ON THE WELL-POSEDNESS OF THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION WITH A MODERATE ANGULAR SINGULARITY

NICOLAS FOURNIER[1], CLÉMENT MOUHOT[2]

Abstract. We prove an inequality on the Kantorovich-Rubinstein distance – which can be seen as a particular case of a Wasserstein metric– between two solutions of the spatially homogeneous Boltzmann equation without angular cutoff, but with a moderate angular singularity. Our method is in the spirit of [7]. We deduce some well-posedness and stability results in the physically relevant cases of hard and moderately soft potentials.

In the case of hard potentials, we relax the regularity assumption of [6], but we need stronger assumptions on the tail of the distribution (namely some exponential decay). We thus obtain the first uniqueness result for measure initial data.

In the case of moderately soft potentials, we prove existence and uniqueness assuming only that the initial datum has finite energy and entropy (for very moderately soft potentials), plus sometimes an additionnal moment condition. We thus improve significantly on all previous results, where weighted Sobolev spaces were involved.

Mathematics Subject Classification (2000): 76P05 Rarefied gas flows, Boltzmann equation [See also 82B40, 82C40, 82D05].

Keywords: Boltzmann equation without cutoff, long-range interaction, uniqueness, Wasserstein distance, Kantorovich-Rubinstein distance.

Contents

1. Introduction 2
2. Main results 6
3. The general inequality 10
4. Application to hard potentials 16
5. Application to soft potentials 21
References 25

[1]Centre de Mathématiques, Faculté de Sciences et Technologies, Université Paris 12, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, France, e-mail: nicolas.fournier@univ-paris12.fr

[2]Ceremade, Université Paris IX-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris, France, e-mail: clement.mouhot@ceremade.dauphine.fr

1. Introduction

1.1. The Boltzmann equation. We consider a spatially homogeneous gas in dimension $d \geq 2$ modeled by the Boltzmann equation. Therefore the time-dependent density $f = f_t(v)$ of particles with velocity $v \in \mathbb{R}^d$ solves

$$\partial_t f_t(v) = \int_{\mathbb{R}^d} dv_* \int_{\mathbb{S}^{d-1}} d\sigma B(|v-v_*|,\theta)\big[f_t(v')f_t(v'_*) - f_t(v)f_t(v_*)\big], \tag{1.1}$$

where

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$$

and θ is the so-called *deviation angle* defined by $\cos\theta = \frac{(v-v_*)}{|v-v_*|}\cdot\sigma$.

The *collision kernel* $B = B(|v-v_*|,\theta) = B(|v'-v'_*|,\theta)$ is given by physics and is related to the microscopic interaction between particles. In dimension $d=3$ it is related to the probabilistic *cross-section* $\hat{B}$ of the distribution of possible outgoing velocities v' and v'_* arising from a collision with two particles with velocities v and v_*, by the formula $B = |v-v_*|\,\hat{B}$. We refer to the review papers of Desvillettes [5] and Villani [18] for more details.

Conservation of mass, momentum and kinetic energy hold at least formally for solutions to (1.1), that is for all $t \geq 0$,

$$\int_{\mathbb{R}^d} f_t(v)\,\varphi(v)\,dv = \int_{\mathbb{R}^d} f_0(v)\,\varphi(v)\,dv, \qquad \varphi = 1, v, |v|^2$$

and we classically may assume without loss of generality that $\int_{\mathbb{R}^d} f_0(v)\,dv = 1$.

1.2. Assumptions on the collision kernel. We shall assume that the collision kernel takes the form

$$B(|v-v_*|,\theta)\,\sin^{d-2}\theta = \Phi(|v-v_*|)\,\beta(d\theta) \tag{\textbf{A1}}$$

for some function $\Phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and some nonnegative measure β on $(0,\pi]$.

In the case of an interaction potential $V(s) = 1/r^s$ in dimension $d=3$, with $s \in (2,\infty)$, one has

$$\Phi(z) = \text{cst}\, z^\gamma, \quad \beta(\theta) \sim \text{cst}\,\theta^{-1-\nu}, \quad \text{with } \gamma = \frac{s-5}{s-1}, \quad \nu = \frac{2}{s-1}. \tag{1.3}$$

On classically names *hard potentials* the case when $\gamma \in (0,1)$ (*i.e.*, $s > 5$ in dimension $d=3$), *Maxwellian molecules* the case when $\gamma = 0$ (*i.e.*, $s=5$ in dimension $d=3$), and *soft potentials* the case when $\gamma \in (-d,0)$ (*i.e.*, $s \in (2,5)$ in dimension $d=3$).

Let us emphasize that $\int_{0+} \beta(d\theta) = +\infty$, which expresses the affluence of *grazing collisions*, but in any case,

$$\int_0^\pi \theta^2 \, \beta(d\theta) < +\infty. \tag{1.4}$$

In this paper we shall deal with a moderate angular singularity, that is we shall assume that the collision kernel satisfies

$$(\mathbf{A2}) \qquad \kappa_1 = \int_0^\pi \theta \, \beta(d\theta) < +\infty,$$

which corresponds to $s \in (3, \infty)$ in (1.3)).

We will also assume that Φ behaves as a power function, namely that for some $\gamma \in (-d, 1]$, there exists some constant C such that for all $z, \tilde{z} \in \mathbb{R}_+$,

$$(\mathbf{A3}(\gamma)) \qquad \Phi(z) \leq C\, z^\gamma; \quad |\Phi(z) - \Phi(\tilde{z})| \leq C\, |z^\gamma - \tilde{z}^\gamma|.$$

Sometimes, we will need a lowerbound: there exists $c > 0$ such that for all $z \in \mathbb{R}_+$,

$$(\mathbf{A4}(\gamma)) \qquad \Phi(z) \geq c\, z^\gamma.$$

In the case of hard potentials, we will also sometimes use an additionnal technical assumption in order to obtain the propagation of some exponential moments:

$$(\mathbf{A5}) \quad \beta(d\theta) = b(\cos\theta)\, d\theta, \text{ where } b \text{ is nondecreasing, convex and } C^1 \text{ on } [-1, 1).$$

In the case of moderately soft potentials, we will sometimes use

$$(\mathbf{A6}(\nu)) \qquad \beta(d\theta) = \beta(\theta)\, d\theta \text{ with } \beta(\theta) \sim_{\theta\to 0} \mathrm{cst}\, \theta^{-1-\nu}$$

for some positive constant.
In practise, all these assumptions are met when one deals with interaction potential $V(s) = 1/r^s$ in dimension $d = 3$, with $s \in (3, \infty)$.

1.3. **Goals, existing results and difficulties.** We study in this paper the well-posedness of the spatially homogeneous Boltzmann equation for singular collision kernel as introduced above. In particular we focus on the questions of uniqueness and stability with respect to the initial condition which were open, for collision kernel with angular cutoff, until the two recent papers [7, 6] (except in the special case of Maxwell molecules, see below).

In the case of a collision kernel with angular cutoff, that is when $\int_0^\pi \beta(d\theta) < +\infty$, there are some optimal existence and uniqueness results: Mischler-Wennberg [13] in the space of L^1 non-negative functions with finite non-increasing kinetic energy (for counter-examples of spurious solutions with increasing kinetic energy, see [20] in the hard spheres case, and [11] in the case of hard potentials with or without angular cutoff), Lu-Mouhot [10] in the space of non-negative measures with finite non-increasing kinetic energy.

However, the case of collision kernels without cutoff is much more difficult. At the same time it is crucial from the physical viewpoint since it corresponds to the fundamental class of the interactions deriving from inverse power-law between particles. This difficulty is not surprising, since there is a difference of nature in the collision process between the two cases: on each compact time interval, each particle collides with infinitely (resp. finitely) many others in the case without (resp. with) cutoff.

Until recently, the only uniqueness result obtained for non cutoff collision kernel was concerning Maxwellian molecules, studied successively by Tanaka [15], Horowitz-Karandikar [9], Toscani-Villani [16]: it was proved in [16] that uniqueness holds for the Boltzmann equation as soon as Φ is constant and (1.4) is met, for any initial (measure) datum with finite mass and energy, that is $\int_{\mathbb{R}^d}(1+|v|^2)\, f_0(dv) < +\infty$.

There has been recently two papers in the case where β is non cutoff and Φ is not constant. The case where Φ is bounded (together with additionnal regularity assumptions) was treated in Fournier [7], for essentially any initial (measure) datum such that $\int_{\mathbb{R}^d}(1+|v|)f_0(dv) < \infty$. More realistic collision kernels have been treated by Desvillettes-Mouhot [6] (including the physical important cases of hard and moderately soft potentials without cutoff), for initial data in some weighted $W^{1,1}$ spaces.

In the present paper, we extend and improve the method of [7]:

- it can deal with the physical collision kernels corresponding to hard and moderately soft potentials, as in [6]: in dimension $d = 3$ we obtain well-posedeness for interaction potentials $1/r^s$ with $s \in (3, \infty)$,
- the proof is simplified as compared to [7]: it is shorter, allows measure initial conditions (for technical reasons, we had to consider only functions in [7]), and it does not refer anymore to probabilistic arguments.

Finally let us compare our results with those in [6], when applied to the case of an interaction potential $V(s) = 1/r^s$ in dimension $d = 3$.

- Our result is much better in the case of moderately soft potentials ($s \in (3, 5)$). Indeed, we assume only that the initial condition f_0 has finite mass, energy and entropy (plus, if $s \in (3, 3.48)$, a moment condition $\int_{\mathbb{R}^d} |v|^q f_0(v)dv < \infty$ for q large enough). All these conditions, together with $f_0 \in L^p(\mathbb{R}^d) \cap$

$W^{1,1}(\mathbb{R}^d, (1+|v|)^2 dv)$ (for some $p > 1$ depending on the collision rate) were assumed in [6].

- Our result is different in the case of hard potentials ($s \in (5, \infty)$). We allow any measure initial f_0 condition such that for some $\varepsilon > 0$, $\int_{\mathbb{R}^d} e^{\varepsilon |v|^\gamma} f_0(dv) < \infty$, where $\gamma = (s-5)/(s-1)$. In [6], the case where $f_0 \in W^{1,1}(\mathbb{R}^d, (1+|v|)^2 dv)$ was treated. We thus assume much less regularity, but much more *localization*.

Let us remark that our result is quasi-optimal when $s \in (3.48, 5)$, since the finiteness of entropy and energy is physically very reasonnable. It might be possible to relax the entropy condition, but it is not clear: one reasonnably has to assume a few regularity on f_0 to get the uniqueness, since the collision rate involves $|v - v_*|^\gamma$ with $\gamma < 0$, and we remark that $|v - v_*|^\gamma f_0(dv) f_0(dv_*)$ is infinite when f_0 contains, e.g., Dirac measures.

Let us emphasize that, as in [6, 7], we are only able to prove well-posedness in the case of a moderate angular singularity (assumption (**A2**)).
To our knowledge, there is no uniqueness result under the general assumption (1.4), except for Maxwellian molecules (see [16]).

1.4. **Notation.** Let us denote by $\text{Lip}(\mathbb{R}^d)$ the set of globally Lipschitz functions $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$, and by $\text{Lip}_1(\mathbb{R}^d)$ the set of functions $\varphi \in \text{Lip}(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\text{Lip}(\mathbb{R}^d)} = \sup_{v \neq \tilde{v}} \frac{|\varphi(v) - \varphi(\tilde{v})|}{|v - \tilde{v}|} \leq 1.$$

Let also $L^p(\mathbb{R}^d)$ denote the Lebesgue space of measurable functions f such that

$$\|f\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} f^p \, dv \right)^{1/p} < +\infty.$$

Let $\mathcal{P}(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$, and

$$\mathcal{P}_1(\mathbb{R}^d) = \left\{ f \in \mathcal{P}(\mathbb{R}^d),\ m_1(f) < \infty \right\} \quad \text{with} \quad m_1(f) := \int_{\mathbb{R}^d} |v| \, f(dv).$$

We denote by $L^\infty([0,T], \mathcal{P}_1(\mathbb{R}^d))$ the set of measurable families $(f_t)_{t \in [0,T]}$ of probability measures on $\mathbb{R}^d$ such that

$$\sup_{[0,T]} m_1(f_t) < +\infty,$$

and by $L^\infty([0,T],\mathcal{P}_1(\mathbb{R}^d))\cap L^1([0,T],L^p(\mathbb{R}^d))$ the set of measurable families $(f_t)_{t\in[0,T]}$ of probability measures on $\mathbb{R}^d$ such that

$$\sup_{[0,T]} m_1(f_t) < +\infty, \qquad \int_0^T \|f_t\|_{L^p(\mathbb{R}^d)}\, dt < +\infty.$$

For $v, v_* \in \mathbb{R}^d$, and $\sigma \in \mathbb{S}^{d-1}$, we write

$$v' = v'(v, v_*, \sigma) = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,$$

and we write

$$\sigma = (\cos\theta, \sin\theta\,\xi) \quad \text{ with } \xi \in \mathbb{S}^{d-2},\ \theta \in [0,\pi],$$

in some orthonormal basis of $\mathbb{R}^d$ with first vector $(v-v_*)/|v-v_*|$.

Finally we denote $x \wedge y = \min\{x,y\}$ and $x_+ = \max\{x,0\}$, and for some set E we write $1\!\!1_E$ the usual indicator function of E.

2. Main results

Let us define the notion of weak (measure) solutions we shall use.

Definition 2.1. *Let B be a collision kernel which satisfies* **(A1-A2)**. *A family $f = (f_t)_{t\in[0,T]} \in L^\infty([0,T],\mathcal{P}_1(\mathbb{R}^d))$ is a weak solution to (1.1) if*

$$\int_0^T dt \int_{\mathbb{R}^d} f_t(dv) \int_{\mathbb{R}^d} f_t(dv_*)\, \Phi(|v-v_*|)\, |v-v_*| < +\infty, \tag{2.1}$$

and if for any $\varphi \in \mathrm{Lip}(\mathbb{R}^d)$, and any $t \in [0,T]$,

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi(v)\, f_t(dv) = \int_{\mathbb{R}^d} f_t(dv) \int_{\mathbb{R}^d} f_t(dv_*)\, A[\varphi](v,v_*), \tag{2.2}$$

where

$$A[\varphi](v,v_*) = \Phi(|v-v_*|) \int_0^\pi \beta(d\theta) \int_{\xi\in\mathbb{S}^{d-2}} [\varphi(v') - \varphi(v)]\, d\xi. \tag{2.3}$$

Note that for any $\sigma \in \mathbb{S}^{d-1}$,

$$|v'-v| = |v-v_*| \sqrt{\frac{1-\cos\theta}{2}} \le \frac{\theta}{2}\, |v-v_*|, \tag{2.4}$$

so that thanks to assumption **(A2)**, (2.1) ensures that all the terms in (2.2) are well-defined.

Let us now introduce the distance on $\mathcal{P}_1(\mathbb{R}^d)$ we shall use. For $g, \tilde{g} \in \mathcal{P}_1(\mathbb{R}^d)$, let $\mathcal{H}(g, \tilde{g})$ be the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal g and second marginal $\tilde{g}$. We then set

$$\begin{aligned} d_1(g,\tilde{g}) &= \inf\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}|v-\tilde{v}|\,G(dv,d\tilde{v}),\quad G\in\mathcal{H}(g,\tilde{g})\right\}\\ &= \min\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}|v-\tilde{v}|\,G(dv,d\tilde{v}),\quad G\in\mathcal{H}(g,\tilde{g})\right\}\\ &= \sup\left\{\int_{\mathbb{R}^d}\varphi(v)\,\big[g(dv)-\tilde{g}(dv)\big],\quad \varphi\in \mathrm{Lip}_1(\mathbb{R}^d)\right\}. \end{aligned} \tag{2.5}$$

This distance is the Kantorovitch-Rubinstein distance, and can be viewed as a particular Wasserstein distance. We refer to Villani [19, Section 7] for more details on this distance, and for proofs that the equalities in (2.5) hold.

Our main result is the following inequality, which will be applied in the sequel to hard and soft potentials separately.

Theorem 2.2. *Let B be a collision kernel which satisfies* **(A1-A2)**. *Let us consider two weak solutions $f, \tilde{f}$ to (1.1) lying in $L^\infty([0,T], \mathcal{P}_1(\mathbb{R}^d))$, and satisfying*

$$\int_0^T dt \int_{\mathbb{R}^d\times\mathbb{R}^d}\Big[f_t(dv)\,f_t(dv_*)+\tilde{f}_t(dv)\,\tilde{f}_t(dv_*)\Big]\,(1+|v|)\,\Phi(|v-v_*|)<+\infty. \tag{2.6}$$

For $s \in [0,T]$, let $R_s \in \mathcal{H}(f_s, \tilde{f}_s)$ be such that

$$d_1(f_s,\tilde{f}_s)=\int_{\mathbb{R}^d\times\mathbb{R}^d}|v-\tilde{v}|\,R_s(dv,d\tilde{v}).$$

Then for all $t \in [0,T]$,

$$\begin{aligned} d_1(f_t,\tilde{f}_t)\le d_1(f_0,\tilde{f}_0)+\kappa_1\frac{|\mathbb{S}^{d-2}|}{2}\int_0^t ds\int_{\mathbb{R}^d\times\mathbb{R}^d}R_s(dv,d\tilde{v})\int_{\mathbb{R}^d\times\mathbb{R}^d}R_s(dv_*,d\tilde{v}_*)&\\ \times\Big[8\,\big(\Phi(|v-v_*|)\wedge\Phi(\tilde{v}-\tilde{v}_*)\big)\,|v-\tilde{v}|&\\ +\big(\Phi(|v-v_*|)-\Phi(\tilde{v}-\tilde{v}_*)\big)_+\,|v-v_*|&\\ +\big(\Phi(|\tilde{v}-\tilde{v}_*|)-\Phi(v-v_*)\big)_+\,|\tilde{v}-\tilde{v}_*|\Big].& \end{aligned} \tag{2.7}$$

The meaning of this inequality can be understood by means of probabilistic arguments, see [7] for details. Consider however two infinite particle systems, whose velocity distributions are f and $\tilde{f}$ respectively. The main ideas are that the first term on the right hand side expresses an increase of the optimal coupling due to simultaneous collisions (in both systems), whose rate is (optimally) the minimum between the two rates. Next, the second and third terms explain that the optimal coupling also increases due to a difference between the rates of collision in the two systems. Note that these two last terms equal zero in case of Maxwellian molecules.

We now give the application of our inequality to the study of hard potentials.

Corollary 2.3. *Let B be a collision kernel which satisfies* **(A1-A2)**, *and* **(A3)**(γ) *for some $\gamma \in (0,1]$.*

(i) *Let $\varepsilon > 0$ be fixed. There exists a constant $K_\varepsilon > 0$ such that for any pair of weak solutions $(f_t)_{t\in[0,T]}$, $(\tilde f_t)_{t\in[0,T]}$ to (1.1), lying in $L^\infty([0,T],\mathcal{P}_1(\mathbb{R}^d))$ and satisfying*

$$C\big(T, f+\tilde f, \varepsilon\big) := \sup_{[0,T]} \int_{\mathbb{R}^d} e^{\varepsilon |v|^\gamma} \big[f_t + \tilde f_t\big](dv) < +\infty, \tag{2.8}$$

there holds for all $t \in [0,T]$:

$$d_1(f_t,\tilde f_t) \le d_1(f_0, g_0) + K_\varepsilon\, C\big(T, f+\tilde f,\varepsilon\big) \int_0^t d_1(f_s,\tilde f_s)\big(1 + \big|\log d_1(f_s,\tilde f_s)\big|\big)\, ds.$$

(ii) *As a consequence for any $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$, there exists at most one weak solution $f \in L^\infty([0,T],\mathcal{P}_1(\mathbb{R}^d))$ to (1.1) starting from f_0 and such that $C(T,f,\varepsilon) < +\infty$.*

(iii) *Let us now give an existence and uniqueness result, assuming (here only) additionnally* **(A4)**(γ) *and* (**A5**). *Consider $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$ such that, for some $\varepsilon_0 > 0$, $K > 0$, we have*

$$\int_{\mathbb{R}^d} e^{\varepsilon_0 |v|^\gamma}\, f_0(dv) \le K < +\infty. \tag{2.9}$$

Then there exists a unique weak solution $(f_t)_{t\in[0,\infty)} \in L^1_{\text{loc}}([0,\infty),\mathcal{P}_1(\mathbb{R}^d))$ starting from f_0. Furthermore, there exist $\varepsilon_1 > 0$ and $\bar K > 0$, depending only on ε_0, K, B, such that for all $T > 0$, $C(T,f,\varepsilon_1) \le \bar K$.

(iv) *Finally let us give a result on the dependence according to the initial datum. Consider a family $(f^n)_{n\ge1}, f^\infty$ of weak solutions to (1.1) such that, for some $\varepsilon > 0$, $T > 0$, we have*

$$\sup_{n\ge1} C(T, f^\infty + f^n, \varepsilon) < +\infty.$$

Then

$$\lim_{n\to\infty} d_1(f_0^n, f_0^\infty) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \sup_{[0,T]} d_1(f_t^n, f_t^\infty) = 0.$$

Let us recall that this result applies in particular to hard potentials in dimension $d = 3$ (that is inverse power-law potentials with $s > 5$). In [6], under very similar conditions on the collision kernel, a well-posedness and stability result was obtained in the space $L^\infty([0,T],W^{1,1}(\mathbb{R}^d,(1+|v|^2)\,dv))$. We thus relax the regularity assumption, but we require more moments.

We finally apply our inequality to the study of soft potentials.

Corollary 2.4. *Let B be a collision kernel which satisfies* **(A1-A2)**, *and* **(A3)**(γ) *for some $\gamma \in (-d,0)$.*

(i) *Let $p \in (d/(d+\gamma),\infty]$ be fixed. There exists a constant $K_p > 0$ such that for any pair of weak solutions $(f_t)_{t\in[0,T]}$, $(\tilde f_t)_{t\in[0,T]}$ to (1.1) on $[0,T]$, lying in $L^\infty([0,T],\mathcal{P}_1(\mathbb{R}^d)) \cap L^1([0,T],L^p(\mathbb{R}^d))$, there holds*

$$\forall\, t\in[0,T], \qquad d_1(f_t,\tilde f_t) \le d_1(f_0,g_0)\, e^{K_p\left[C(t,f,p)+C(t,\tilde f,p)+t\right]},$$

where

$$\forall\, t\in[0,T], \qquad C(t,f,p) = \int_0^t \|f_s\|_{L^p(\mathbb{R}^d)}\,ds.$$

Uniqueness and stability thus hold in $L^\infty([0,T],\mathcal{P}_1(\mathbb{R}^d))\cap L^1([0,T],L^p(\mathbb{R}^d))$.

(ii) *Let $p\in(d/(d+\gamma),\infty]$. For any initial condition $f_0\in\mathcal{P}_1(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$, local existence and uniqueness hold, that is there exists*

$$T_* = T_*\big(\|f_0\|_{L^p(\mathbb{R}^d)},B\big) > 0$$

such that there exists a unique weak solution $(f_t)_{t\in[0,T_*)}$ to (1.1) which furthermore belongs to

$$L^\infty_{\text{loc}}\big([0,T_*),\mathcal{P}_1(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)\big).$$

(iii) *Assume now furthermore that $\gamma\in(-1,0)$,* **(A4)**(γ), *and* **(A6)**(ν) *for some $\nu\in(-\gamma,1)$. Consider an initial datum $f_0\in\mathcal{P}_1(\mathbb{R}^d)$ with finite energy and entropy, that is*

$$\int_{\mathbb{R}^d} f_0(v)(|v|^2+|\log f_0(v)|)dv < \infty. \tag{2.10}$$

Assume also that for some $q>\gamma^2/(\nu+\gamma)$, $f_0\in L^1(\mathbb{R}^d,|v|^q dv)$. Then there exists a unique weak solution $(f_t)_{t\in[0,\infty)}$ to (1.1), which furthermore belongs to

$$L^\infty_{\text{loc}}\big([0,\infty),\mathcal{P}_1(\mathbb{R}^d)\cap L^1(\mathbb{R}^d,(|v|^q+|v|^2)\,dv)\big)\cap L^1_{\text{loc}}\big([0,\infty),L^p(\mathbb{R}^d)\big)$$

for some (explicit) $p\in(d/(d+\gamma),d/(d-\nu))$.

Let us recall that point (iii) applies, in dimension $d=3$, to the case of moderately soft potentials, that is inverse power-law potentials with $s\in(3,5)$. In such a case, one has $\gamma=(s-5)/(s-1)$ and $\nu=2/(s-1)\in(-\gamma,1)$. We observe that for

$s \in (s_0, 5)$, with $s_0 = 2\sqrt{5} - 1 \simeq 3.472$, the choice $q = 2$ is possible, so that our conditions reduce to the finiteness of entropy and energy.
On the contrary, for $s > 3$ close to 3, q has to be chosen very large, e.g., for $s = 3.01$, we have to take $q \simeq 200$.

A similar result was obtained in [6, Theorem 1.3], assuming that $f_0 \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, |v|^q\,dv) \cap W^{1,1}(\mathbb{R}^d, (1+|v|^2)\,dv)$, with $p > d/(d+\gamma)$ and $q > \gamma^2/(\nu+\gamma)$. We thus relax a large part of these conditions.

The rest of the paper is dedicated to the proof of these results: we establish Theorem 2.2 in Section 3. Applications to hard and soft potentials are studied in Sections 4 and 5 respectively.

3. The general inequality

As a preliminary step, we shall parameterize precisely the post-collisional velocities. We follow here the approach of [8], which was strongly inspired by Tanaka [15], and we extend it to any dimension $d \geq 2$.

The first step is to define a parameterization of the sphere orthogonal to some given vector $X \in \mathbb{R}^d$. This parameterization shall not be smooth of course. We identify in the sequel $\mathbb{S}^0 = \{-1, +1\}$.
For $X \in \mathbb{R}^d \backslash \{0\}$, we set S_X to be the symmetry with respect to the hyperplane

$$H_X = \left(e_d - \frac{X}{|X|} \right)^{\perp}$$

(where $e_d = (0, \dots, 0, 1)$) if $e_d \neq X/|X|$, and $S_X = \mathrm{Id}$ else. We set

$$C_X = \big\{ U \in \mathbb{R}^d \ ; \ |U| = |X| \text{ and } \langle U, X \rangle = 0 \big\}.$$

Then we parameterize C_X by $\mathbb{S}^{d-2}$ as follows: we set

$$\forall\, \xi = (\xi_1, ..., \xi_{d-1}) \in \mathbb{S}^{d-2}, \quad \Pi(\xi) = (\xi_1, \dots, \xi_{d-1}, 0) \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$$

and

$$\Gamma(X, \xi) = |X|\, S_X\big(\Pi(\xi)\big).$$

It is easy to check that for a given X, the map $\xi \in \mathbb{S}^{d-2} \mapsto \Gamma(X, \xi)$ is a bijection onto C_X and is a unitary parameterization. Therefore, for $\xi \in \mathbb{S}^{d-2}$, $\theta \in [0, \pi]$, and $X, v, v_* \in \mathbb{R}^d$, one may write

$$v' = v'(v, v_*, \theta, \xi) = v + \frac{\cos\theta - 1}{2}\,(v - v_*) + \frac{\sin\theta}{2}\,\Gamma(v - v_*, \xi)$$

and for all $\varphi \in \mathrm{Lip}(\mathbb{R}^d)$, recalling (2.3)

$$A[\varphi](v,v_*) = \Phi(|v-v_*|) \int_0^\pi \beta(d\theta) \int_{\mathbb{S}^{d-2}} d\xi \left[\varphi\big(v'(v,v_*,\theta,\xi)\big) - \varphi(v)\right].$$

A problem of this parameterization is its lack of smoothness. To overcome this difficulty, we shall prove the following fine version of a Lemma due to Tanaka [15], whose proof may be found in [8, Lemma 2.6] in dimension 3.

Lemma 3.1. *There exists a measurable map $\xi_0 : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-2} \mapsto \mathbb{S}^{d-2}$ such that for any $X, Y \in \mathbb{R}^d \setminus \{0\}$, the map $\xi \mapsto \xi_0(X,Y,\xi)$ is a bijection with jacobian 1 from $\mathbb{S}^{d-2}$ into itself (when $d \geq 3$), and*

$$\forall\, \xi \in \mathbb{S}^{d-2}, \quad \big|\Gamma(X,\xi) - \Gamma\big(Y, \xi_0(X,Y,\xi)\big)\big| \leq 3\,|X-Y|. \tag{3.1}$$

This implies that for all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^d$, all $\theta \in [0,\pi]$, all $\xi \in \mathbb{S}^{d-2}$, we have

$$\begin{aligned} &\big|v'(v,v_*,\theta,\xi) - v'(\tilde{v},\tilde{v}_*,\theta,\xi_0(v-v_*,\tilde{v}-\tilde{v}_*,\xi)\big| \\ &\qquad \leq |v-\tilde{v}| + 2\,\theta\,\big(|v-\tilde{v}| + |v_*-\tilde{v}_*|\big). \end{aligned} \tag{3.2}$$

Proof of Lemma 3.1. The case $d = 2$ is trivial, therefore we assume $d \geq 3$.
Let us consider $X, Y \in \mathbb{R}^d \setminus \{0\}$. If $X/|X| = Y/|Y|$, it is enough to choose $\xi_0(X,Y,\xi) = \xi$. Indeed in this case $S_X = S_Y$ so that

$$\big|\Gamma(X,\xi) - \Gamma(Y,\xi)\big| = \big||X| - |Y|\big| \leq |X-Y|.$$

Now assume that $X/|X| \neq Y/|Y|$. Then let us define $R_{X,Y}$ to be the axial rotation of $\mathbb{R}^d$ transforming $X/|X|$ into $Y/|Y|$ around a line perpendicular to the plane determined by X and Y. Let us then define ξ_0 by the identity

$$\Gamma\big(Y, \xi_0(X,Y,\xi)\big) = \frac{|Y|}{|X|}\, R_{X,Y}\big(\Gamma(X,\xi)\big) \in C_Y.$$

For any $X, Y \in \mathbb{R}^d \setminus \{0\}$, the application $\xi \mapsto \xi_0(X,Y,\xi)$ is the restriction to $\mathbb{S}^{d-2}$ of the following orthogonal linear transformation on $\mathbb{R}^{d-1}$

$$\forall\, Z \in \mathbb{R}^{d-1}, \quad O_{X,Y}(Z) = \Pi^{-1} \circ S_Y \circ R_{X,Y} \circ S_X \circ \Pi(Z).$$

Therefore it has unit jacobian. Finally let us check the control (3.1):

$$\begin{aligned} \big|\Gamma(X,\xi) - \Gamma\big(Y,\xi_0(X,Y,\xi)\big)\big| &= \left|\Gamma(X,\xi) - \frac{|Y|}{|X|}\, R_{X,Y}\,\Gamma(X,\xi)\right| \\ \leq \left|\Gamma(X,\xi)\left(1 - \frac{|Y|}{|X|}\right)\right| &+ \frac{|Y|}{|X|}\,\big|\Gamma(X,\xi) - R_{X,Y}\,\Gamma(X,\xi)\big)\big| \\ &\leq |X-Y| + |Y|\,\left|\frac{Y}{|Y|} - \frac{X}{|X|}\right| \leq 3\,|X-Y|. \end{aligned}$$

□

Since the transformaction $\xi_0(X,Y,\cdot)$ has unit jacobian, one may finally rewrite (2.3), for all $\varphi \in \mathrm{Lip}(\mathbb{R}^d)$, all $X,Y\in\mathbb{R}^d$ (which may depend on v,v_*,θ), as

$$A[\varphi](v,v_*) = \Phi(|v-v_*|)\int_0^\pi \beta(d\theta)\int_{\mathbb{S}^{d-2}} d\xi\,\big[\varphi\big(v'\big(v,v_*,\theta,\xi_0(X,Y,\xi)\big)\big)-\varphi(v)\big]. \tag{3.3}$$

We may finally give the

Proof of Theorem 2.2. We denote

$$h_t^\varphi := \int_{\mathbb{R}^d}\varphi(v)\,\big(f_t-\tilde f_t\big)(dv)$$

for $\varphi\in\mathrm{Lip}_1(\mathbb{R}^d)$, $t\in[0,T]$. We also set $h_t = d_1(f_t,\tilde f_t)$, and we recall that

$$h_t = \int_{\mathbb{R}^d\times\mathbb{R}^d}|v-\tilde v|\,R_t(dv,d\tilde v) = \sup_{\varphi\,\in\,\mathrm{Lip}_1(\mathbb{R}^d)} h_t^\varphi.$$

Step 1. Let us thus consider $\varphi\in\mathrm{Lip}_1(\mathbb{R}^d)$. Using (2.2), that $R_t\in\mathcal{H}(f_t,\tilde f_t)$ and (3.3), we immediately obtain, using the map ξ_0 built in Lemma 3.1,

$$\begin{aligned}
\frac{d}{dt}h_t^\varphi &= \int_{\mathbb{R}^d\times\mathbb{R}^d} f_t(dv)\,f_t(dv_*)\,A[\varphi](v,v_*) - \int_{\mathbb{R}^d\times\mathbb{R}^d}\tilde f_t(d\tilde v)\,f_t(d\tilde v_*)\,A[\varphi](\tilde v,\tilde v_*)\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv,d\tilde v)\int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*,d\tilde v_*)\Big(A[\varphi](v,v_*)-A[\varphi](\tilde v,\tilde v_*)\Big)\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv,d\tilde v)\int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*,d\tilde v_*)\int_0^\pi\beta(d\theta)\int_{\mathbb{S}^{d-2}}d\xi\\
&\qquad \Big(\Phi(|v-v_*|)\,\big[\varphi\big(v'(v,v_*,\theta,\xi)\big)-\varphi(v)\big]\\
&\qquad -\Phi(|\tilde v-\tilde v_*|)\,\big[\varphi\big(v'(\tilde v,\tilde v_*,\theta,\xi_0\big(v-v_*,\tilde v-\tilde v_*,\xi\big)\big)-\varphi(\tilde v)\big]\Big).
\end{aligned}\tag{3.4}$$

We now use the shortened notation

$$v' = v'(v,v_*,\theta,\xi) \text{ and } \tilde v' = v'\big(\tilde v,\tilde v_*,\theta,\xi_0\big(v-v_*,\tilde v-\tilde v_*,\xi\big)\big).$$

Noting that for all $x,y\in\mathbb{R}$, $x = x\wedge y + (x-y)_+$, we easily deduce from (3.4) that

$$\begin{aligned}
\frac{d}{dt}h_t^\varphi &= \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv,d\tilde v)\int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*,d\tilde v_*)\int_0^\pi\beta(d\theta)\int_{\mathbb{S}^{d-2}}d\xi\\
&\qquad\Big(\big[\Phi(|v-v_*|)\wedge\Phi(|\tilde v-\tilde v_*|)\big]\times\big[\varphi(v')-\varphi(\tilde v')-\varphi(v)+\varphi(\tilde v)\big]\\
&\qquad +\big[\Phi(|v-v_*|)-\Phi(|\tilde v-\tilde v_*|)\big]_+\times\big[\varphi(v')-\varphi(v)\big]\\
&\qquad +\big[\Phi(|\tilde v-\tilde v_*|)-\Phi(|v-v_*|)\big]_+\times\big[\varphi(\tilde v)-\varphi(\tilde v')\big]\Big)\\
&=: I_1^\varphi(t)+I_2^\varphi(t)+I_3^\varphi(t),
\end{aligned}$$

where the last equality stands for a definition. Using that $\varphi \in \mathrm{Lip}_1(\mathbb{R}^d)$, (2.4), and **(A2)**, we get

$$I_2^\varphi(t) + I_3^\varphi(t) \le \kappa_1 \frac{|\mathbb{S}^{d-2}|}{2} \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv, d\tilde v) \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*, d\tilde v_*) \Big(\big[\Phi(|v-v_*|) - \Phi(|\tilde v - \tilde v_*|)\big]_+ |v-v_*| + \big[\Phi(|\tilde v-\tilde v_*|) - \Phi(|v - v_*|)\big]_+ |\tilde v-\tilde v_*| \Big).$$

Next, using again that $\varphi \in \mathrm{Lip}_1(\mathbb{R}^d)$, we get that for all $\varepsilon \in (0,\pi)$,

$$\begin{aligned} I_1^\varphi(t) \le & \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv, d\tilde v) \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*, d\tilde v_*) \int_0^\varepsilon \beta(d\theta) \int_{\mathbb{S}^{d-2}} d\xi \\ & \qquad \big[\Phi(|v-v_*|) \wedge \Phi(|\tilde v - \tilde v_*|)\big] \times \big[|v'-v| + |\tilde v' - \tilde v|\big] \\ & + \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv, d\tilde v) \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*, d\tilde v_*) \int_\varepsilon^\pi \beta(d\theta) \int_{\mathbb{S}^{d-2}} d\xi \\ & \qquad \big[\Phi(|v-v_*|) \wedge \Phi(|\tilde v - \tilde v_*|)\big] \times \big[|v'-\tilde v'| - |v - \tilde v|\big] \\ & + \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv, d\tilde v) \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*, d\tilde v_*) \int_\varepsilon^\pi \beta(d\theta) \int_{\mathbb{S}^{d-2}} d\xi \\ & \qquad \big[\Phi(|v-v_*|) \wedge \Phi(|\tilde v - \tilde v_*|)\big] \times \big[|v-\tilde v| - \big(\varphi(v) - \varphi(\tilde v)\big)\big] \\ =: & \; J_1^\varphi(t,\varepsilon) + J_2^\varphi(t,\varepsilon) + J_3^\varphi(t,\varepsilon), \end{aligned}$$

where the last equality stands for a definition. First for $J_2^\varphi(t,\varepsilon)$, using (3.2) and **(A2)**, we immediately get, by symmetry, that

$$\begin{aligned} J_2^\varphi(t) \le & \; 2\left(\int_\varepsilon^\pi \theta\,\beta(d\theta)\right) |\mathbb{S}^{d-2}| \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv, d\tilde v) \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*, d\tilde v_*) \\ & \qquad \big[\Phi(|v-v_*|) \wedge \Phi(|\tilde v - \tilde v_*|)\big] \big[|v-\tilde v| + |v_* - \tilde v_*|\big] \\ \le & \; 4\,\kappa_1\,|\mathbb{S}^{d-2}| \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv, d\tilde v) \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*, d\tilde v_*) \\ & \qquad \big[\Phi(|v-v_*|) \wedge \Phi(|\tilde v - \tilde v_*|)\big] |v-\tilde v|. \end{aligned}$$

Next, setting

$$\alpha_\varepsilon = |\mathbb{S}^{d-2}| \int_0^\varepsilon \theta\,\beta(d\theta),$$

it is not hard to obtain, using (2.4), the fact that $R_t \in \mathcal{H}(f_t, \tilde{f}_t)$ and a symmetry argument, that

$$\begin{aligned} J_1^\varphi(t,\varepsilon) \le \frac{\alpha_\varepsilon}{2} \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv, d\tilde{v}) \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*, d\tilde{v}_*) \\ \times \left[\Phi(|v-v_*|)\wedge\Phi(|\tilde{v}-\tilde{v}_*|)\right]\left(|v|+|\tilde{v}|+|v_*|+|\tilde{v}_*|\right) \\ \le \alpha_\varepsilon \int_{\mathbb{R}^d} f_t(dv)\int_{\mathbb{R}^d} f_t(dv_*)\,\Phi(|v-v_*|)\,|v| + \alpha_\varepsilon \int_{\mathbb{R}^d}\tilde{f}_t(d\tilde{v})\int_{\mathbb{R}^d}\tilde{f}_t(d\tilde{v}_*)\,\Phi(|\tilde{v}-\tilde{v}_*|)\,|\tilde{v}| \\ \le C\big(t,f,\tilde{f}\big)\,\alpha_\varepsilon, \end{aligned}$$

where the constant $C\big(t,f,\tilde{f}\big)$ belongs to $L^1([0,T])$ due to (2.6).

Finally for $J_3^\varphi(t,\varepsilon)$ we notice that the integrand is nonnegative (since $\varphi \in \mathrm{Lip}_1(\mathbb{R}^d)$) and does not depend on θ, φ. Hence, denoting

$$S_\varepsilon := |\mathbb{S}^{d-2}| \int_\varepsilon^\pi \beta(d\theta) < +\infty,$$

we have, for any $A > 0$,

$$J_3^\varphi(t,\varepsilon) \le K_1^\varphi(t,\varepsilon,A) + K_2^\varphi(t,\varepsilon,A),$$

where

$$\begin{aligned} K_1^\varphi(t,\varepsilon,A) &= A\,S_\varepsilon \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv,d\tilde{v})\int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*,d\tilde{v}_*) \\ &\qquad \left[|v-\tilde{v}| - \big(\varphi(v)-\varphi(\tilde{v})\big)\right] \\ K_2^\varphi(t,\varepsilon,A) &= S_\varepsilon \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv,d\tilde{v})\int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*,d\tilde{v}_*) \\ &\qquad \left[\Phi(|v-v_*|)\wedge\Phi(|\tilde{v}-\tilde{v}_*|)\right] 1\!\!1_{\{\Phi(|v-v_*|)\wedge\Phi(|\tilde{v}-\tilde{v}_*|)>A\}}\,|v-\tilde{v}|. \end{aligned}$$

Using that $R_t \in \mathcal{H}(f_t, \tilde{f}_t)$, and that it achieves the Wasserstein distance, we get

$$K_1^\varphi(t,\varepsilon,A) = A\,S_\varepsilon\left[d_1(f_t,\tilde{f}_t) - h_t^\varphi\right].$$

Next, we obtain

$$\begin{aligned} K_2^\varphi(t,\varepsilon,A) &\le S_\varepsilon \int_{\mathbb{R}^d} f_t(dv)\int_{\mathbb{R}^d} f_t(dv_*)\,|v|\,\Phi(|v-v_*|)\,1\!\!1_{\{\Phi(|v-v_*|)>A\}} \\ &\quad + S_\varepsilon \int_{\mathbb{R}^d}\tilde{f}_t(d\tilde{v})\int_{\mathbb{R}^d}\tilde{f}_t(d\tilde{v}_*)\,|\tilde{v}|\,\Phi(|\tilde{v}-\tilde{v}_*|)\,1\!\!1_{\{\Phi(|\tilde{v}-\tilde{v}_*|)>A\}} \\ &\le S_\varepsilon\,C_A\big(t,f,\tilde{f}\big). \end{aligned}$$

Due to (2.6), we observe that

$$\lim_{A\to\infty} \int_0^T C_A\big(t,f,\tilde f\big)\,dt = 0.$$

Step 2. Gathering all the previous estimates, we observe that for any $\varphi \in \mathrm{Lip}_1(\mathbb{R}^d)$, $t\in[0,T]$, $\varepsilon>0$, $A>0$, we have

$$\frac{d}{dt} h_t^\varphi \;\;\le\;\; H_t + \Gamma_{\varepsilon,A}(t) + A\,S_\varepsilon\,\big[h_t - h_t^\varphi\big], \tag{3.5}$$

where

$$\Gamma_{\varepsilon,A}(t) := \alpha_\varepsilon\,C\big(t,f,\tilde f\big) + S_\varepsilon\,C_A\big(t,f,\tilde f\big),$$

and

$$\begin{aligned} H_t := \kappa_1\,\frac{|\mathbb{S}^{d-2}|}{2} \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv,d\tilde v) \int_{\mathbb{R}^d\times\mathbb{R}^d} R_t(dv_*,d\tilde v_*) \Big(8\,\big[\Phi(|v-v_*|)\wedge\Phi(\tilde v-\tilde v_*)\big]\,|v-\tilde v| \\ + \big[\Phi(|v-v_*|)-\Phi(\tilde v-\tilde v_*)\big]_+\,|v-v_*| + \big[\Phi(|\tilde v-\tilde v_*|)-\Phi(v-v_*)\big]_+\,|\tilde v-\tilde v_*|\Big). \end{aligned}$$

Recall that $h_t = \sup_{\varphi\in\mathrm{Lip}_1(\mathbb{R}^d)} h_t^\varphi$, and that our aim is to prove that

$$h_t \le h_0 + \int_0^t H_s\,ds. \tag{3.6}$$

We immediately deduce from (3.5) that

$$h_t^\varphi\,e^{A\,S_\varepsilon\,t} \le h_0^\varphi + \int_0^t e^{A\,S_\varepsilon\,s}\,\big[H_s + \Gamma_{\varepsilon,A}(s)\big]\,ds + A\,S_\varepsilon \int_0^t h_s\,e^{A\,S_\varepsilon\,s}\,ds.$$

Then we take the supremum over $\varphi\in\mathrm{Lip}_1(\mathbb{R}^d)$ and we use the generalized Gronwall Lemma which states that

$$u_t \le g_t + a\int_0^t u_s\,ds$$

implies that

$$u_t \le g_0\,e^{at} + \int_0^t e^{a\,(t-s)}\,\frac{dg_s}{ds}\,ds,$$

which yields

$$h_t\,e^{A\,S_\varepsilon\,t} \le h_0\,e^{A\,S_\varepsilon\,t} + e^{A\,S_\varepsilon\,t}\int_0^t \big[H_s + \Gamma_{\varepsilon,A}(s)\big]\,ds,$$

so that for all $t\in[0,T]$,

$$h_t \le h_0 + \int_0^t H_s\,ds + \int_0^T \Gamma_{\varepsilon,A}(t)\,dt.$$

This inequality holding for any $\varepsilon > 0$, $A > 0$, we easily conclude that (3.6) holds, since

$$\lim_{A\to\infty} \int_0^T \Gamma_{\varepsilon,A}(t)\,dt = \alpha_\varepsilon \int_0^T C\big(t,f,\tilde f\big)\,dt$$

with

$$\int_0^T C\big(t,f,\tilde f\big)\,dt < +\infty$$

and

$$\alpha_\varepsilon = |\mathbb{S}^{d-2}| \int_0^\varepsilon \theta\,\beta(d\theta) \xrightarrow[\varepsilon\to 0]{} 0$$

due to **(A2)**. □

4. Application to hard potentials

4.1. **Propagation of exponential moments.** We first prove a lemma on the propagation (and appearance) of exponential moment, which is a variant of results first obtained in [2, 3] (and also developed in [14, 12]).

Lemma 4.1. *Let B be a collision kernel satisfying assumptions* **(A1-A2-A5)** *and* **(A3)-(A4)**(γ) *for some $\gamma \in (0,1]$. Let $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$.*

(i) *Assume that for some $\varepsilon_0 > 0$, some $s \in (0,2)$,*

$$\int_{\mathbb{R}^d} e^{\varepsilon_0 |v|^s}\, f_0(dv) \le C_{\varepsilon_0,s} < +\infty.$$

Then there exists $\varepsilon_1 > 0$ and a constant $C > 0$, depending only on s, ε_0, $C_{\varepsilon_0,s}$, such that for any $T > 0$, any weak solution $(f_t)_{t\in[0,T]}$ to (1.1) *satisfies*

$$\sup_{[0,T]} \int_{\mathbb{R}^d} e^{\varepsilon_1 |v|^s}\, f_t(dv) \le C < +\infty.$$

(ii) *Assume now only that $e_0 = \int_{\mathbb{R}^d} |v|^2 f_0(dv) < \infty$. For any $s \in (0,\gamma/2)$, any $\tau > 0$, there exists $\varepsilon > 0$ and $C > 0$, depending only on s, τ, and an upperbound of e_0 such that for any $T > 0$, any weak solution $(f_t)_{t\in[0,T]}$ to* (1.1) *satisfies*

$$\sup_{t\in[\tau,T]} \int_{\mathbb{R}^d} e^{\varepsilon |v|^s}\, f_t(dv) \le C < +\infty.$$

Proof of Lemma 4.1. We first recall that for any $t \in [0,T]$,

$$\int_{\mathbb{R}^d} |v|^2 f_t(dv) = \int_{\mathbb{R}^d} |v|^2 f_0(dv) =: e_0, \tag{4.1}$$

and we observe that for all $v \in \mathbb{R}^d$, all $t \geq 0$, since $\gamma \in (0,1]$ and since $f_t \in \mathcal{P}_1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |v - v_*|^\gamma f_t(dv_*) \geq |v|^\gamma - \int_{\mathbb{R}^d} |v_*|^\gamma f_t(dv_*) \geq |v|^\gamma - e_0^{\gamma/2}. \tag{4.2}$$

Let us fix $0 < s < 2$. We define for any $p \in \mathbb{R}_+$

$$m_p(t) := \int_{\mathbb{R}^d} |v|^{sp} \, f_t(dv).$$

Step 1. The evolution equation (2.2) yields

$$\frac{dm_p}{dt} = \int_{\mathbb{R}^d \times \mathbb{R}^d} \Phi(|v - v_*|) \, K_p(v, v_*) \, f_t(dv) \, f_t(dv_*), \tag{4.3}$$

where, using **(A5)** and a symmetry argument,

$$K_p(v, v_*) := \frac{1}{2} \int_0^\pi \int_{\mathbb{S}^{d-2}} \left(|v'|^{sp} + |v'_*|^{sp} - |v|^{sp} - |v_*|^{sp} \right) b(\cos\theta) \, d\theta \, d\xi.$$

Let us split $b = b^c_\eta + b^r_\eta$ for some $\eta \in (0, \pi)$ with

$$b^c_\eta(\cos\theta) = b(\cos\theta) \, 1\!\!1_{\theta \geq \eta} + \left[b(\cos\eta) + b'(\cos\eta) \, (\cos\theta - \cos\eta) \right] 1\!\!1_{0 \leq \theta \leq \eta}$$

for $\theta \in (0, \pi]$. Due to **(A5)**, we know that $b^c_\eta \leq b$, so that $b^r_\eta \geq 0$. We can split correspondingly $K_p = K^{c,\eta}_p + K^{r,\eta}_p$. We also easily check that for each $\eta \in (0, \pi)$, b^c_η is convex, non-decreasing, and bounded on $[-1, 1)$. We are thus in a position to apply [3, Corollary 1], which yields that for $p > 2/s$,

$$K^{c,\eta}_p(v, v_*) \leq \alpha_p(\eta) \left(|v|^2 + |v_*|^2 \right)^{sp/2} - K(\eta) \left(|v|^{sp} + |v_*|^{sp} \right) \tag{4.4}$$

where $(\alpha_p(\eta))_p$ is strictly decreasing and satifies

$$\forall \, p > 2/s, \quad 0 < \alpha_p < \frac{C(\eta)}{sp + 1} \tag{4.5}$$

for some constant $C(\eta)$ depending on an upper bound of b^c_η, and some constant $K(\eta)$ depending on a lower bound of the mass of β^c_η. Therefore K can be made uniform according to η as $\eta \to 0$.

For the other part of the collision kernel we use for instance [6, Lemma 2.1] and assumption **(A2)** to deduce that (as soon as $sp \geq 2$)

$$K^{r,\eta}_p(v, v_*) \leq \delta(\eta) \left(|v|^{sp} + |v_*|^{sp} \right) \tag{4.6}$$

with

$$\delta(\eta) \leq \text{cst} \int_0^\pi \theta \, b^r_\eta(\cos\theta) \, d\theta \to 0$$

as $\eta \to 0$, due to **(A2)**.

Combining (4.4,4.5,4.6) and fixing carefully η we thus find for all $p > 2/s$

$$K_p(v, v_*) \le \bar{\alpha}_p \left(|v|^2 + |v_*|^2\right)^{sp/2} - \bar{K}\left(|v|^{sp} + |v_*|^{sp}\right)$$

for some constant $\bar{K} > 0$ and where $(\bar{\alpha}_p)_p$ is strictly decreasing and satisfies, for some constant $\bar{C} > 0$,

$$\forall\, p > 2/s, \quad 0 < \bar{\alpha}_p < \frac{\bar{C}}{sp+1}.$$

We of course deduce that for p large enough, say $p \ge p_0 > 2/s$,

$$K_p(v, v_*) \le \bar{\alpha}_p \left[\left(|v|^2 + |v_*|^2\right)^{sp/2} - |v|^{sp} - |v_*|^{sp}\right] - \bar{K}\left(|v|^{sp} + |v_*|^{sp}\right), \tag{4.7}$$

changing if necessary the value of $\bar{K} > 0$.
We now insert (4.7) in (4.3). Using **(A4)**(γ) and (4.2), we get, for $p \ge p_0$,

$$\frac{dm_p}{dt} \le \bar{\alpha}_p\, Q_p - K'\left(m_{p+\gamma/s} - e_0^{\gamma/2} m_p\right) \tag{4.8}$$

for some new constant $K' > 0$ and with

$$Q_p := \int_{\mathbb{R}^d \times \mathbb{R}^d} \left[\left(|v|^2 + |v_*|^2\right)^{sp/2} - |v|^{sp} - |v_*|^{sp}\right] \Phi(|v - v_*|)\, f_t(dv)\, f_t(dv_*).$$

Step 2. Using **(A3)**(γ) and following line by line the proof of [14, Lemma 4.7] from [14, eq. (4.13)] which is the same as (4.8) here to [14, eq. (4.19)] (this proof is itself essentially based on [3]), we obtain the following conclusion. Set $k_p = [sp/4 + 1/2]$ (here $[\cdot]$ stands for the integer part). Set also, with the usual Gamma function,

$$z_p := \frac{m_p}{\Gamma(p + 1/2)} \quad \text{and} \quad Z_p := \max_{k=1,..,k_p} \left\{ z_{(2k+\gamma)/s}\, z_{p-2k/s},\; z_{2k/s}\, z_{p-2k/s+\gamma/s} \right\}.$$

Then for some constants $A' > 0$, $A'' > 0$, $A''' > 0$, for all $p \ge p_0$,

$$\frac{dz_p}{dt} \le A'\, p^{\gamma/s - 1/2}\, Z_p - A''\, p^{\gamma/s}\, z_p^{1 + \frac{\gamma}{sp}} + A''' z_p. \tag{4.9}$$

Step 3. Next, point (i) can be checked following the ideas of [12, Proposition 3.2] (for $\gamma = 1$) and using that classically, for any $p \ge 0$, $\sup_{t \in [0,\infty)} m_p \le C_p$, for some constant depending only on B and $m_p(0)$, see e.g., [18, Theorem 1-(ii)] and [6, Lemma 2.1].

Step 4. Finally, point (ii) can be proved following line by line the proof of [14, Lemma 4.7].

□

4.2. **Proof of Corollary 2.3.** We first recall the following variant of a classical lemma used by Yudovitch [21] in his Cauchy theorem for bidimensional incompressible non-viscious flow. See [4, Lemme 5.2.1, p. 89] for a proof.

Lemma 4.2. *Consider a nonnegative bounded function ρ on $[0,T]$, a real number $a \in [0,\infty)$, and a strictly positive, continuous and non-decreasing function $\mu = \mu(x)$ on $(0,\infty)$. Assume furthermore that*

$$\int_0^1 \frac{dx}{\mu(x)} = +\infty,$$

and that for all $t \in [0,T]$,

$$\rho(t) \le a + \int_0^t \mu(\rho(s))\, ds.$$

Then

(i) *if $a = 0$, then $\rho(t) = 0$ for all $t \in [0,T]$;*

(ii) *if $a > 0$, then*

$$\forall\, t \in [0,T], \quad m(a) - m(\rho(t)) \le t$$

where

$$m(x) = \int_x^1 \frac{dy}{\mu(y)}.$$

We may now give the

Proof of Corollary 2.3. We thus consider $\gamma \in (0,1]$, and we assume **(A1)-(A2)-(A3)**(γ). We also consider some $\varepsilon > 0$ fixed.

Step 1. Let us first prove point (i). Let us consider two weak solutions $(f_t)_{t\in[0,T]}$, $(\tilde f_t)_{t\in[0,T]}$ to (1.1), lying in $L^\infty([0,T],\mathcal{P}_1(\mathbb{R}^d))$ and satisfying (2.8). We are in position to apply Theorem 2, since **(A3)**(γ) and (2.8) clearly guarantee that (2.6) holds. We thus know that (2.7) holds. Using **(A3)**(γ), simple computations show that

$$\big(\Phi(|v-v_*|) \wedge \Phi(|\tilde v - \tilde v_*|)\big)\, |v - \tilde v| \le C\big[|v|^\gamma + |v_*|^\gamma\big]\, |v - \tilde v|,$$

while

$$\begin{aligned}
&\big[\Phi(|v-v_*|) - \Phi(|\tilde v - \tilde v_*|)\big]_+ |v - v_*| \\
&\le C\,|(|v-v_*|^\gamma - |\tilde v - \tilde v_*|^\gamma)|\, \big[|v-v_*| \wedge |\tilde v - \tilde v_*| + |(|v-v_*| - |\tilde v - \tilde v_*|)|\,\big] \\
&\le C\gamma\,(|v-v_*| \wedge |\tilde v - \tilde v_*|)^{\gamma-1}\, \big|\big(|v-v_*| - |\tilde v - \tilde v_*|\big)\big|\, (|v-v_*| \wedge |\tilde v - \tilde v_*|) \\
&\qquad\qquad + C\big[|v-v_*|^\gamma + |\tilde v - \tilde v_*|^\gamma\big]\, \big[|v - \tilde v| + |v_* - \tilde v_*|\big] \\
&\le C(1+\gamma)\big[|v|^\gamma + |v_*|^\gamma + |\tilde v|^\gamma + |\tilde v_*|^\gamma\big]\, \big[|v - \tilde v| + |v_* - \tilde v_*|\big].
\end{aligned}$$

We hence obtain by inserting these inequalities in (2.7) and using symmetry properties, that for some constant $D>0$,

$$d_1(f_t,\tilde f_t) \le d_1(f_0,\tilde f_0) + D\int_0^t ds \int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv,d\tilde v)\int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv_*,d\tilde v_*) \\ \times \left[|v|^\gamma+|v_*|^\gamma+|\tilde v|^\gamma+|\tilde v_*|^\gamma\right]|v-\tilde v|.$$

Recall now that $R_s\in\mathcal{H}(f_s,\tilde f_s)$ achieves the Wasserstein distance. It is thus clear (recall that $C(T,f+\tilde f,\varepsilon)$ was defined in (2.8)) that

$$\sup_{[0,T]}\int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv_*,d\tilde v_*)\left[|v_*|^\gamma+|\tilde v_*|^\gamma\right] \le A_\varepsilon\, C(T,f+\tilde f,\varepsilon)$$

for some constant A_ε. We thus get

$$d_1(f_t,\tilde f_t) \le d_1(f_0,\tilde f_0) + D\,A_\varepsilon\, C\big(T,f+\tilde f,\varepsilon\big)\int_0^t d_1(f_s,\tilde f_s)\,ds \\ +D\int_0^t ds\int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv,d\tilde v)\left[|v|^\gamma+|\tilde v|^\gamma\right]|v-\tilde v|.$$

Next, for any $s\in[0,T]$ and $a>0$, we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv,d\tilde v)\left[|v|^\gamma+|\tilde v|^\gamma\right]|v-\tilde v| \le 2\,a^\gamma\, d_1(f_s,\tilde f_s) \\ +\int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv,d\tilde v)\left[|v|^\gamma+|\tilde v|^\gamma\right]\left[|v|+|\tilde v|\right]\left(\mathbb{1}_{\{|v|>a\}}+\mathbb{1}_{\{|\tilde v|>a\}}\right) \\ \le 2\,a^\gamma\, d_1(f_s,\tilde f_s) + L_\varepsilon\int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv,d\tilde v)\left[e^{-\varepsilon a^\gamma/2}\,e^{\varepsilon|v|^\gamma}+e^{-\varepsilon a^\gamma/2}\,e^{\varepsilon|\tilde v|^\gamma}\right] \\ \le 2\,a^\gamma\, d_1(f_s,\tilde f_s) + L_\varepsilon\, e^{-\varepsilon a^\gamma/2}\, C\big(T,f+\tilde f,\varepsilon\big),$$

for some constant L_ε such that

$$\left[|v|^\gamma+|\tilde v|^\gamma\right]\left[|v|+|\tilde v|\right]\left[e^{\varepsilon|v|^\gamma/2}+e^{\varepsilon|\tilde v|^\gamma/2}\right] \le L_\varepsilon\left[e^{\varepsilon|v|^\gamma}+e^{\varepsilon|\tilde v|^\gamma}\right].$$

Choosing a such that

$$a^\gamma = \left|2\,\log d_1(f_s,\tilde f_s)/\varepsilon\right|,$$

we finally get

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv,d\tilde v)\left[|v|^\gamma+|\tilde v|^\gamma\right]|v-\tilde v| \le \frac{4}{\varepsilon}\, d_1(f_s,\tilde f_s)\left|\log d_1(f_s,\tilde f_s)\right| \\ +L_\varepsilon\, C\big(T,f+\tilde f,\varepsilon\big)\, d_1(f_s,\tilde f_s).$$

We finally obtain, setting $K_\varepsilon = D\,(A_\varepsilon + L_\varepsilon + 4/\varepsilon)$, that

$$d_1(f_t,\tilde f_t) \le d_1(f_0,\tilde f_0) + K_\varepsilon\, C\big(T, f+\tilde f, \varepsilon\big) \int_0^t d_1(f_s,\tilde f_s)\,\big(1+\big|\log d_1(f_s,\tilde f_s)\big|\big)\,ds.$$

Step 2. Points (ii) and (iv) are immediate consequences of point (i) and Lemma 4.2 applied with $\mu(x) = x\,\big(1+|\log x|\big)$.

Step 3. Finally, we check point (iii). We thus assume **(A1)-(A2)-(A3)**(γ)-**(A4)**(γ)-**(A5)**, and consider an initial condition $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$ satisfying (2.9) for some $\varepsilon_0 > 0$. Then we know from Lemma 4.1-(i) that any weak solution starting from f_0 satisfies (2.8) for some $\varepsilon_1 > 0$. We thus deduce the uniqueness part from point (ii).
Next, we approximate f_0 by a sequence of initial conditions f_0^n with finite entropy satisfying (2.9) uniformly (in n), and such that $d_1(f_0, f_0^n)$ tends to 0. Then, using for example the existence result of Villani [17, Theorem 1], we know that for each n, there exists a weak solution $(f_t^n)_{t\ge0}$ to (1.1) starting from f_0^n. Due to Lemma 4.1-(i), we deduce that there exists $\varepsilon_1 > 0$ such that for all $T > 0$, $\sup_n C(T, f_t^n, \varepsilon_1) < \infty$. It is then not hard to deduce from point (i) and Lemma 4.2 that there exists $(f_t)_{t\ge0}$ such that for all $T>0$, $C(T,f,\varepsilon_1)<\infty$ and $\lim_n \sup_{[0,T]} d_1(f_t^n, f_t) = 0$. An easy consequence is that $(f_t)_{t\ge0}$ is a weak solution to (1.1) starting from f_0. □

5. Application to soft potentials

The application to soft potentials is easier, since we shall apply the standard Gronwall Lemma instead of that of Yudovitch.

Proof of Corollary 2.4. We consider $\gamma \in (-d,0)$, and assume that **(A1)-(A2)-(A3)**(γ).

We observe at once that for $\alpha \in (-d,0)$, and for $q \in (d/(d+\alpha), \infty]$, there exists a constant $C_{\alpha,q}$ such that for any $g \in \mathcal{P}_1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, any $v \in \mathbb{R}^d$,

$$\begin{aligned}\int_{\mathbb{R}^d} g(v_*)\,|v-v_*|^\alpha\,dv_* &\le \int_{|v_*-v|<1} g(v_*)\,|v-v_*|^\alpha\,dv_* + \int_{|v_*-v|\ge1} g(v_*)\,dv_* \\ &\le C_{\alpha,q}\,\|g\|_{L^q(\mathbb{R}^d)} + 1.\end{aligned} \tag{5.1}$$

Step 1. We first prove point (i). Let thus $p \in (d/(d+\gamma), \infty]$. We consider two solutions $f, \tilde f$ as in the statement. In order to apply Theorem 2.2, we have to check that (2.6) holds. But using (5.1), since $p > d/(d+\gamma)$, we get for $t \in [0,T]$

$$\begin{aligned}
&\int_{\mathbb{R}^d} f_t(dv) \int_{\mathbb{R}^d} f_t(dv_*)\,(1+|v|)\,|v-v_*|^\gamma \\
&\le \left[\int_{\mathbb{R}^d} f_t(dv)\,(1+|v|)\right] \sup_{v\in\mathbb{R}^d} \int_{\mathbb{R}^d} f_t(dv_*)\,|v-v_*|^\gamma \\
&\le \left[\int_{\mathbb{R}^d} f_t(dv)\,(1+|v|)\right] \big(C_{\gamma,p}\|f_t\|_{L^p(\mathbb{R}^d)}+1\big)\,.
\end{aligned}$$

The same estimate holds for $\tilde f$, and therefore we conclude that the estimate (2.6) holds using that f and $\tilde f$ belong to $L^\infty([0,T],\mathcal{P}_1(\mathbb{R}^d)) \cap L^1([0,T],L^p(\mathbb{R}^d))$.

Hence we deduce that (2.7) holds. Simple computations using **(A3)**(γ) show that

$$\Big(\Phi(|v-v_*|)\wedge\Phi(|\tilde v-\tilde v_*|)\Big)\,|v-\tilde v| \le C|v-v_*|^\gamma\,|v-\tilde v|,$$

while

$$\begin{aligned}
&\Big(\Phi(|v-v_*|)-\Phi(|\tilde v-\tilde v_*|)\Big)_+\,|v-v_*| \\
&\le C\Big|\Big(|v-v_*|^\gamma-|\tilde v-\tilde v_*|^\gamma\Big)\Big|\,\Big(|v-v_*|\wedge|\tilde v-\tilde v_*|+\Big|\Big(|v-v_*|-|\tilde v-\tilde v_*|\Big)\Big|\Big) \\
&\le C|\gamma|\,\left(|v-v_*|\wedge|\tilde v-\tilde v_*|\right)^{\gamma-1}\,\Big|\left(|v-v_*|-|\tilde v-\tilde v_*|\right)\Big|\,\left(|v-v_*|\wedge|\tilde v-\tilde v_*|\right) \\
&+C\Big(|v-v_*|^\gamma\vee|\tilde v-\tilde v_*|^\gamma\Big)\,\Big|\left(|v-v_*|-|\tilde v-\tilde v_*|\right)\Big| \\
&\le C(1+|\gamma|)\,\left(|v-v_*|^\gamma+|\tilde v-\tilde v_*|^\gamma\right)\,\left(|v-\tilde v|+|v_*-\tilde v_*|\right).
\end{aligned}$$

Inserting these inequalities in (2.7) and using a symmetry argument, we obtain that for some constant $D>0$,

$$\begin{aligned}
d_1(f_t,\tilde f_t) \le d_1(f_0,\tilde f_0) + D\int_0^t ds \int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv,d\tilde v) \int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv_*,d\tilde v_*) \\
\Big[|v-v_*|^\gamma+|\tilde v-\tilde v_*|^\gamma\Big]\,|v-\tilde v|.
\end{aligned}$$

Recall now that $R_s \in \mathcal{H}(f_s,\tilde f_s)$ and achieves the Wasserstein distance. Hence,

$$\begin{aligned}
&\sup_{v,\tilde v}\int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv_*,d\tilde v_*)\Big[|v-v_*|^\gamma+|\tilde v-\tilde v_*|^\gamma\Big] \\
&\le \sup_v \int_{\mathbb{R}^d} f_t(dv_*)\,|v-v_*|^\gamma + \sup_{\tilde v}\int_{\mathbb{R}^d} \tilde f_t(d\tilde v_*)\,|\tilde v-\tilde v_*|^\gamma \\
&\le C_{\gamma,p}\,\|f_t\|_{L^p(\mathbb{R}^d)} + C_{\gamma,p}\,\|\tilde f_t\|_{L^p(\mathbb{R}^d)} + 2 \\
&\le C_{\gamma,p}\left[\|f_t\|_{L^p(\mathbb{R}^d)}+\|\tilde f_t\|_{L^p(\mathbb{R}^d)}\right]+2,
\end{aligned}$$

where we used (5.1). Since

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} R_s(dv, d\tilde v)\, |v-\tilde v| = d_1(f_s, \tilde f_s)$$

we obtain finally, choosing $K_p := D\,(C_{\gamma,p}+2)$,

$$d_1(f_t, \tilde f_t) \le d_1(f_0, \tilde f_0) + K_p\left[\|f_t\|_{L^p(\mathbb{R}^d)} + \|\tilde f_t\|_{L^p(\mathbb{R}^d)} + 1\right] \int_0^t d_1(f_s, \tilde f_s)\, ds.$$

The Gronwall Lemma then allows us to conclude the proof.

Step 2. We now check point (ii). We only have to prove the existence of solutions, since uniqueness follows from point (i). Using some results of Villani [17, Theorems 1 and 3], we know that for $\gamma \in (-d, 0)$, for any $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$ such that

$$\int f_0(v)\left(|v|^2 + |\log f_0(v)|\right) dv < +\infty,$$

there exists a weak solution $f \in L^\infty([0,\infty), (1+|v|^2)\, dv)$ to (1.1) starting from f_0. Then the existence result of point (ii) follows immediately from point (i) together with the following *a priori* estimates, which guarantee that if $f_0 \in \mathcal{P}_1(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$, then this bound propagates locally (in time): first there exists $C = C(B)$ such that (see [6, Proposition 3.2] and its proof) for any $p \in (d/(d+\gamma), \infty]$, any weak solution to (1.1) satisfies

$$\frac{d}{dt}\|f_t\|_{L^p} \le C\left(1 + \|f_t\|^2_{L^p(\mathbb{R}^d)}\right),$$

so that for $0 \le t < T_* := \frac{1}{C}(\pi/2 - \arctan ||f_0||_{L^p})$, we have

$$\|f_t\|_{L^p} \le \tan\left(\arctan \|f_0\|_{L^p} + C\,t\right). \tag{5.2}$$

Next, we easily check, using (2.2), (2.4) and **(A2)** that

$$\frac{d}{dt}\int_{\mathbb{R}^d} |v|\, f_t(dv) \quad\le\quad \kappa_1\,\frac{|\mathbb{S}^{d-2}|}{2}\int_{\mathbb{R}^d} f_t(dv)\int_{\mathbb{R}^d} f_t(dv_*)\, |v-v_*|^{1+\gamma}.$$

If $1+\gamma \ge 0$, we immediately conclude, since $|v-v_*|^{1+\gamma} \le 1 + |v| + |v_*|$, that

$$\frac{d}{dt}\int_{\mathbb{R}^d} |v|\, f_t(dv) \le \kappa_1\,\frac{|\mathbb{S}^{d-2}|}{2}\left(1 + 2\int_{\mathbb{R}^d} |v|\, f_t(dv)\right),$$

so that for $t \ge 0$, we have

$$\int_{\mathbb{R}^d} |v| f_t(dv) \le e^{\kappa_1|\mathbb{S}^{d-2}|t}\left(\int_{\mathbb{R}^d} |v| f_0(dv) + 1\right).$$

If $1+\gamma \le 0$, we use (5.1) (with $\alpha = 1+\gamma$ and $q = p$, which is valid since $p > d/(d+\gamma) > d/(d+\alpha)$), and we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}^d} |v|\, f_t(dv) \le \kappa_1 \,\frac{|\mathbb{S}^{d-2}|}{2}\, \big(C_{1+\gamma,p}\, \|f_t\|_{L^p(\mathbb{R}^d)} + 1\big) =: A\, \|f_t\|_{L^p(\mathbb{R}^d)} + A',$$

so that for $0 \le t < T_*$, we have, recalling (5.2),

$$\int_{\mathbb{R}^d} |v|\, f_t(dv) \le \int_{\mathbb{R}^d} |v|\, f_0(dv) + A \int_0^t \tan\left(\arctan \|f_0\|_{L^p} + C\, s\right)\, ds + A'\, t.$$

Step 3. We now assume additionnally that $\gamma \in (-1,0)$, (**A4**)(γ), and (**A6**)(ν) for some $\nu \in (-\gamma, 1)$. We consider an initial datum f_0 with finite energy and entropy (2.10), and such that for some $q > q_0 = \gamma^2/(\nu+\gamma)$, $f_0 \in L^1(\mathbb{R}^d, |v|^q dv)$. Applying the result of Villani [17, Theorem 1], we know that there exists a weak solution $(f_t)_{t\in[0,\infty)}$ to (1.1).
To conclude the proof, it suffices to apply point (i), and to check that for any weak solution $(f_t)_{t\in[0,\infty)}$ to (1.1) starting from f_0,

(a) $f \in L^\infty_{loc}([0,\infty), L^1(\mathbb{R}^d, (|v|^2 + |v|^q)dv))$,

(b) there exists $p > p_0 := d/(d+\gamma)$ such that $f \in L^1_{loc}([0,\infty), L^p(\mathbb{R}^d))$.

Point (a) follows from a straightforward application of (2.2), using (**A1**)-(**A2**)-(**A3**)(γ) and that $\gamma \in (-1,0)$, and concluding with the Gronwall Lemma.

To check point (b), we follow the line of [6, Proposition 3.3] (see (3.2), (3.3) and (3.4) in [6]), which was relying on exploiting the entropy production and its regularization property obtained by Alexandre-Desvillettes-Villani-Wennberg [1].
Exactly as in [6, (3.2)], we get that for any $\alpha > 0$,

$$\int_0^T \|(1+|v|)^{\gamma-\alpha} f_t(v)\|_{L^{d/(d-\nu)}(\mathbb{R}^d)}\, dt \le C\,(1+T) \tag{5.3}$$

for any $\alpha > 0$ any $T > 0$ and some constant $C > 0$ (depending on α). Using point (a), we also now that for all $T > 0$,

$$C_T := \sup_{[0,T]} \|(1+|v|)^q f_t(v)\|_{L^1(\mathbb{R}^d)} < \infty. \tag{5.4}$$

By interpolation between estimates (5.3) and (5.4), we see that for all $T > 0$, for another constant C_T depending on T,

$$\int_0^T \|f\|_{L^p(\mathbb{R}^d)}\, dt \le C_T \tag{5.5}$$

for any $1 < p < d/(d-\nu)$ as soon as, for instance

$$q > (\alpha - \gamma)\frac{p-1}{1-p(d-\nu)/d}. \tag{5.6}$$

Since $p_0 = d/(d+\gamma) < d/(d-\nu)$ (because $\gamma > -\nu$) and since by assumption,

$$q > q_0 = (-\gamma)\frac{p_0 - 1}{1 - p_0(d-\nu)/d} = \frac{\gamma^2}{\gamma+\nu},$$

we clearly have (5.6) when choosing with $\alpha > 0$ small enough and $p > p_0$ close enough. This concludes the proof of point (b). □

References

[1] R. Alexandre, L. Desvillettes, C. Villani, B. Wennberg, *Entropy dissipation and long-range interactions*, Arch. Rational Mech. Anal. 152, no. 4, 327-355, 2000.

[2] A. V. Bobylev, *Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems*, J. Statist. Phys. 88, no. 5-6, 1183–1214, 1997.

[3] A. V. Bobylev, I. M. Gamba, V. A. Panferov, *Moment inequalities and high-energy tails for Boltzmann equations with inelastic interactions*, J. Statist. Phys. 116, no. 5-6, 1651–1682, 2004.

[4] J.Y. Chemin, *Fluides parfaits incompressibles*, Astérisque, no. 230, 1995.

[5] L. Desvillettes, *Boltzmann's Kernel and the Spatially Homogeneous Boltzmann Equation*, Rivista di Matematica dell'Universita di Parma, vol. 6 , n. 4, 1–22 (special issue), 2001.

[6] L. Desvillettes, C. Mouhot, *Regularity, stability and uniqueness for the spatially homogeneous Boltzmann equation with long-range interactions*, arXiv eprint math.AP/0606307 (2006).

[7] N. Fournier, *Uniqueness for a class of spatially homogeneous Boltzmann equations without angular cutoff*, J. Statist. Phys. 125, no. 4, 927–946, 2006.

[8] N. Fournier, S. Méléard, *A stochastic particle numerical method for 3D Boltzmann equations without cutoff*, Math. Comp. 71, no. 238, 583–604, 2002.

[9] J. Horowitz, R.L. Karandikar, *Martingale problems associated with the Boltzmann equation*, Seminar on Stochastic Processes, 1989 (San Diego, CA, 1989), 75–122, Progr. Probab., 18, Birkhuser Boston, Boston, MA, 1990.

[10] X. Lu, C. Mouhot, *About measures solutions of the spatially homogeneous Boltzmann equation*, work in progress.

[11] X. Lu, B. Wennberg, *Solutions with increasing energy for the spatially homogeneous Boltzmann equation*, Nonlinear Anal. Real World Appl. 3, no. 2, 243–258, 2002.

[12] S. Mischler, C. Mouhot, M. Rodriguez Ricard, *Cooling process for inelastic Boltzmann equations for hard spheres. I. The Cauchy problem*, J. Stat. Phys. 124, no. 2-4, 655–702, 2006.

[13] S. Mischler, B. Wennberg, *On the spatially homogeneous Boltzmann equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 16, no. 4, 467–501, 1999.

[14] C. Mouhot, *Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials*, Comm. Math. Phys. 261, no. 3, 629–672, 2006.

[15] H. Tanaka, *Probabilistic treatment of the Boltzmann equation of Maxwellian molecules*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 46, no. 1, 67–105, 1978-1979.

[16] G. Toscani, C. Villani, *Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas*, J. Statist. Phys. 94, no. 3-4, 619–637, 1999.

[17] C. Villani, *On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations*, Arch. Rational Mech. Anal. 143, no. 3, 273–307, 1998.
[18] C. Villani, *A review of mathematical topics in collisional kinetic theory*, Handbook of mathematical fluid dynamics, Vol. I, 71–305, North-Holland, Amsterdam, 2002.
[19] C. Villani, *Topics in optimal transportation,* Graduate Studies in Mathematics, 58. American Mathematical Society, Providence, RI, 2003.
[20] B. Wennberg, *An example of nonuniqueness for solutions to the homogeneous Boltzmann equation*, J. Statist. Phys. 95, no. 1-2, 469–477, 1999.
[21] V. Yudovich, *Non stationary flow of an ideal incompressible liquid*, Zhurn. Vych. Mat 3, 1032–1066, 1963 (in russian).