# On some stochastic coalescents

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**Abstract**

We consider infinite systems of macroscopic particles characterized by their masses. Each pair of particles with masses $x$ and $y$ coalesce at a given rate $K(x, y)$. We assume that $K$ satisfies a sort of Hölder property with index $\lambda \in (0, 1]$, and that the initial condition admits a moment of order $\lambda$. We show the existence of such infinite particle systems, as strong Markov processes enjoying a Feller property. We also show that the obtained processes are the only possible limits when making the number of particles tend to infinity in a sequence of finite particle systems with the same dynamics.

*Key words* : Coalescence, Stochastic interacting particle systems.

*MSC 2000* : 60K35, 60J25.

## 1 Introduction

Coalescence is a widespread phenomenon: it arises in physics (droplets, smoke), chemistry (polymer), astrophysics (formation of galaxies), biology (hematology, population theory), and mathematics (graphs and trees).

We consider a possibly infinite system of particles characterized by their masses. The total mass of the system is supposed to be equal to 1. The only interactions taken into account are the following: two particles with masses $x$ and $y$ are assumed to merge into a single particle with mass $x + y$ at some given rate $K(x, y) \geq 0$, which we will refer to as the *coagulation kernel*. Two different situations have to be separated.

(a) Assume first that the particles are microscopic, and that the rate of coalescence is infinitesimal. Then the system can be described by $\{c(t, m)\}_{t\geq 0, m\in(0,\infty)}$, where $c(t, m)$ is the concentration of particles with mass $m$ at time $t$. In such a case, $\{c(t, m)\}_{m\in(0,\infty)}$ solves a nonlinear deterministic integro-differential equation, known as the Smoluchowski coagulation equation, see Laurençot-Mischler [7] and Leyvraz [8] for recent reviews on this topic.

(b) When the particles are macroscopic and when the rate of coagulation is not infinitesimal, then the study can not be reduced to the investigation of a deterministic quantity.

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We refer to the review of Aldous [3] on stochastic coalescence and on the links between the microscopic and macroscopic scales. The present work is dedicated to the study of case (b).

When the initial state consists in a finite number of macroscopic particles, the stochastic coalescent obviously exists, and is known as the Marcus-Lushnikov process [10, 9]. When there are initially infinitely many particles, stochastic coalescence with constant, additive, and multiplicative kernels have been extensively studied, see Kingman [6], Aldous [1], Aldous-Pitman [2]. Much less seems to be known for general kernels. The only work dealing with general kernels seems to be that of Evans-Pitman [5], where the existence of such processes satisfying a Feller property is proved under some quite restrictive conditions on the coagulation kernel and the initial state.

The aim of this work is to generalize the results of [5], and to investigate existence, uniqueness, and Feller property of stochastic coalescents with general kernels. Corollary 2.5 below answers partially to Open Problem 13 of Aldous [3].

# 2 Notations and main results

Consider the following state space

$$\mathcal{S} = \left\{ m = (m_k)_{k\geq 1}, \quad m_1 \geq m_2 \geq ... \geq 0, \ \sum_{k=1}^{\infty} m_k \leq 1 \right\}.$$

We think a state $m \in \mathcal{S}$ as the sequence of ordered masses of the particles in an infinite system. We endow $\mathcal{S}$ with the pointwise convergence topology, which makes $\mathcal{S}$ compact (due to the Fatou Lemma), and which can be metrized by the distance

$$d(m, \tilde{m}) = \sum_{k\geq 1} 2^{-k} |m_k - \tilde{m}_k|. \tag{2.1}$$

In the sequel, a *coagulation kernel* $K$ will be a function on $[0,1]^2$ such that for any $x, y \in [0,1]$, $K(x,y) = K(y,x) \geq 0$. The number $K(x,y)$ represents the rate at which two particles with masses $x$ and $y$ will aggregate.

For $1 \leq i < j$, the coalescence between the $i$-th and $j$-th larger particles is described by the map $c_{ij} : \mathcal{S} \mapsto \mathcal{S}$, with

$$c_{ij}(m) = \text{reordered}(m_1, ..., m_{i-1}, m_i + m_j, m_{i+1}, ..., m_{j-1}, m_{j+1}, ...). \tag{2.2}$$

For any $k \geq 1$, $[c_{ij}(m)]_k$ stands for the $k$-th element of the sequence $c_{ij}(m)$.

Our aim is to study a $\mathcal{S}$-valued Markov process $(M(t))_{t\geq 0}$, starting from some given $M(0) \in \mathcal{S}$, with generator $\mathcal{L}$ given, for any $\Phi : \mathcal{S} \mapsto \mathbb{R}$ sufficiently regular, any $m \in \mathcal{S}$, by

$$\mathcal{L}\Phi(m) = \sum_{1\leq i<j<\infty} K(m_i, m_j) \left[\Phi(c_{ij}(m)) - \Phi(m)\right]. \tag{2.3}$$

We will prove in Section 3 that $\mathcal{L}$ is at least well-defined, as soon as $K$ is bounded, on the following class of functionnals:

$$\mathcal{C} = \{\Phi : \mathcal{S} \mapsto \mathbb{R}, \quad \exists\, c > 0,\ \forall\, m, \tilde{m} \in \mathcal{S},\ |\Phi(m) - \Phi(\tilde{m})| \leq c d(m, \tilde{m})\}. \tag{2.4}$$

For example, for any $n \geq 1$, any Lipschitz function $f : [0,1]^n \mapsto \mathbb{R}$, $\Phi(m) = f(m_1, ..., m_n)$ belongs to $\mathcal{C}$.

We need to introduce some subsets of $\mathcal{S}$. First,

$$\mathcal{S}_0 = \Big\{ m = (m_k)_{k\geq 1} \in \mathcal{S}, \quad \sum_{k\geq 1} m_k = 1, \ \exists\, n \geq 1, \ m = (m_1, ..., m_n, 0, ...) \Big\}$$

contains all states with finitely many particles. Next, for $\lambda \in (0,1]$,

$$\mathcal{S}_\lambda = \left\{ m = (m_k)_{k\geq 1} \in \mathcal{S}, \quad \sum_{k\geq 1} m_k = 1, \quad \sum_{k=1}^{\infty} m_k^\lambda < \infty \right\}$$

stands the set of states with total mass 1 and with a moment of order $\lambda$. Note that for any $0 < \lambda < \mu < 1$,

$$\mathcal{S}_0 \subset \mathcal{S}_\lambda \subset \mathcal{S}_\mu \subset \mathcal{S}_1.$$

**Definition 2.1** *(i) We say that a $\mathcal{S}$-valued (or $\mathcal{S}_\lambda$-valued) process $(M(t))_{t\geq 0}$ is càdlàg if a.s., the map $t \mapsto M(t)$ is càdlàg for the pointwise convergence topology on $\mathcal{S}$.*
*(ii) For a coagulation kernel $K$ bounded by $\bar{K}$, we consider a Poisson measure $N(dt, d(i,j), dz)$ on $[0,\infty) \times \{(i,j) \in \mathbb{N}^2, i < j\} \times [0, \bar{K}]$ with intensity measure $dt \left( \sum_{1\leq k<l<\infty} \delta_{(k,l)} \right) dz$, and denote by $\{\mathcal{F}_t\}_{t\geq 0}$ the associated canonical filtration.*
*(iii) Let $M(0) \in \mathcal{S}$. A càdlàg $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathcal{S}$-valued process $(M(t))_{t\geq 0}$ is said to solve $(SDE(K, M(0), N))$ if a.s., for all $t \geq 0$,*

$$M(t) = M(0) + \int_0^t \int_{i<j} \int_0^{\bar{K}} [c_{ij}(M(s-)) - M(s-)] \mathbb{1}_{\{z \leq K(M_i(s-), M_j(s-))\}} N(ds, d(i,j), dz). \qquad (2.5)$$

**Remark 2.2** *Equation (2.5) has to be understood as: for all $t \geq 0$, all $k \geq 1$,*

$$M_k(t) = M_k(0) + \int_0^t \int_{i<j} \int_0^{\bar{K}} \{[c_{ij}(M(s-))]_k - M_k(s-)\} \mathbb{1}_{\{z \leq K(M_i(s-), M_j(s-))\}} N(ds, d(i,j), dz), \qquad (2.6)$$

*and the above integrals are a.s. well-defined and finite for any càdlàg $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathcal{S}$-valued process $(M(t))_{t\geq 0}$.*

The convergence of the integrals will be checked in Section 4.

**Remark 2.3** *Consider a bounded coagulation kernel $K$, and let $N$ be as in Definition 2.1. Consider an initial condition $m \in \mathcal{S}_0$. Then there exists a unique solution $(M^m(t))_{t\geq 0}$ to $(SDE(K, m, N))$. Furthermore, $(M(t))_{t\geq 0}$ is a strong Markov $\mathcal{S}_0$-valued process. Its infinitesimal generator is given by $\mathcal{L}$ on all bounded measurable functions $\Phi : \mathcal{S} \mapsto \mathbb{R}$. (Notice that $\mathcal{L}\Phi(m)$ is well-defined for any $m \in \mathcal{S}_0$ and any bounded measurable $\Phi : \mathcal{S} \mapsto \mathbb{R}$). Such a Markov process is unique (in law), and is called the $(K, m)$-Marcus-Lushnikov process.*

The proof of this remark is almost immediate, since in this case, the total rate of jump of the system is bounded from above by the constant $\frac{n(n-1)}{2}\bar{K}$, where $n \in \mathbb{N}$ is such that $m_k = 0$ for all $k > n$. We refer to Aldous [3] and the references therein.

To state our result, we introduce, for $\lambda \in (0,1]$, for $m$ and $\tilde{m}$ in $\mathcal{S}_\lambda$,

$$||m||_\lambda = \sum_{k\geq 1} m_k^\lambda, \quad d_\lambda(m,\tilde{m}) = \sum_{k\geq 1} |m_k^\lambda - \tilde{m}_k^\lambda|. \tag{2.7}$$

**Theorem 2.4** *Consider a coagulation kernel $K$ bounded by $\bar{K}$ and satisfying, for some $\kappa \geq 0$, some $\lambda \in (0,1]$,*

$$|K(x,y) - K(u,v)| \leq \kappa \left(|x^\lambda - u^\lambda| + |y^\lambda - v^\lambda|\right), \quad \forall\, x,y,u,v \in [0,1]. \tag{2.8}$$

*Let $N$ be as in Definition 2.1.*
*(i) For $M(0) \in \mathcal{S}_\lambda$, there exists a unique $\mathcal{S}_\lambda$-valued solution $(M(t))_{t\geq 0}$ to $(SDE(K, M(0), N))$.*
*(ii) For any pair of initial conditions $M(0) \in \mathcal{S}_\lambda$ and $\tilde{M}(0) \in \mathcal{S}_\lambda$, denote by $(M(t))_{t\geq 0}$ and $(\tilde{M}(t))_{t\geq 0}$ the associated $\mathcal{S}_\lambda$-valued solutions to $(SDE(K, M(0), N))$ and $(SDE(K, \tilde{M}(0), N))$. For all $t \geq 0$,*

$$E\left[\sup_{[0,t]} d_\lambda(M(s), \tilde{M}(s))\right] \leq d_\lambda(M(0), \tilde{M}(0)) e^{4\kappa(||M(0)||_\lambda + ||\tilde{M}(0)||_\lambda)t}. \tag{2.9}$$

Remark that if $\lambda = 1$, then $||M(0)||_\lambda = ||\tilde{M}(0)||_\lambda = 1$. Notice also that if $K$ is constant, then (2.8) holds for each $\lambda \in (0,1]$ with $\kappa = 0$, so that $d_\lambda$ is non-expanding along solutions to (2.5). As a corollary, we obtain the following result.

**Corollary 2.5** *Consider a bounded coagulation kernel $K$ satisfying (2.8), for some $\lambda \in (0,1]$, $\kappa \geq 0$.*
*(i) For any $m \in \mathcal{S}_\lambda$, there exists a unique (in law) càdlàg $\mathcal{S}_\lambda$-valued process $(M^m(t))_{t\geq 0}$ such that for any sequence $m^n \in \mathcal{S}_0$ satisfying $\lim_n d_\lambda(m, m^n) = 0$, the sequence of $(K, m^n)$-Marcus-Lushnikov processes $(M^{m^n}(t))_{t\geq 0}$ defined in Remark 2.3 converges in law to $(M^m(t))_{t\geq 0}$ in $\mathbb{D}([0,\infty), \mathcal{S}_\lambda)$, $\mathcal{S}_\lambda$ being endowed with the distance $d_\lambda$.*
*(ii) The process $(M^m(t))_{t\geq 0}$ is a strong Markov process enjoying the following Feller property: for all $t \geq 0$, the map $m \mapsto law(M^m(t))$ is continuous from $\mathcal{S}_\lambda$ into $\mathcal{P}(\mathcal{S}_\lambda)$ for the weak convergence topology on $\mathcal{P}(\mathcal{S}_\lambda)$, $\mathcal{S}_\lambda$ being endowed with the distance $d_\lambda$.*
*(iii) The infinitesimal generator of $(M^m(t))_{t\geq 0}$ on functions $\Phi \in \mathcal{C}$ is given by $\mathcal{L}$.*

When $K \equiv 1$, $(M(t))_{t\geq 0}$ is known as the Kingman coalescent, [6]. The case where $K(x,y) = x + y$ (resp. $K(x,y) = xy$) is called the additive (resp. multiplicative) coalescent, see Aldous-Pitman, [2], Aldous [1]. When the initial state contains infinitely many particles, the only work dealing with general kernels seems to be that of Evans and Pitman [5]. They have shown the existence of $(M(t))_{t\geq 0}$ as a Feller process in the situation where $K(0,0) = 0$ and where $K$ is Lipschitz on $[0,1]^2$, for initial values in the set $\{m \in \mathcal{S}_1, \sum_{k\geq 1} k m_k < \infty\}$. Hence, Theorem 2.4 seems to improve consequently their result: it applies to a wider class of kernels (such as $K(x,y) = x^\lambda + y^\lambda$ or $K(x,y) = (xy)^\lambda$, for all $\lambda \in (0,1]$), and even in the case of a Lipschitz kernel, we require a weaker assumption on the initial condition.

After some preliminaries about the distances $d$ and $d_\lambda$ presented in Section 3, we prove Theorem 2.4 in Section 4.

## 3 Preliminaries

We devote this section to some technical issues: we investigate the action of coalescence on the distances $d$ and $d_\lambda$. To this aim, we start with a lemma, which will often allow us to neglect the *reordering* after a coalescence. Recall that a *finite permutation* $\sigma$ of $\mathbb{N} = \{1, 2, ...\}$ is a bijection from $\mathbb{N}$ into $\mathbb{N}$ such that $\sigma(n) = n$ for all $n$ sufficiently large.

**Lemma 3.1** *Consider a finite permutation $\sigma$ of $\mathbb{N}$ and two nonincreasing nonnegative sequences $m = (m_1, m_2, ...)$ and $\tilde{m} = (\tilde{m}_1, \tilde{m}_2, ...)$. Consider also a nonincreasing nonnegative sequence $(a_k)_{k\geq 1}$. Then*

$$\sum_{k\geq 1} a_k |m_k - \tilde{m}_k| \leq \sum_{k\geq 1} a_k |m_k - \tilde{m}_{\sigma(k)}|.$$

**Proof** It clearly suffices to check that for any $n \in \mathbb{N}$, any $m_1 \geq ... \geq m_n \geq 0$, $\tilde{m}_1 \geq ... \geq \tilde{m}_n \geq 0$, and any permutation $\sigma$ of $\{1, ..., n\}$, $\sum_{i=1}^n a_i |m_i - \tilde{m}_i| \leq \sum_{i=1}^n a_i |m_i - \tilde{m}_{\sigma(i)}|$. We work by induction on $n$.

*Step 1.* If $n = 2$, we just have to consider the case where $\sigma(1) = 2$, $\sigma(2) = 1$. Recalling that $a_1 \geq a_2 \geq 0$, and assuming for example that $m_1 \geq \tilde{m}_1$,

$$\begin{aligned} a_1|m_1 - \tilde{m}_1| \ + \ a_2|m_2 - \tilde{m}_2| &= a_1(m_1 - \tilde{m}_2) + a_1(\tilde{m}_2 - \tilde{m}_1) + a_2|m_2 - \tilde{m}_2| \\ &= a_1|m_1 - \tilde{m}_2| - a_1|\tilde{m}_1 - \tilde{m}_2| + a_2|m_2 - \tilde{m}_2| \\ &\leq a_1|m_1 - \tilde{m}_2| + a_2(-|\tilde{m}_1 - \tilde{m}_2| + |m_2 - \tilde{m}_2|) \\ &\leq a_1|m_1 - \tilde{m}_2| + a_2|m_2 - \tilde{m}_1|. \end{aligned}$$

*Step 2.* Assume now that the property holds for permutations of $\{1, ..., n-1\}$, and consider a permutation $\sigma$ of $\{1, ..., n\}$. If $\sigma(n) = n$, the inductive assumption allows us to conclude immediately. Else, consider the inverse bijection $\tau = \sigma^{-1}$, and denote by $k = \tau(n)$. Define the permutation $\tilde{\sigma}$ of $\{1, ..., n-1\}$ as

$$\tilde{\sigma}(i) = \sigma(i) \text{ for } i \neq k, \quad \tilde{\sigma}(k) = \sigma(n).$$

Due to Step 1, since $m_k \geq m_n$, $\tilde{m}_{\sigma(n)} \geq \tilde{m}_n$ and $a_k \geq a_n$,

$$a_n|m_n - \tilde{m}_n| + a_k|m_k - \tilde{m}_{\sigma(n)}| \leq a_n|m_n - \tilde{m}_{\sigma(n)}| + a_k|m_k - \tilde{m}_n|,$$

which can be rewritten as

$$a_n|m_n - \tilde{m}_n| + a_k|m_k - \tilde{m}_{\tilde{\sigma}(k)}| \leq a_n|m_n - \tilde{m}_{\sigma(n)}| + a_k|m_k - \tilde{m}_{\sigma(k)}|.$$

Hence,

$$\begin{aligned} \sum_{i=1}^n a_i|m_i - \tilde{m}_{\sigma(i)}| &= a_n|m_n - \tilde{m}_{\sigma(n)}| + a_k|m_k - \tilde{m}_{\sigma(k)}| + \sum_{i=1, i\neq k}^{n-1} a_i|m_i - \tilde{m}_{\sigma(i)}| \\ &\geq a_n|m_n - \tilde{m}_n| + a_k|m_k - \tilde{m}_{\tilde{\sigma}(k)}| + \sum_{i=1, i\neq k}^{n-1} a_i|m_i - \tilde{m}_{\tilde{\sigma}(i)}| \\ &\geq a_n|m_n - \tilde{m}_n| + \sum_{i=1}^{n-1} a_i|m_i - \tilde{m}_{\tilde{\sigma}(i)}|. \end{aligned}$$

We may conclude, using the inductive assumption. □

The next statement contains some estimates that will be of constant use.

**Corollary 3.2** *Consider $m, \tilde{m} \in \mathcal{S}$ and $1 \le i < j < \infty$. Recall that $c_{ij}(m)$ and and $d$ were defined by (2.2) and (2.1). Then*

$$d(c_{ij}(m), m) \le \tfrac{3}{2} 2^{-i} m_j, \quad \textstyle\sum_{1 \le k < l < \infty} d(c_{kl}(m), m) \le \tfrac{3}{2}, \tag{3.1}$$

$$d(c_{ij}(m), c_{ij}(\tilde{m})) \le (2^i + 2^j) d(m, \tilde{m}). \tag{3.2}$$

*Let now $\lambda \in (0,1]$. Recall that $|| \ ||_\lambda$ and $d_\lambda$ were defined in (2.7). Then, for any $m, \tilde{m}$ in $\mathcal{S}_\lambda$, for any $1 \le i < j < \infty$,*

$$||c_{ij}(m)||_\lambda \le ||m||_\lambda, \tag{3.3}$$

$$d_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) \le d_\lambda(m, \tilde{m}), \tag{3.4}$$

$$d_\lambda(c_{ij}(m), \tilde{m}) \le d_\lambda(m, \tilde{m}) + 2 m_j^\lambda. \tag{3.5}$$

**Proof** First note that for $m \in \mathcal{S}$, $c_{ij}(m)$ is a nonincreasing sequence, and that there exists a finite permutation $\sigma$ of $\mathbb{N}$ (with $\sigma(n) = n$ for all $n \ge i+1$) such that $([c_{ij}(m)]_{\sigma(k)})_{k \ge 1} = (c_k)_{k \ge 1}$, the sequence $c$ being defined by $(c_k)_{k \ge 1} = (m_1, ..., m_{i-1}, m_i + m_j, m_{i+1}, ..., m_{j-1}, m_{j+1}, ...)$. We may also build the corresponding $\tilde{\sigma}$ and $(\tilde{c}_k)_{k \ge 1}$ for $\tilde{m}$.
Using Lemma 3.1 with $a_k = 2^{-k}$, we obtain

$$\begin{aligned} d(c_{ij}(m), m) &= \sum_{k \ge 1} 2^{-k} |[c_{ij}(m)]_k - m_k| \le \sum_{k \ge 1} 2^{-k} |[c_{ij}(m)]_{\sigma(k)} - m_k| \\ &\le \sum_{k \ge 1} 2^{-k} |c_k - m_k| = 2^{-i} m_j + \sum_{k=j}^{\infty} 2^{-k} |m_{k+1} - m_k| \\ &\le 2^{-i} m_j + 2^{-j} m_j \le \frac{3}{2} 2^{-i} m_j, \end{aligned}$$

since $|m_{k+1} - m_k| = m_k - m_{k+1}$ and $i < j$. Since $\sum_{j \ge 1} m_j \le 1$, the second assertion in (3.1) follows immediately.
To prove (3.2), recall that $\sigma(n) = \tilde{\sigma}(n) = n$ for $n \ge i+1$, and write

$$\begin{aligned} d(c_{ij}(m), c_{ij}(\tilde{m})) &= \sum_{k=1}^{i} 2^{-k} |[c_{ij}(m)]_k - [c_{ij}(\tilde{m})]_k| + \sum_{k=i+1}^{j-1} 2^{-k} |m_k - \tilde{m}_k| \\ &\quad + \sum_{k=j}^{\infty} 2^{-k} |m_{k+1} - \tilde{m}_{k+1}| \qquad (3.6) \\ &\le \sum_{k=1}^{i} |[c_{ij}(m)]_k - [c_{ij}(\tilde{m})]_k| + 2 \sum_{k=i+1}^{\infty} 2^{-k} |m_k - \tilde{m}_k|. \end{aligned}$$

Next, due to Lemma 3.1 (with $a_k = 1$ and with the permutation $\tilde{\sigma} \circ \sigma^{-1}$ of $\{1, ..., i\}$),

$$\begin{aligned} \sum_{k=1}^{i} |[c_{ij}(m)]_k - [c_{ij}(\tilde{m})]_k| &\le \sum_{k=1}^{i} |[c_{ij}(m)]_k - [c_{ij}(\tilde{m})]_{\tilde{\sigma} \circ \sigma^{-1}(k)}| \\ &= \sum_{k=1}^{i} |c_{\sigma^{-1}(k)} - \tilde{c}_{\sigma^{-1}(k)}| = \sum_{k=1}^{i} |c_k - \tilde{c}_k| \\ &= \sum_{k=1}^{i-1} |m_k - \tilde{m}_k| + |m_i + m_j - \tilde{m}_i - \tilde{m}_j| \\ &\le |m_j - \tilde{m}_j| + 2^i \sum_{k=1}^{i} 2^{-k} |m_k - \tilde{m}_k|. \qquad (3.7) \end{aligned}$$

Gathering (3.6) and (3.7), we deduce (3.2).
Next, (3.3) is immediate, since $\lambda \in (0,1]$, so that $(m_i+m_j)^\lambda \le m_i^\lambda + m_j^\lambda$.
We now use Lemma 3.1, with $a_k = 1$ for all $k$ and with the permutation $\tilde\sigma \circ \sigma^{-1}$, to deduce that

$$\begin{aligned}
&d_\lambda(c_{ij}(m), c_{ij}(\tilde m)) = \sum_{k\ge1} \left|[c_{ij}(m)]_k^\lambda - [c_{ij}(\tilde m)]_k^\lambda\right| \\
&\le \sum_{k\ge1} \left|[c_{ij}(m)]_k^\lambda - [c_{ij}(\tilde m)]_{\tilde\sigma\circ\sigma^{-1}(k)}^\lambda\right| \\
&= \sum_{k\ge1} \left|c_{\sigma^{-1}(k)}^\lambda - \tilde c_{\sigma^{-1}(k)}^\lambda\right| = \sum_{k\ge1} |c_k^\lambda - \tilde c_k^\lambda| \\
&= \sum_{k\ge1} |m_k^\lambda - \tilde m_k^\lambda| + |(m_i+m_j)^\lambda - (\tilde m_i + \tilde m_j)^\lambda| - |m_i^\lambda - \tilde m_i^\lambda| - |m_j^\lambda - \tilde m_j^\lambda| \\
&\le \sum_{k\ge1} |m_k^\lambda - \tilde m_k^\lambda| = d_\lambda(m, \tilde m).
\end{aligned}$$

We used here that for any $0 \le x \le y$, $0 \le u \le v$, since $\lambda \in (0,1]$,

$$\begin{aligned}
|(x+y)^\lambda - (u+v)^\lambda| &\le |(x+y)^\lambda - (x+v)^\lambda| + |(x+v)^\lambda - (u+v)^\lambda| \\
&\le |y^\lambda - v^\lambda| + |x^\lambda - u^\lambda|.
\end{aligned}$$

We finally check (3.5), using Lemma 3.1, with $a_k = 1$ for all $k$.

$$\begin{aligned}
d_\lambda(c_{ij}(m), \tilde m) &\le \sum_{k\ge1} \left|[c_{ij}(m)]_{\sigma(k)}^\lambda - \tilde m_k^\lambda\right| = \sum_{k\ge1} |c_k^\lambda - \tilde m_k^\lambda| \\
&\le \sum_{k=1}^{i-1} |m_k^\lambda - \tilde m_k^\lambda| + |(m_i+m_j)^\lambda - \tilde m_i^\lambda| \\
&\quad + \sum_{k=i+1}^{j-1} |m_k^\lambda - \tilde m_k^\lambda| + \sum_{k\ge j} |m_{k+1}^\lambda - \tilde m_k^\lambda| \\
&\le \sum_{k\ge1} |m_k^\lambda - \tilde m_k^\lambda| + \left\{|(m_i+m_j)^\lambda - \tilde m_j^\lambda| - |m_i^\lambda - \tilde m_i^\lambda|\right\} \\
&\quad + \sum_{k\ge j} \left\{|m_{k+1}^\lambda - \tilde m_k^\lambda| - |m_k^\lambda - \tilde m_k^\lambda|\right\} \\
&\le d_\lambda(m, \tilde m) + |(m_i+m_j)^\lambda - m_i^\lambda| + \sum_{k\ge j}(m_k^\lambda - m_{k+1}^\lambda) \\
&\le d_\lambda(m, \tilde m) + m_j^\lambda + m_j^\lambda.
\end{aligned}$$

This ends the proof. □

We conclude this section with a Lemma concerning the operator $\mathcal{L}$.

**Lemma 3.3** *Consider $\Phi$ in $\mathcal{C}$ (recall (2.4)).*
*(a) If $K$ is bounded, $\mathcal{L}\Phi$ is well-defined and bounded on $\mathcal{S}$.*
*(b) If $K$ is continuous on $[0,1]^2$, $\mathcal{L}\Phi$ is continuous at any point $m \in \mathcal{S}_1$.*

**Proof** Point (a) is straightforward from (3.1), since for $\Phi \in \mathcal{C}$, for $m \in \mathcal{S}$,

$$|\mathcal{L}\Phi(m)| \le c\bar K \sum_{1\le i<j<\infty} d(c_{ij}(m), m) \le 3c\bar K/2.$$

We now study the continuity of $\mathcal{L}\Phi$. First, let $1 \le i < j < \infty$ be fixed. The map $m \mapsto c_{ij}(m)$ is continuous on $\mathcal{S}$ due to (3.2). Since $K$ is continuous and

$\Phi \in \mathcal{C}$, $m \mapsto \Delta_{ij}(m) = K(m_i, m_j)[\Phi(c_{ij}(m)) - \Phi(m)]$ is continuous on $\mathcal{S}$. Let now $m \in \mathcal{S}_1$, and let $m^l \in \mathcal{S}$ go to $m$ pointwise. Inferring the dominated convergence Theorem, we just have to prove that

$$\lim_{k\to\infty} \limsup_{l\to\infty} \sum_{1\leq i<j<\infty} |\Delta_{ij}(m^l)| \mathbb{1}_{\{i+j\geq k\}} = 0. \tag{3.8}$$

Using (3.1), we obtain

$$\begin{aligned}\sum_{1\leq i<j<\infty} |\Delta_{ij}(m^l)| \mathbb{1}_{\{i+j\geq k\}} &\leq c\bar{K} \sum_{1\leq i<j<\infty} d(c_{ij}(m^l), m^l) \mathbb{1}_{\{i+j\geq k\}} \\ &\leq \frac{3c\bar{K}}{2} \sum_{1\leq i<j<\infty} 2^{-i} m_j^l \mathbb{1}_{\{j\geq k/2\}} \leq \frac{3c\bar{K}}{2} \sum_{j\geq k/2} m_j^l.\end{aligned}$$

To deduce that (3.8) holds, it suffices to note that, since $\sum_{i\geq 1} m_i = 1$ and since $\lim_l m^l = m$ (pointwise),

$$\limsup_l \sum_{j\geq k/2} m_j^l \leq 1 - \liminf_l \sum_{j<k/2} m_j^l = 1 - \sum_{j<k/2} m_j = \sum_{j\geq k/2} m_j,$$

which tends to 0 when $k \to \infty$. □

# 4 Existence and Feller property

This section is dedicated to the proofs of Theorem 2.4 and Corollary 2.5. We first check that (2.5) always makes sense.

**Proof of Remark 2.2.** Consider any càdlàg $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process $(M(t))_{t\geq 0}$. For each $k \geq 1$, set

$$C_k(t) = \int_0^t ds \sum_{i<j} E\Big[K(M_i(s), M_j(s)) \big| [c_{ij}(M(s))]_k - M_k(s) \big| \Big].$$

Then the integrals in the right-hand side of (2.6) are well-defined for all $k \geq 1$ if $C_k(t) < \infty$ for all $k \geq 1$, all $t \geq 0$. But, recalling (2.1) and using (3.1),

$$\sum_{k\geq 1} 2^{-k} C_k(t) \leq \bar{K} \int_0^t ds E\left[\sum_{i<j} d(M(s), c_{ij}(M(s)))\right] \leq \bar{K}\frac{3}{2}t.$$

□

The heart of the proof lies in the following estimate.

**Lemma 4.1** *Assume (2.8), for some $\lambda \in (0,1]$, $\kappa \geq 0$. Consider a Poisson measure $N$ as in Definition 2.1, and $M(0), \tilde{M}(0) \in \mathcal{S}_\lambda$. Assume that there exist some $\mathcal{S}_\lambda$-valued solutions $(M(t))_{t\geq 0}$ and $(\tilde{M}(t))_{t\geq 0}$ to $(SDE(K, M(0), N))$ and $(SDE(K, \tilde{M}(0), N))$. Recall (2.7).*
*1. Almost surely, $t \mapsto ||M(t)||_\lambda$ (and $t \mapsto ||\tilde{M}(t)||_\lambda$) is nonincreasing.*
*2. For any $t \geq 0$,*

$$E\left[\sup_{[0,t]} d_\lambda(M(s), \tilde{M}(s))\right] \leq d_\lambda(M(0), \tilde{M}(0)) e^{4\kappa(||M(0)||_\lambda + ||\tilde{M}(0)||_\lambda)t}.$$

**Proof** The proof of 1 follows from (3.3). To prove 2, note that, since $M$ and $\tilde{M}$ solve (2.5) with the same Poisson measure $N$, we have, for any $t \geq 0$,

$$d_\lambda(M(t), \tilde{M}(t)) = d_\lambda(M(0), \tilde{M}(0)) + A_t + B_t + C_t,$$

where

$$\begin{aligned}
A_t &:= \int_0^t \int_{i<j} \int_0^{\bar{K}} \Big\{ d_\lambda(c_{ij}(M(s-)), c_{ij}(\tilde{M}(s-))) - d_\lambda(M(s-), \tilde{M}(s-)) \Big\} \\
&\qquad \mathbb{1}_{\{z \leq K(M_i(s-), M_j(s-)) \wedge K(\tilde{M}_i(s-), \tilde{M}_j(s-))\}} N(ds, d(i,j), dz), \\
B_t &:= \int_0^t \int_{i<j} \int_0^{\bar{K}} \Big\{ d_\lambda(c_{ij}(M(s-)), \tilde{M}(s-)) - d_\lambda(M(s-), \tilde{M}(s-)) \Big\} \\
&\qquad \mathbb{1}_{\{K(\tilde{M}_i(s-), \tilde{M}_j(s-)) \leq z \leq K(M_i(s-), M_j(s-))\}} N(ds, d(i,j), dz), \\
C_t &:= \int_0^t \int_{i<j} \int_0^{\bar{K}} \Big\{ d_\lambda(M(s-), c_{ij}(\tilde{M}(s-))) - d_\lambda(M(s-), \tilde{M}(s-)) \Big\} \\
&\qquad \mathbb{1}_{\{K(M_i(s-), M_j(s-)) \leq z \leq K(\tilde{M}_i(s-), \tilde{M}_j(s-))\}} N(ds, d(i,j), dz).
\end{aligned}$$

Due to (3.4), we know that $A_t \leq 0$ for all $t \geq 0$ a.s. Next, using (3.5) and (2.8), we conclude that, setting $(x)_+ = \max(x, 0)$,

$$\begin{aligned}
E\left[\sup_{[0,t]} B_s\right] &\leq \int_0^t E\left[\sum_{i<j} 2M_j^\lambda(s)\Big(K(M_i(s), M_j(s)) - K(\tilde{M}_i(s), \tilde{M}_j(s))\Big)_+\right] ds \\
&\leq 2\kappa \int_0^t E\left[\sum_{i<j} M_j^\lambda(s)\left[|M_i^\lambda(s) - \tilde{M}_i^\lambda(s)| + |M_j^\lambda(s) - \tilde{M}_j^\lambda(s)|\right]\right] ds \\
&\leq 2\kappa \int_0^t E\left[\sum_{i\geq 1} |M_i^\lambda(s) - \tilde{M}_i^\lambda(s)| \sum_{j=i+1}^{\infty} M_j^\lambda(s)\right] ds \\
&\quad + 2\kappa \int_0^t E\left[\sum_{j\geq 2} |M_j^\lambda(s) - \tilde{M}_j^\lambda(s)| \sum_{i=1}^{j-1} M_j^\lambda(s)\right] ds \\
&\leq 4\kappa \int_0^t E\left[d_\lambda(M(s), \tilde{M}(s)) ||M(s)||_\lambda\right] ds. \\
&\leq 4\kappa ||M(0)||_\lambda \int_0^t E\left[d_\lambda(M(s), \tilde{M}(s))\right] ds.
\end{aligned}$$

We used here that for $m \in \mathcal{S}_\lambda$, $\sum_{i=1}^{j-1} m_j^\lambda \leq \sum_{i=1}^{j-1} m_i^\lambda \leq ||m||_\lambda$, and Point 1. Using finally the same computation for $C_t$, we conclude that

$$\begin{aligned}
E\left[\sup_{[0,t]} d_\lambda(M(s), \tilde{M}(s))\right] &\leq d_\lambda(M(0), \tilde{M}(0)) \\
&\quad + 4\kappa \left\{||M(0)||_\lambda + ||\tilde{M}(0)||_\lambda\right\} \int_0^t E\left[d_\lambda(M(s), \tilde{M}(s))\right] ds.
\end{aligned}$$

The Gronwall Lemma allows us to conclude. □

**Proof of Theorem 2.4.** Point (ii) has already been checked in Lemma 4.1, from which the uniqueness part in point (i) follows immediately. We thus just have to prove the existence part. We thus consider $\lambda \in (0,1]$ to be fixed, we assume that (2.8) holds, that $M(0) \in \mathcal{S}_\lambda$, and we consider a Poisson measure

$N$ as in Definition 2.1.

*Step 1.* For each $n \geq 1$, consider the initial condition

$$M^n(0) = \left(\frac{M_1(0)}{\alpha_n}, ..., \frac{M_n(0)}{\alpha_n}, 0, ...\right), \quad \alpha_n = \sum_{i=1}^{n} M_i(0).$$

Due to Remark 2.3, there exists a unique $\mathcal{S}_0$-valued solution $(M^n(t))_{t\geq 1}$ to $(SDE(K, M^n(0), N))$. Since $M(0) \in \mathcal{S}_\lambda$, easy computations show that

$$\lim_n d_\lambda(M(0), M^n(0)) = 0, \tag{4.1}$$

$$\sum_{n\geq 1} d_\lambda(M^n(0), M^{n+1}(0)) < \infty, \tag{4.2}$$

$$a := \sup_{n\geq 1} ||M^n(0)||_\lambda = \sup_{n\geq 1, t\geq 0} ||M^n(t)||_\lambda < \infty. \tag{4.3}$$

One may then apply Lemma 4.1 to get, for each $T \geq 0$,

$$E\left[\sup_{[0,T]} d_\lambda(M^n(s), M^{n+1}(s))\right] \leq C_T d_\lambda(M^n(0), M^{n+1}(0))$$

where $C_T = \sup_n e^{4\kappa T(||M^n(0)||_\lambda + ||M^{n+1}(0)||_\lambda)} < \infty$. Thus, due to (4.2),

$$\sum_{n\geq 1} E\left[\sup_{[0,T]} d_\lambda(M^n(s), M^{n+1}(s))\right] < \infty.$$

We deduce that there exists a càdlàg adapted $\mathcal{S}_\lambda$-valued process $(M(t))_{t\geq 0}$ such that for all $T \geq 0$,

$$\lim_n E\left[\sup_{[0,T]} d_\lambda(M^n(s), M(s))\right] = 0. \tag{4.4}$$

The fact that $(M(t))_{t\geq 0}$ is $\mathcal{S}_\lambda$-valued relies on the strong convergence (4.4).

*Step 2.* One may then pass to the limit in (2.5): it suffices to show that $\Delta_n(t) \to 0$ as $n \to \infty$, for any $t > 0$, where

$$\begin{aligned}
\Delta_n(t) &= E\Big[\int_0^t \int_{i<j} \int_0^{\bar{K}} N(ds, d(i,j), dz) \sum_{k\geq 1} 2^{-k} \\
&\quad \Big| ([c_{ij}(M(s-))]_k - M_k(s-)) \, 1\!\!1_{\{z \leq K(M_i(s-), M_j(s-))\}} \\
&\quad - ([c_{ij}(M^n(s-))]_k - M^n_k(s-)) \, 1\!\!1_{\{z \leq K(M^n_i(s-), M^n_j(s-))\}} \Big| \Big] \\
&\leq A_n(t) + B_n(t),
\end{aligned}$$

where $A_n(t) = \sum_{i<j} A_n^{ij}(t)$, with

$$\begin{aligned}
A_n^{ij}(t) &= \int_0^t ds E\Big[K(M_i(s), M_j(s)) \sum_{k\geq 1} 2^{-k} \\
&\quad \Big| ([c_{ij}(M(s))]_k - M_k(s)) - ([c_{ij}(M^n(s))]_k - M^n_k(s)) \Big| \Big],
\end{aligned}$$

and

$$\begin{aligned}
B_n(t) &= \int_0^t ds E\Big[\sum_{i<j} \Big| K(M_i(s), M_j(s)) - K(M^n_i(s), M^n_j(s)) \Big| \\
&\quad \sum_{k\geq 1} 2^{-k} \Big| [c_{ij}(M^n(s))]_k - M^n_k(s) \Big| \Big].
\end{aligned}$$

First, recalling (2.1), using (2.8) and (3.1),

$$\begin{aligned} B_n(t) &\le \kappa \int_0^t E\Big[\sum_{i<j} \{|(M_i^n(s))^\lambda - (M_i(s))^\lambda| + |(M_j^n(s))^\lambda - (M_j(s))^\lambda|\} \\ &\qquad\qquad d(M^n(s), c_{ij}(M^n(s)))\Big] ds \\ &\le \frac{3\kappa}{2} \int_0^t E\Big[\sum_{i<j} \{|(M_i^n(s))^\lambda - (M_i(s))^\lambda| + |(M_j^n(s))^\lambda - (M_j(s))^\lambda|\} \\ &\qquad\qquad 2^{-i} M_j^n(s)\Big] ds \\ &\le 3\kappa \int_0^t E\Big[d_\lambda(M(s), M^n(s))\Big] ds, \end{aligned}$$

which tends to 0 due to (4.4). Next, we prove that $A_n(t)$ tends to 0 using the Lebesgue dominated convergence Theorem. It suffices to show that:
(a) for each $1 \le i < j$, $A_n^{ij}(t)$ tends to 0 as $n$ tends to infinity,
(b) $\lim_{k\to\infty} \limsup_{n\to\infty} \sum_{i+j\ge k} A_n^{ij}(t) = 0$.

For each $i < j$, using (3.2) and since $d(m, \tilde{m}) \le d_\lambda(m, \tilde{m})$ for each $m, \tilde{m} \in \mathcal{S}_\lambda$, we obtain

$$\begin{aligned} A_n^{ij}(t) &\le \bar{K} \int_0^t ds E\Big[d(M(s), M^n(s)) + d(c_{ij}(M(s)), c_{ij}(M^n(s)))\Big] \\ &\le \bar{K}(2^i + 2^j + 1) \int_0^t ds E\Big[d(M(s), M^n(s))\Big] \\ &\le \bar{K}(2^i + 2^j + 1) \int_0^t ds E\Big[d_\lambda(M(s), M^n(s))\Big] \to 0 \end{aligned}$$

as $n$ tends to infinity, due to (4.4). Thus (a) holds. On the other hand, we obtain, using Corollary 3.2,

$$\begin{aligned} A_n^{ij}(t) &\le \bar{K} \int_0^t ds E\Big[d(M(s), c_{ij}(M(s))) + d(M^n(s), c_{ij}(M^n(s)))\Big] \\ &\le \frac{3\bar{K}}{2} \int_0^t ds 2^{-i} E\Big[M_j(s) + M_j^n(s)\Big]. \end{aligned} \tag{4.5}$$

Since $\sum_{j\ge1} \int_0^t E[M_j(s)] ds \le t$ and since $\sum_{i\ge1} 2^{-i} = 1$, (b) reduces to

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sum_{j\ge k} \int_0^t ds E\big[M_j^n(s)\big] = 0.$$

But for each $k \ge 1$, since $M^n(s)$ and $M(s)$ belong to $\mathcal{S}_1$, and since the map $m \mapsto \sum_1^{k-1} m_j$ is continuous for the pointwise convergence topology,

$$\begin{aligned} \limsup_{n\to\infty} \int_0^t ds E\left[\sum_{j\ge k} M_j^n(s)\right] &= \int_0^t ds \left(1 - \lim_{n\to\infty} E\left[\sum_{j=1}^{k-1} M_j^n(s)\right]\right) \\ &= \int_0^t ds \left(1 - E\left[\sum_{j=1}^{k-1} M_j(s)\right]\right) \\ &= \int_0^t ds E\left[\sum_{j=k}^{\infty} M_j(s)\right], \end{aligned}$$

which tends to 0 as $k$ tends to infinity, due to the dominated convergence Theorem (for each $s$, a.s., $\sum_{j=k}^{\infty} M_j(s)$ tends to 0 because $M(s) \in \mathcal{S}_1$, and $\sum_{j=k}^{\infty} M_j(s) \leq 1 \in L^1(\Omega \times [0,t], P \otimes ds)$). Thus (b) holds, and $A_n(t)$ tends to 0 as $n$ tends to infinity. □

**Proof of Corollary 2.5** Let $N$ be a Poisson measure as in Definition 2.1. For each $m \in \mathcal{S}_\lambda$, denote by $(M^m(t))_{t\geq 0}$ the unique $\mathcal{S}_\lambda$-valued solution to $(SDE(K, m, N))$.
Point (i) is clear from Theorem 2.4-(ii). The fact that $(M^m(t))_{t\geq 0}$ is a strong Markov process is strightforward, since it solves a time-homogeneous Poisson-driven stochastic differential equation for which (pathwise) uniqueness holds. The announced Feller property follows immediately from Theorem 2.4 (point (ii)). Finally, it is straightforward that for $\Phi \in \mathcal{C}$ and $m \in \mathcal{S}_\lambda$,

$$\frac{d}{dt} E\left[\Phi(M^m(t))\right]\Big|_{t=0} = \mathcal{L}\Phi(m). \tag{4.6}$$

Indeed, it is clear from (2.5) that for all $t \geq 0$, a.s.,

$$E\left[\Phi(M^m(t))\right] = \Phi(m) + \int_0^t ds E\left[\mathcal{L}\Phi(M^m(s))\right]. \tag{4.7}$$

Furthermore, a.s., $\lim_{s\to 0} M^m(s) = m$ for the pointwise convergence, since $s \mapsto M^m(s)$ is a.s. right continuous. Due to Lemma 3.3 and since $m \in \mathcal{S}_\lambda \subset \mathcal{S}_1$, we deduce that $\mathcal{L}\Phi(M^m(s))$ is bounded and tends to $\mathcal{L}\Phi(m)$ a.s. as $s$ tends to 0. Hence, due to the Lebesgue Theorem,

$$\frac{d}{dt} E\left[\Phi(M^m(t))\right]\Big|_{t=0} = \lim_{t\to 0} \frac{1}{t} \int_0^t ds E\left[\mathcal{L}\Phi(M^m(s))\right] = \mathcal{L}\Phi(m), \tag{4.8}$$

which concludes the proof. □

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