right] \leq C \left((1+T)\epsilon^2 + \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right] \right).$$

Inserting (6.3) in (6.2), we obtain

$$\sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n, \epsilon}) \right] \leq C(1+T)^3 \left(\epsilon^2 + \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right] \right),$$

which concludes the proof. \square

Finally, we give the proof of Corollary 2.8.

Proof of Corollary 2.8 It suffices to apply Theorem 2.7, Lemma A.4 with $\gamma = 2$ and $q = p - \gamma$, and Lemma A.5. \square

6.4. Other Wasserstein distances. The first part of the following result is Proposition 2.9 and in the second part, we give some estimates about $\mathbb{E}[W_\gamma^\gamma(f_t, \mu_t^{n, \epsilon})]$ for $\gamma \in (1, 2)$.

Proposition 6.3. *Adopt the same notation as in Theorem 2.7.*

(i) *If we assume that $f_0 \in \mathcal{P}_4(\mathbb{R})$ and if $\int_0^\pi \theta \beta(\theta) d\theta < \infty$, then*

$$\sup_{[0, T]} \mathbb{E}[W_1(f_t, \mu_t^{n, \epsilon})] \leq C_T \left(\epsilon + \frac{1}{\sqrt{n}} \right),$$

where C_T depends only on T , f_0 and β .

(ii) *If $f_0 \in \mathcal{P}_p(\mathbb{R})$ for some even $p \geq 4$ and if $\int_0^\pi \theta^\gamma \beta(\theta) d\theta < \infty$ for some $\gamma \in (1, 2)$, then*

$$\sup_{[0, T]} \mathbb{E}[W_\gamma^\gamma(f_t, \mu_t^{n, \epsilon})] \leq C_T \left(\epsilon^\gamma + \frac{1}{n^{\frac{p-\gamma}{2p-2}}} \right),$$

where C_T depends only on T , f_0 , β , p and γ .

Proof. Let $\gamma \in [1, 2)$ be fixed. We assume that $\int_0^\pi \theta^\gamma \beta(\theta) d\theta < \infty$ and we set $V_t^i = V_t^{i,n,\epsilon}$ and $\bar{V}_t^i = \bar{V}_t^{i,\epsilon}$ for the whole proof to lighten notation.

Step 1: we first prove that

$$I_{\gamma,\epsilon}(x) := \int_{|\theta|>\epsilon} [|\cos \theta - x \sin \theta|^\gamma - 1] \beta(\theta) d\theta \leq C(1 + |x|^\gamma).$$

To this end, we set $J_{\gamma,\epsilon}(x) = \int_{|\theta|>\epsilon} [1 - x\theta|^\gamma - 1] \beta(\theta) d\theta$. Using the inequality $|a^\gamma - b^\gamma| \leq C|a - b|(a^{\gamma-1} + b^{\gamma-1})$, we get

$$\begin{aligned} |I_{\gamma,\epsilon}(x) - J_{\gamma,\epsilon}(x)| &\leq C \int_{|\theta|>\epsilon} |(\cos \theta - 1) - x(\sin \theta - \theta)| \\ &\quad (|\cos \theta - x \sin \theta|^{\gamma-1} + |1 - x\theta|^{\gamma-1}) \beta(\theta) d\theta \\ &\leq C \int_{|\theta|>\epsilon} \theta^2 (1 + |x|)(1 + |x|^{\gamma-1}) \beta(\theta) d\theta \\ &\leq C(1 + |x|)(1 + |x|^{\gamma-1}) \\ &\leq C(1 + |x|^\gamma). \end{aligned}$$

Using the fact that β is even, we can write

$$J_{\gamma,\epsilon}(x) = \int_{|\theta|>\epsilon} [|1 - x\theta|^\gamma - 1 + \gamma x\theta] \beta(\theta) d\theta = J_{\gamma,\epsilon}^1(x) + J_{\gamma,\epsilon}^2(x)$$

with

$$J_{\gamma,\epsilon}^1(x) = \int_{|\theta|>\epsilon, |x\theta|<1/2} [|1 - x\theta|^\gamma - 1 + \gamma x\theta] \beta(\theta) d\theta$$

and

$$J_{\gamma,\epsilon}^2(x) = \int_{|\theta|>\epsilon, |x\theta|>1/2} [|1 - x\theta|^\gamma - 1] \beta(\theta) d\theta.$$

By Taylor's formula, we get, observing that $|x^2\theta^2| \leq |x|^\gamma |\theta|^\gamma$ if $|x\theta| < 1/2$,

$$J_{\gamma,\epsilon}^1(x) \leq C \int_{|\theta|>\epsilon, |x\theta|<1/2} x^2 \theta^2 \beta(\theta) d\theta \leq C|x|^\gamma \int_{-\pi}^\pi |\theta|^\gamma \beta(\theta) d\theta.$$

Next, since $|x\theta| > 1/2$ implies $1 + |x\theta|^\gamma \leq (1 + 2^\gamma)|x\theta|^\gamma$,

$$J_{\gamma,\epsilon}^2(x) \leq C \int_{|\theta|>\epsilon, |x\theta|>1/2} [1 + |x\theta|^\gamma] \beta(\theta) d\theta \leq C|x|^\gamma \int_{-\pi}^\pi |\theta|^\gamma \beta(\theta) d\theta.$$

We thus have $J_{\gamma,\epsilon}(x) \leq C|x|^\gamma$ and hence $I_{\gamma,\epsilon}(x) \leq C(1 + |x|^\gamma)$.

Step 2: using Step 1, we now prove

$$\mathbb{E} \left[W_\gamma^\gamma(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon}) \right] \leq C e^{Ct} \int_0^t \mathbb{E} \left[W_\gamma^\gamma(f_s^\epsilon, \mu_s^{n,\epsilon}) \right] ds,$$

for all $t \geq 0$. We have $\mathbb{E} \left[W_\gamma^\gamma(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon}) \right] \leq \mathbb{E}(|\Delta_t|^\gamma)$, where $\Delta_t = \bar{V}_t^1 - V_t^1$. It holds

$$\Delta_t = \int_0^t \int_0^1 \int_{|\theta|>\epsilon} [(\cos \theta - 1) \Delta_{s-} - \sin \theta \delta_{s-}(\alpha)] N^1(ds d\alpha d\theta) - b_\epsilon \int_0^t \Delta_s ds,$$

where $\delta_t(\alpha) = (F_t^{n,\epsilon})^{-1}(\alpha) - (F_t^\epsilon)^{-1}(\alpha)$. By Itô's formula,

$$\begin{aligned} \mathbb{E}[|\Delta_t|^\gamma] &= \mathbb{E}\left[\int_0^t \int_0^1 \int_{|\theta|>\epsilon} [|\cos\theta\Delta_s - \sin\theta\delta_s(\alpha)|^\gamma - |\Delta_s|^\gamma] \beta(\theta) ds d\alpha d\theta\right] \\ &\quad - \gamma b_\epsilon \int_0^t \mathbb{E}[|\Delta_s|^\gamma] ds \\ &\leq \mathbb{E}\left[\int_0^t \int_0^1 |\Delta_s|^\gamma \int_{|\theta|>\epsilon} [|\cos\theta - \sin\theta \frac{\delta_s(\alpha)}{\Delta_s}|^\gamma - 1] \beta(\theta) ds d\alpha d\theta\right]. \end{aligned}$$

Using Step 1 and then Proposition 6.1, we get

$$\begin{aligned} \mathbb{E}[|\Delta_t|^\gamma] &\leq C \mathbb{E}\left[\int_0^t \int_0^1 |\Delta_s|^\gamma \left(1 + \frac{|\delta_s(\alpha)|^\gamma}{|\Delta_s|^\gamma}\right) d\alpha ds\right] \\ &\leq C \int_0^t \mathbb{E}[|\Delta_s|^\gamma] ds + C \int_0^t \mathbb{E}\left[W_\gamma^\gamma(f_s^\epsilon, \mu_s^{n,\epsilon})\right] ds. \end{aligned}$$

We conclude by Grönwall's lemma.

Step 3: using very similar arguments as in the proof of Theorem 2.7 and observing that $W_\gamma(f_t, f_t^\epsilon) \leq W_2(f_t, f_t^\epsilon)$, we get

$$\mathbb{E}\left[W_\gamma^\gamma(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon})\right] \leq C(1+t)^{\gamma/2} \left[\epsilon^\gamma + \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^n)\right]\right],$$

for all $t \geq 0$, where μ_t^n is the empirical measure of a sequence of i.i.d. random variables with law f_t .

Step 4: the aim of this step is to prove that

$$(6.4) \quad \sup_{[0,T]} \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^{n,\epsilon})\right] \leq C_T \left(\epsilon^\gamma + \sup_{[0,T]} \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^n)\right]\right).$$

Using the triangular inequality, Theorem 2.5, Step 2 and Step 3, we have

$$\begin{aligned} \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^{n,\epsilon})\right] &\leq C \left(W_\gamma^\gamma(f_t, f_t^\epsilon) + \mathbb{E}\left[W_\gamma^\gamma(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon})\right] + \mathbb{E}\left[W_\gamma^\gamma(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon})\right]\right) \\ &\leq C_T \left(\epsilon^\gamma + \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^n)\right] + C_T \int_0^t \mathbb{E}\left[W_\gamma^\gamma(f_s^\epsilon, \mu_s^{n,\epsilon})\right] ds\right). \end{aligned}$$

Using again the triangular inequality and Theorem 2.5, we get

$$\begin{aligned} \mathbb{E}\left[W_\gamma^\gamma(f_s^\epsilon, \mu_s^{n,\epsilon})\right] &\leq C \left(\mathbb{E}\left[W_\gamma^\gamma(f_s^\epsilon, f_s)\right] + \mathbb{E}\left[W_\gamma^\gamma(f_s, \mu_s^{n,\epsilon})\right]\right) \\ &\leq C(1+T)^{\gamma/2} \epsilon^\gamma + C \mathbb{E}\left[W_\gamma^\gamma(f_s, \mu_s^{n,\epsilon})\right]. \end{aligned}$$

We thus have

$$\mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^{n,\epsilon})\right] \leq C_T \left(\epsilon^\gamma + \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^n)\right] + \int_0^t \mathbb{E}\left[W_\gamma^\gamma(f_s, \mu_s^{n,\epsilon})\right] ds\right),$$

and we conclude with the help of Grönwall's lemma.

Step 5: we can now prove (i). Since $f_0 \in \mathcal{P}_4(\mathbb{R})$, Lemma A.5 implies that $\sup_{t \geq 0} \int_{\mathbb{R}} v^4 f_t(dv) < \infty$, whence $\sup_{t \geq 0} \mathbb{E}\left[W_1(f_t, \mu_t^n)\right] \leq \frac{C}{\sqrt{n}}$ by Lemma A.3. Inserting this in (6.4), we easily conclude.

Step 6: we finally prove (ii). Since $f_0 \in \mathcal{P}_p(\mathbb{R})$, Lemma A.5 implies that $\sup_{t \geq 0} \int_{\mathbb{R}} v^p f_t(dv) < \infty$, whence $\sup_{t \geq 0} \mathbb{E} \left[W_\gamma^\gamma(f_t, \mu_t^n) \right] \leq \frac{C}{n^{\frac{p-\gamma}{2p-2}}}$ by Lemma A.4. Inserting this in (6.4), we easily conclude. \square

7. NUMERICAL RESULTS

We consider here the cross section $\beta(\theta) = |\theta|^{-1-\nu}$, with $0 < \nu < 2$. Let f_0 be a probability measure admitting a moment of order 4. We fix an integer n , a small parameter $\epsilon > 0$, and we take the same notation as in Section 2.4.

We simulate two systems of particles : the system $(V_t^{i,n,\epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ described in Section 2.4 (system with diffusion) and the following system without diffusion: for $i \in \{1, \dots, n\}$ and $t \geq 0$,

$$\tilde{V}_t^{i,n,\epsilon} = V_0^i + \int_0^t \int_0^1 \int_{|\theta| > \epsilon} \left[(\cos \theta - 1) \tilde{V}_{s-}^{i,n,\epsilon} - \sin \theta \tilde{V}_{s-}^{j,n,\epsilon} \right] N^{i,n}(ds d\theta dj).$$

The algorithm is the following (we write in *italic* the parts which only concern the system with diffusion).

- We set $t = 0$, and for $i = 1, \dots, n$, we simulate $V(i) \sim f_0$ and set $T^{up}(i) = 0$.
- While $t < T^{final}$ (where T^{final} is the time that we want to reach), we simulate an exponential random variable T with parameter $n \int_{|\theta| > \epsilon} \beta(\theta) d\theta$ and we put $t = t + T$. We choose randomly two integers i and j in $\{1, \dots, n\}$. *For our system with diffusion, we update the particles i and j by setting*

$$V(i) = V(i) \exp(-b_\epsilon(t - T^{up}(i))) + G(i),$$

and

$$V(j) = V(j) \exp(-b_\epsilon(t - T^{up}(j))) + G(j),$$

where $G(i)$ (*resp.* $G(j)$) has a centered Gaussian law with variance $1 - \exp(-2b_\epsilon(t - T^{up}(i)))$ (*resp.* $1 - \exp(-2b_\epsilon(t - T^{up}(j)))$), where b_ϵ is defined in (5.1), and we set $T^{up}(i) = T^{up}(j) = t$.

Next, we simulate a random variable Θ with density $\beta_\epsilon / \|\beta_\epsilon\|_1$, where $\beta_\epsilon(\theta) = \beta(\theta) \mathbb{1}_{|\theta| > \epsilon}$, and then, for the two systems, we put $V(i) = \cos \Theta V(i) - \sin \Theta V(j)$.

- *Only for the system with diffusion, we update all particles with a Gaussian term: for $i \in \{1, \dots, n\}$,*

$$V(i) = V(i) \exp(-b_\epsilon(t - T^{up}(i))) + G(i),$$

where $G(i)$ has a centered Gaussian law with variance $1 - \exp(-2b_\epsilon(t - T^{up}(i)))$.

For our simulation, we take $T^{final} = 0.1$. Our initial data is $f_0 = (\delta_{-1} + \delta_1)/2$. The goal here is to see what system is more efficient. For this, we need a reference curve. We obtain it by simulating $n = 10^7$ particles with $\epsilon = 0.03$, and by using a smoothing procedure.

We see that the system with diffusion term is much more efficient when ν is close to 2. For ν smaller, the difference not clear.

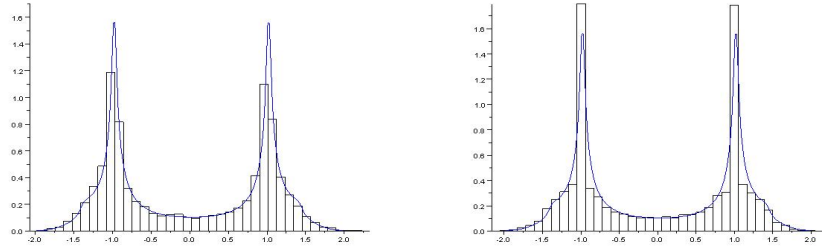


FIGURE 1. $\nu = 0.5$. Left graphic: system with diffusion, $n = 10^4$, $\epsilon = 0.1$. Right graphic: system without diffusion, $n = 2 \cdot 10^4$, $\epsilon = 0.1$. Both simulations need approximately 0.05s.

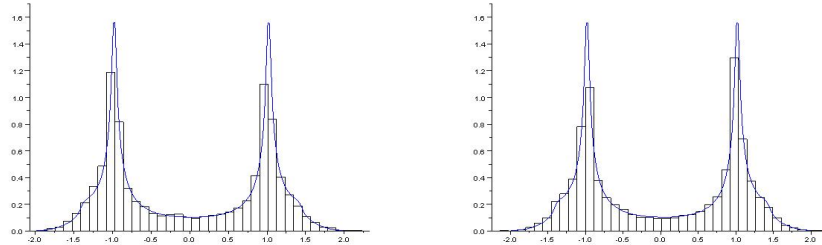


FIGURE 2. $\nu = 0.5$. Left graphic: system with diffusion, $n = 10^4$, $\epsilon = 0.1$. Right graphic: system without diffusion, $n = 10^4$, $\epsilon = 0.02$. Both simulations need approximately 0.05s.

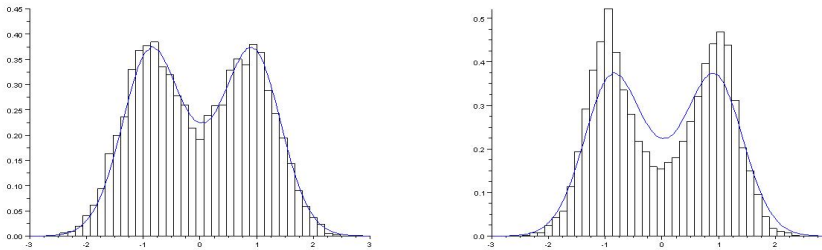


FIGURE 3. $\nu = 1.5$. Left graphic: system with diffusion, $n = 10^4$, $\epsilon = 0.1$. Right graphic: system without diffusion, $n = 2 \cdot 10^4$, $\epsilon = 0.1$. Both simulations need approximately 0.14s.

APPENDIX A. APPENDIX

A.1. Wasserstein distance between a Poisson integral and a Gaussian law.

We start with a result of Rio [27, Theorem 4.1], which gives some very precise rate of convergence for the standard central limit theorem in Wasserstein distance.

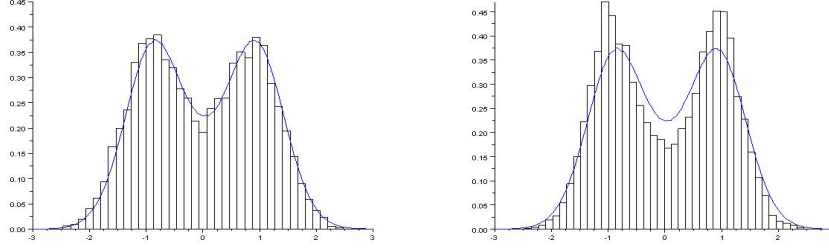


FIGURE 4. $\nu = 1.5$. Left graphic: system with diffusion, $n = 10^4$, $\epsilon = 0.1$. Right graphic: system without diffusion, $n = 10^4$, $\epsilon = 0.06$. Both simulations need approximately 0.14s.

Theorem A.1. *There exists a constant C_0 such that for any positive integer n , for any sequence $(X_i)_{i \geq 0}$ of real independent centered random variables in L^4 ,*

$$W_2^2(\eta_n, \mathcal{N}(0, 1)) \leq C_0 v_n^{-2} \sum_{i=1}^n \mathbb{E}(|X_i|^4),$$

where $\eta_n = \mathcal{L}(v_n^{-1/2} S_n)$, $S_n = \sum_{i=1}^n X_i$, $v_n = \text{Var}(S_n)$.

Using this result, we can estimate the Wasserstein distance between a compensated Poisson integral and a centered Gaussian law with the same variance. The following result is very close to [13, Corollary 6].

Corollary A.2. *If E is a Polish space endowed with a non-negative σ -finite measure ν , if N is a Poisson measure on $[0, T] \times E$ with intensity measure $dt\nu(dz)$ and if $H : [0, T] \times E \mapsto \mathbb{R}$ is a deterministic function such that $\int_0^t \int_E (H^2(s, z) + H^4(s, z))\nu(dz)ds < \infty$, then setting*

$$X_t = \int_0^t \int_E H(s, z) \tilde{N}(ds, dz), \quad q_t = \int_0^t \int_E H^2(s, z) \nu(dz) ds,$$

we have

$$W_2^2(\mathcal{L}(X_t), \mathcal{N}(0, q_t)) \leq C_0 \frac{\int_0^t \int_E H^4(s, z) \nu(dz) ds}{q_t},$$

where C_0 is a universal constant (the same as in Theorem A.1).

Proof. For $n \geq 1$, $i \in \{1, \dots, n\}$, we set

$$X_i^n = \sqrt{n} \int_{(i-1)t/n}^{it/n} \int_E H(s, z) \tilde{N}(ds, dz) \quad \text{and} \quad S_n = \sum_{i=1}^n X_i^n.$$

We have $X_t = \frac{S_n}{\sqrt{n}}$. The random variables X_i^n are independent, centered,

$$\mathbb{E}[(X_i^n)^2] = n \int_{(i-1)t/n}^{it/n} \int_E H^2(s, z) \nu(dz) ds, \quad v_n = \text{Var}(S_n) = \sum_{i=1}^n \mathbb{E}[(X_i^n)^2] = nq_t.$$

It classically holds that

$$\mathbb{E}[(X_i^n)^4] = n^2 \int_{(i-1)t/n}^{it/n} \int_E H^4(s, z) \nu(dz) ds + 3n^2 \left(\int_{(i-1)t/n}^{it/n} \int_E H^2(s, z) \nu(dz) ds \right)^2.$$

Hence

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}[(X_i^n)^4] &= n^2 \int_0^t \int_E H^4(s, z) \nu(dz) ds \\ &\quad + 3n^2 \sum_{i=1}^n \left(\int_{(i-1)t/n}^{it/n} \int_E H^2(s, z) \nu(dz) ds \right)^2. \end{aligned}$$

By Theorem A.1,

$$\begin{aligned} W_2^2(\mathcal{L}(X_t), \mathcal{N}(0, q_t)) &= W_2^2\left(\mathcal{L}\left(\frac{1}{\sqrt{n}} S_n\right), \mathcal{N}(0, q_t)\right) \\ &= q_t W_2^2\left(\mathcal{L}\left(\frac{1}{\sqrt{v_n}} S_n\right), \mathcal{N}(0, 1)\right) \\ &\leq C_0 \frac{q_t}{v_n^2} \sum_{i=1}^n \mathbb{E}[(X_i^n)^4] \\ &\leq C_0 \frac{\int_0^t \int_E H^4(s, z) \nu(dz) ds}{q_t} \\ &\quad + 3 \frac{C_0}{q_t} \sum_{i=1}^n \left(\int_{(i-1)t/n}^{it/n} \int_E H^2(s, z) \nu(dz) ds \right)^2. \end{aligned}$$

Setting $F(t) = \int_0^t |f(s)| ds$ with $f(s) = \int_E H^2(s, z) \nu(dz)$ and observing that F is continuous (and so uniformly continuous on $[0, T]$ for all $T \geq 0$), we obtain that $\sum_{i=1}^n \left(\int_{(i-1)t/n}^{it/n} \int_E H^2(s, z) \nu(dz) ds \right)^2 \rightarrow 0$ when $n \rightarrow +\infty$. Since the last formula holds for all $n \geq 1$, we easily conclude. \square

A.2. Rate of convergence of empirical measures. We first give a classical result about the Wasserstein distance W_1 .

Lemma A.3. *Let μ be a probability measure in $\mathcal{P}_4(\mathbb{R})$. We consider n i.i.d. random variables $(X_i)_{i \in \{1, \dots, n\}}$ with law μ and we set $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$. Then there exists a constant C depending only on $\int_{\mathbb{R}} x^4 \mu(dx)$ such that*

$$\mathbb{E}\left[W_1(\mu, \mu_n)\right] \leq \frac{C}{\sqrt{n}}.$$

Proof. If we set $F(x) = \mu((-\infty, x])$ and $F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{X_i \leq x}$, we have (see Villani [36, p 75])

$$\begin{aligned} \mathbb{E}\left(W_1(\mu, \mu_n)\right) &= \mathbb{E}\left(\int_{-\infty}^{\infty} |F(x) - F_n(x)| dx\right) \\ &= \mathbb{E}\left(\int_{-\infty}^{\infty} \left|F(x) - \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{X_i \leq x}\right| dx\right). \end{aligned}$$

If $Y \sim \mathcal{B}(n, p)$, $\mathbb{E}\left(\left|\frac{Y}{n} - p\right|\right) \leq \sqrt{\mathbb{E}\left(\left(\frac{Y}{n} - p\right)^2\right)} = \frac{1}{\sqrt{n}} \sqrt{p(1-p)}$. Hence, since for each x , $\sum_{i=1}^n \mathbb{1}_{X_i \leq x} \sim \mathcal{B}(n, F(x))$,

$$\mathbb{E}\left(W_1(\mu, \mu_n)\right) \leq \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} \sqrt{F(x)(1-F(x))} dx.$$

But $A := \int_{\mathbb{R}} x^4 \mu(dx) < \infty$ implies that for $x \geq 1$, $(1 - F(x)) = \mu([x, +\infty)) \leq A/x^4$ and for $x \leq -1$, $F(x) = \mu((-\infty, x]) \leq A/x^4$, so that $\int_{-\infty}^{\infty} \sqrt{F(x)(1 - F(x))} dx < \infty$. \square

We now deduce similar estimates for other Wasserstein distances.

Lemma A.4. *Let μ be a probability measure admitting a moment of order $q + \gamma$, with $\gamma > 1$ and $q > 0$. We consider n i.i.d. random variables $(X_i)_{i \in \{1, \dots, n\}}$ with law μ and we set $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$. There exists a constant C depending on γ , q and on the moment of μ of order $q + \gamma$ such that*

$$\mathbb{E} \left[W_{\gamma}^{\gamma}(\mu, \mu_n) \right] \leq \frac{C}{n^{\frac{q}{2(q+\gamma-1)}}}.$$

Proof. Let us denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space on which X_1, \dots, X_n are defined. For a fixed $\omega \in \Omega$, we consider two random variables X and Y^{ω} defined on the probability space $([0, 1], \mathcal{B}([0, 1]), d\alpha)$ with $\mathcal{L}_{\alpha}(X) = \mu$ and $\mathcal{L}_{\alpha}(Y^{\omega}) = \mu_n(\omega)$ such that $W_1(\mu, \mu_n(\omega)) = \mathbb{E}_{\alpha}(|X - Y^{\omega}|)$. Then we have, for any $A > 0$,

$$W_{\gamma}^{\gamma}(\mu, \mu_n(\omega)) \leq \mathbb{E}_{\alpha}(|X - Y^{\omega}|^{\gamma}) \leq A^{\gamma-1} W_1(\mu, \mu_n(\omega)) + \mathbb{E}_{\alpha}(|X - Y^{\omega}|^{\gamma} \mathbb{1}_{|X - Y^{\omega}| > A}).$$

We observe that

$$\mathbb{E}_{\alpha}(|X - Y^{\omega}|^{\gamma} \mathbb{1}_{|X - Y^{\omega}| > A}) \leq \frac{\mathbb{E}_{\alpha}(|X - Y^{\omega}|^{q+\gamma})}{A^q}.$$

But, setting $m_p(\mu) = \int_{\mathbb{R}} |x|^p \mu(dx)$,

$$\mathbb{E}_{\alpha}(|X - Y^{\omega}|^{q+\gamma}) \leq C \left(\mathbb{E}_{\alpha}(|X|^{q+\gamma}) + \mathbb{E}_{\alpha}(|Y^{\omega}|^{q+\gamma}) \right) = C(m_{q+\gamma}(\mu) + m_{q+\gamma}(\mu_n(\omega))).$$

One easily checks that $\mathbb{E} \left[m_{q+\gamma}(\mu_n) \right] = m_{q+\gamma}(\mu)$. Using Lemma A.3, we finally get

$$\mathbb{E} \left[W_{\gamma}^{\gamma}(\mu, \mu_n) \right] \leq C \left(\frac{A^{\gamma-1}}{\sqrt{n}} + \frac{m_{q+\gamma}(\mu)}{A^q} \right).$$

Choosing $A = n^{\frac{1}{2(q+\gamma-1)}}$ completes the proof. \square

A.3. Moments of a solution to (1.1). In many places of the proof, we need to upperbound $\int_{\mathbb{R}} v^4 f_t(dv)$ for any $t \geq 0$ where $(f_t)_{t \geq 0}$ solves (1.1). We also need to upperbound higher moments.

Lemma A.5. *For $f_0 \in \mathcal{P}_4(\mathbb{R})$, consider the unique solution $(f_t)_{t \geq 0}$ to (1.1). For any $t \geq 0$, we have*

$$\int_{\mathbb{R}} v^4 f_t(dv) \leq \int_{\mathbb{R}} v^4 f_0(dv) + 3 \int_{\mathbb{R}} v^2 f_0(dv).$$

If $f_0 \in \mathcal{P}_p(\mathbb{R})$ with p even, then there exists a constant C depending on p , β and on $\int_{\mathbb{R}} v^p f_0(dv)$ such that for any $t \geq 0$,

$$\int_{\mathbb{R}} v^p f_t(dv) \leq C.$$

Proof. We only treat the case $p = 4$, see Truesdell [33] and Desvillettes [8] for the general case. If we take $\varphi(v) = v^4$, we find, recalling (2.2) and using that β is even,

$$K_\beta^\varphi(v, v_*) = \int_{-\pi}^{\pi} \left[(\cos^4 \theta - 1)v^4 + \sin^4 \theta v_*^4 + 6 \cos^2 \theta \sin^2 \theta v^2 v_*^2 \right] \beta(\theta) d\theta.$$

Setting $m_k(\mu) = \int_{\mathbb{R}} v^k \mu(dv)$ for μ a probability measure on \mathbb{R} and $k \in \mathbb{N}$, we thus get, using (2.5),

$$m_4(f_t) = m_4(f_0) + \int_0^t \int_{-\pi}^{\pi} \left[-(1 - \cos^4 \theta - \sin^4 \theta) m_4(f_s) + 6 \cos^2 \theta \sin^2 \theta m_2^2(f_s) \right] \beta(\theta) d\theta ds.$$

Recalling that $m_2(f_s) = m_2(f_0) =: \mathcal{E}$ for any $s \geq 0$, observing that $(\cos^2 \theta + \sin^2 \theta)^2 = 1$, which gives $2 \cos^2 \theta \sin^2 \theta = 1 - \cos^4 \theta - \sin^4 \theta$ and setting $c = \int_{-\pi}^{\pi} (1 - \cos^4 \theta - \sin^4 \theta) \beta(\theta) d\theta$, we have

$$m_4(f_t) = m_4(f_0) - c \int_0^t m_4(f_s) ds + 3c\mathcal{E}^2 t,$$

whence

$$m_4(f_t) = (m_4(f_0) - 3\mathcal{E}^2) \exp(-ct) + 3\mathcal{E}^2 \leq m_4(f_0) + 3\mathcal{E}^2,$$

as desired. \square

A.4. Well-posedness for a P.D.E. To conclude this paper, we state the following result.

Proposition A.6. *For $t \geq 0$ and $(v, v_*) \in \mathbb{R}^2$, we consider two finite non-negative measures $q(t, v, dh)$ and $r(t, v, v_*, dh)$ on \mathbb{R} such that $\Lambda_q := \sup_{t,v} q(t, v, \mathbb{R}) < \infty$, $\Lambda_r := \sup_{t,v,v_*} r(t, v, v_*, \mathbb{R}) < \infty$ and for all $T \geq 0$, all $(v, v_*) \in \mathbb{R}^2$,*

$$(A.1) \quad \sup_{[0,T]} \int_{\mathbb{R}} (h^2 + 2vh) q(t, v, dh) \leq C_T(1 + v^2),$$

and

$$(A.2) \quad \sup_{[0,T]} \int_{\mathbb{R}} (h^2 + 2vh) r(t, v, v_*, dh) \leq C_T(1 + v^2 + v_*^2),$$

Let also $a \geq 0$ and $b \in \mathbb{R}$ be fixed. Then, for any $f_0 \in \mathcal{P}_2(\mathbb{R})$, there exists a unique $(f_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ such that for all $\varphi \in C_b^2(\mathbb{R})$, all $t \geq 0$,

$$(A.3) \quad \begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \varphi(v) f_t(dv) &= a \int_{\mathbb{R}} \varphi''(v) f_t(dv) + b \int_{\mathbb{R}} v \varphi'(v) f_t(dv) \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\varphi(v+h) - \varphi(v) \right] q(t, v, dh) f_t(dv) \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\varphi(v+h) - \varphi(v) \right] r(t, v, v_*, dh) f_t(dv) f_t(dv_*). \end{aligned}$$

Proof. We denote by $\mathcal{M}(\mathbb{R})$ the set of finite signed measures on \mathbb{R} . If $\mu \in \mathcal{M}(\mathbb{R})$, we set $|\mu|_{TV} = \sup_{\varphi \in L^\infty, \|\varphi\|_\infty \leq 1} \int_{\mathbb{R}} \varphi(v) \mu(dv)$. Using the Lusin Theorem (see e.g. [1, Theorem 9.11]), we have $|\mu|_{TV} = \sup_{\varphi \in C_b, \|\varphi\|_\infty \leq 1} \int_{\mathbb{R}} \varphi(v) \mu(dv)$. We also have $|\mu|_{TV} = \int_{\mathbb{R}} |\mu|(dv)$ where $|\mu| = \mu_+ + \mu_-$ and if μ has a density f with respect to the Lebesgue measure, $|\mu|_{TV} = \int_{\mathbb{R}} |f(v)| dv$.

Preliminaries. For $\epsilon > 0$, we set $G_\epsilon(v) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{v^2}{2\epsilon}}$. Let $\mu \in \mathcal{M}(\mathbb{R})$. We claim that $\lim_{\epsilon \rightarrow 0} |\mu * G_\epsilon|_{TV} = |\mu|_{TV}$. Observe that this is not obvious, since it does not hold true, generally, that $\lim_{\epsilon \rightarrow 0} |\mu * G_\epsilon - \mu|_{TV} = 0$ (choose e.g. $\mu = \delta_0$). First, we have

$$\begin{aligned} |\mu * G_\epsilon|_{TV} &= \int_{\mathbb{R}} |\mu * G_\epsilon(v)| dv = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} G_\epsilon(v-w) \mu(dw) \right| dv \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} G_\epsilon(v-w) dv |\mu(dw)| = |\mu|_{TV}. \end{aligned}$$

Next, let $\alpha > 0$. There exists a function $\varphi \in C_b$ with $\|\varphi\|_\infty \leq 1$ such that $\int_{\mathbb{R}} \varphi(v) \mu(dw) \geq |\mu|_{TV} - \alpha$. We have, since $\mu * G_\epsilon$ clearly converges weakly (in the sense of measures) to μ ,

$$|\mu * G_\epsilon|_{TV} \geq \int_{\mathbb{R}} \varphi(v) (\mu * G_\epsilon)(v) dv \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{R}} \varphi(v) \mu(dw) \geq |\mu|_{TV} - \alpha.$$

Making α tend to zero, we get $\liminf_{\epsilon \rightarrow 0} |\mu * G_\epsilon|_{TV} \geq |\mu|_{TV}$.

Uniqueness. We consider two solutions $(f_t)_{t \geq 0}$ and $(g_t)_{t \geq 0}$, with $f_0 = g_0$ and for $t \geq 0$ we set $\mu_t = f_t - g_t$. For any $\varphi \in C_b^2(\mathbb{R})$, any $t \geq 0$

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \varphi(v) \mu_t(dv) &= a \int_{\mathbb{R}} \varphi''(v) \mu_t(dv) + b \int_{\mathbb{R}} v \varphi'(v) \mu_t(dv) \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(v+h) - \varphi(v)] q(t, v, dh) \mu_t(dv) \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(v+h) - \varphi(v)] r(t, v, v_*, dh) [f_t(dv) f_t(dv_*) \\ &\quad \quad \quad - g_t(dv) g_t(dv_*)]. \end{aligned}$$

We first observe that $f_t(dv) f_t(dv_*) - g_t(dv) g_t(dv_*) = f_t(dv) \mu_t(dv_*) + g_t(dv_*) \mu_t(dv)$. We have

$$\begin{aligned} \partial_t (\mu_t * G_\epsilon)(v) &= \frac{d}{dt} \int_{\mathbb{R}} G_\epsilon(v-w) \mu_t(dw) \\ &= a \int_{\mathbb{R}} G_\epsilon''(v-w) \mu_t(dw) - b \int_{\mathbb{R}} w G_\epsilon'(v-w) \mu_t(dw) \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} [G_\epsilon(v-w-h) - G_\epsilon(v-w)] \mu_t(dw) q(t, w, dh) \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} [G_\epsilon(v-w-h) - G_\epsilon(v-w)] [f_t(dw) \mu_t(dw_*) \\ &\quad \quad \quad + g_t(dw_*) \mu_t(dw)] r(t, w, w_*, dh). \end{aligned}$$

For $\eta > 0$, we consider a function Γ_η of class C^2 such that for any $x \in \mathbb{R}$, $(|x| - \eta)_+ \leq \Gamma_\eta(x) \leq |x|$, $\Gamma_\eta''(x) \geq 0$ and $\|\Gamma_\eta'\|_\infty \leq 1$. We also assume that $\Gamma_{\eta_1} \geq \Gamma_{\eta_2}$ if $\eta_1 \leq \eta_2$. Observing that $\int_{\mathbb{R}} G_\epsilon''(v-w) \mu_t(dw) = (\mu_t * G_\epsilon)''(v)$, we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \Gamma_\eta((\mu_t * G_\epsilon)(v)) dv &= \int_{\mathbb{R}} \Gamma_\eta'((\mu_t * G_\epsilon)(v)) \partial_t (\mu_t * G_\epsilon)(v) dv \\ &= A_t + B_t + C_t + D_t, \end{aligned}$$

where

$$A_t = a \int_{\mathbb{R}} \Gamma'_\eta((\mu_t * G_\epsilon)(v)) (\mu_t * G_\epsilon)''(v) dv,$$

$$B_t = -b \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma'_\eta((\mu_t * G_\epsilon)(v)) w G'_\epsilon(v-w) \mu_t(dw) dv,$$

$$C_t = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma'_\eta((\mu_t * G_\epsilon)(v)) [G_\epsilon(v-w-h) - G_\epsilon(v-w)] \mu_t(dw) q(t, w, dh) dv,$$

and

$$D_t = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma'_\eta((\mu_t * G_\epsilon)(v)) [G_\epsilon(v-w-h) - G_\epsilon(v-w)] [f_t(dw) \mu_t(dw_*) + g_t(dw_*) \mu_t(dw)] r(t, w, w_*, dh) dv.$$

Using an integration by parts and recalling that $\Gamma''_\eta(x) \geq 0$ for any $x \in \mathbb{R}$, we have

$$A_t = -a \int_{\mathbb{R}} \Gamma''_\eta((\mu_t * G_\epsilon)(v)) ((\mu_t * G_\epsilon)'(v))^2 dv \leq 0.$$

First writing $w = v + w - v$ and then using an integration by parts (observe that $\int_{\mathbb{R}} G'_\epsilon(v-w) \mu_t(dw) = (\mu_t * G_\epsilon)'(v)$), we have

$$\begin{aligned} B_t &\leq -b \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma'_\eta((\mu_t * G_\epsilon)(v)) v G'_\epsilon(v-w) \mu_t(dw) dv \\ &\quad + \|\Gamma'_\eta\|_\infty |b| \int_{\mathbb{R}} \int_{\mathbb{R}} |w-v| |G'_\epsilon(v-w)| |\mu_t(dw)| dv \\ &\leq b \int_{\mathbb{R}} \Gamma_\eta((\mu_t * G_\epsilon)(v)) dv + |b| \int_{\mathbb{R}} |v| |G'_\epsilon(v)| dv \int_{\mathbb{R}} |\mu_t|(dw) \\ &\leq C \left(\int_{\mathbb{R}} |(\mu_t * G_\epsilon)(v)| dv + |\mu_t|_{TV} \right) \leq C |\mu_t|_{TV}. \end{aligned}$$

We used the preliminaries and the fact that $\int_{\mathbb{R}} |v| |G'_\epsilon(v)| dv \leq C$. Using next that $\int_{\mathbb{R}} G_\epsilon(v-w-h) dv = \int_{\mathbb{R}} G_\epsilon(v-w) dv = 1$, we have

$$\begin{aligned} C_t + D_t &\leq 2 \|\Gamma'_\eta\|_\infty \Lambda_q \int_{\mathbb{R}} |\mu_t|(dw) + 2 \|\Gamma'_\eta\|_\infty \Lambda_r \left(\int_{\mathbb{R}} |\mu_t|(dw_*) + \int_{\mathbb{R}} |\mu_t|(dw) \right) \\ &\leq C |\mu_t|_{TV}. \end{aligned}$$

We thus get

$$\frac{d}{dt} \int_{\mathbb{R}} \Gamma_\eta(\mu_t * G_\epsilon(v)) dv \leq C |\mu_t|_{TV}.$$

Using the monotone convergence Theorem (recall that $\Gamma_\eta(x)$ increases to $|x|$ as η decreases to 0) and recalling that $\mu_0 = 0$, we have

$$\begin{aligned} |\mu_t * G_\epsilon|_{TV} &= \lim_{\eta \rightarrow 0} \int_{\mathbb{R}} \Gamma_\eta((\mu_t * G_\epsilon)(v)) dv \\ &\leq \lim_{\eta \rightarrow 0} \int_{\mathbb{R}} \Gamma_\eta((\mu_0 * G_\epsilon)(v)) dv + C \int_0^t |\mu_s|_{TV} ds \\ &\leq C \int_0^t |\mu_s|_{TV} ds. \end{aligned}$$

Making ϵ tend to 0 and using the preliminaries, we get,

$$|\mu_t|_{TV} \leq C \int_0^t |\mu_s|_{TV} ds,$$

and we deduce that $|\mu_t|_{TV} = 0$ by Grönwall's lemma.

Existence. For $(Q_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ given, we consider the following linear P.D.E. with unknown $(g_t^Q)_{t \geq 0}$: for all $\varphi \in C_b^2(\mathbb{R})$, all $t \geq 0$,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \varphi(v) g_t^Q(dv) &= a \int_{\mathbb{R}} \varphi''(v) g_t^Q(dv) + b \int_{\mathbb{R}} v \varphi'(v) g_t^Q(dv) \\ (A.4) \quad &+ \int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(v+h) - \varphi(v)] q(t, v, dh) g_t^Q(dv) \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(v+h) - \varphi(v)] r(t, v, v_*, dh) g_t^Q(dv) Q_t(dv_*). \end{aligned}$$

For $t \geq 0$ and $(v, v_*) \in \mathbb{R}^2$, we consider the following probability measures

$$\eta_{t,v}^q(dh) := \frac{q(t, v, dh)}{\Lambda_q} + \left(1 - \frac{q(t, v, \mathbb{R})}{\Lambda_q}\right) \delta_0(dh)$$

and

$$\eta_{t,v,v_*}^r(dh) := \frac{r(t, v, v_*, dh)}{\Lambda_r} + \left(1 - \frac{r(t, v, v_*, \mathbb{R})}{\Lambda_r}\right) \delta_0(dh),$$

and we set $F_{t,v}^q(x) := \eta_{t,v}^q((-\infty, x])$ and $F_{t,v,v_*}^r(x) := \eta_{t,v,v_*}^r((-\infty, x])$. We also set $H^q(t, v, u) := (F_{t,v}^q)^{-1}(u)$, $H^r(t, v, v_*, u) := (F_{t,v,v_*}^r)^{-1}(u)$ and we consider the following S.D.E.

$$\begin{aligned} (A.5) \quad V_t &= V_0 + \int_0^t \int_0^1 H^q(s, V_{s-}, u) N^q(dsdu) + \int_0^t \int_{\mathbb{R}} \int_0^1 H^r(s, V_{s-}, v_*, u) N^r(dsdu) \\ &+ b \int_0^t V_s ds + \sqrt{2a} B_t, \end{aligned}$$

where N^q is a Poisson measure with intensity measure $\Lambda_q dsdu$, N^r is a Poisson measure with intensity measure $\Lambda_r dsdu Q_s(dv_*)$ and B is a Brownian motion. There is existence and uniqueness for this S.D.E. because the Poisson measures N^q and N^r are finite, and because the drift and diffusion coefficients are Lipschitz-continuous (see Ikeda-Watanabe [17]). Using Itô's formula and taking expectations, we get,

for any $\varphi \in C_b^2(\mathbb{R})$,

$$\begin{aligned}
\mathbb{E}[\varphi(V_t)] &= \mathbb{E}[\varphi(V_0)] + \int_0^t \mathbb{E} \left[\int_0^1 [\varphi(V_s + H^q(s, V_s, u)) - \varphi(V_s)] \Lambda_q du \right] ds \\
&\quad + \int_0^t \mathbb{E} \left[\int_{\mathbb{R}} \int_0^1 [\varphi(V_s + H^r(s, V_s, v_*, u)) - \varphi(V_s)] \Lambda_r Q_s(dv_*) du \right] ds \\
&\quad + b \int_0^t \mathbb{E}[\varphi'(V_s) V_s] ds + a \int_0^t \mathbb{E}[\varphi''(V_s)] ds \\
&= \mathbb{E}[\varphi(V_0)] + \int_0^t \mathbb{E} \left[\int_{\mathbb{R}} [\varphi(V_s + h) - \varphi(V_s)] q(s, V_s, dh) \right] ds \\
&\quad + \int_0^t \mathbb{E} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(V_s + h) - \varphi(V_s)] r(s, V_s, v_*, dh) Q_s(dv_*) \right] ds \\
&\quad + b \int_0^t \mathbb{E}[\varphi'(V_s) V_s] ds + a \int_0^t \mathbb{E}[\varphi''(V_s)] ds.
\end{aligned}$$

Setting $g_t^Q = \mathcal{L}(V_t)$, we thus realize that $(g_t^Q)_{t \geq 0}$ solves (A.4).

If $(Q_t)_{t \geq 0}$ and $(R_t)_{t \geq 0}$ are in $L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$, then by the same kind of arguments as in the uniqueness proof, we have for any $t \in [0, T]$, denoting $\mu_t = g_t^Q - g_t^R$,

$$|\mu_t|_{TV} \leq C \int_0^t (|\mu_s|_{TV} + |Q_s - R_s|_{TV}) ds,$$

whence by Grönwall's Lemma,

$$\sup_{[0, T]} |\mu_t|_{TV} \leq C_T \int_0^T |Q_s - R_s|_{TV} ds.$$

We consider $f_0 \in \mathcal{P}_2(\mathbb{R})$. For $t \geq 0$, we set $f_t^0 = f_0$ and $f_t^{k+1} = f_t^{f^k}$. Then we have

$$\sup_{[0, T]} |f_t^{k+1} - f_t^k|_{TV} \leq C_T \int_0^T |f_s^k - f_s^{k-1}|_{TV} ds.$$

We classically conclude that $(f_t^k)_{t \geq 0}$ converges as k tends to infinity to some $(f_t)_{t \geq 0}$ solving (A.3). Using (A.1), (A.2) and (A.3) with $\varphi(v) = v^2$, we see that $(f_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$.

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