Well-posedness of Smoluchowski's coagulation equation for a class of homogeneous kernels

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Abstract

The uniqueness and existence of measure-valued solutions to Smoluchowski's coagulation equation are considered for a class of homogeneous kernels. Denoting by $\lambda \in (-\infty, 2] \setminus \{0\}$ the degree of homogeneity of the coagulation kernel a, measure-valued solutions are shown to be unique under the sole assumption that the moment of order λ of the initial datum is finite. A similar result was already available for the kernels $a(x, y) = 2$, $x + y$ and xy, and is extended here to a much wider class of kernels by a different approach. The uniqueness result presented herein also seems to improve previous results for several explicit kernels. Furthermore, a comparison principle and a contraction property are obtained for the constant kernel.

Key words: Smoluchowski's coagulation equation, homogeneous coagulation kernel, measure-valued solution, existence, uniqueness.

1 Introduction

We investigate the uniqueness and existence of *measure-valued* solutions to the Smoluchowski coagulation equation. We first recall that Smoluchowski's coagulation equation provides a mean-field description of a system of an infinite number of particles growing by successive mergers, each particle being fully identified by its mass $x \in (0, \infty)$. Denoting by $c(t, x) \geq 0$ the concentration of particles of mass $x \in (0, \infty)$ at time $t \geq 0$, the dynamics of c is given by

$$\partial_t c(t, x) = \frac{1}{2} \int_0^x a(y, x - y)\ c(t, y)\ c(t, x - y)\ dy - c(t, x) \int_0^\infty a(x, y)\ c(t, y)\ dy \qquad (1.1)$$

for $(t, x) \in (0, \infty)^2$. We further recall that the only mechanism taken into account in this model is the coalescence of two particles to form a larger one (binary coagulation) and that the *coagulation kernel* $a(x, y) = a(y, x) \geq 0$ models the likelihood that two particles with respective masses x and y merge into a single one (with mass $x + y$). We refer to the review papers [4, 1, 15, 13] for more information on the physical and mathematical properties of (1.1) and its stochastic counterparts. The main issue we wish to consider in this note is the uniqueness of weak solutions (in a suitable sense, see Definition 2.1 below) to (1.1). Since the pioneering papers [17, 16], several uniqueness results have already been obtained and it turns out that almost all of them may be formulated within the general framework developed in [19, Theorem 2.1]: assume that there is a *subadditive*

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function $\varphi : (0,\infty) \to [0,\infty)$ (i.e. $\varphi(x+y) \le \varphi(x) + \varphi(y)$ for $(x,y) \in (0,\infty)^2$) such that $a(x,y) \le \varphi(x)\varphi(y)$ for $(x,y) \in (0,\infty)^2$ and that the initial datum $c(0)$ is a non-negative Radon measure on $(0,\infty)$ satisfying $\langle c(0), \varphi \rangle < \infty$. Then, if $T \in (0,\infty]$, there is at most one solution c to (1.1) with this initial datum such that $s \mapsto \langle c(s), \varphi \rangle$ belongs to $\mathcal{C}([0,T))$ and $s \mapsto \langle c(s), \varphi^2 \rangle$ belongs to $L^1(0,t)$ for each $t \in [0,T)$. In particular, for continuous and bounded initial data, the cases $\varphi =$ const. and $a(x,y) = xy$ are studied in [17] and [16], respectively, while, for integrable initial data, the cases $\varphi(x) = (1+x)^{1/2}$ and $\varphi(x) = x^\alpha + x^\beta$, $-1 \le \alpha \le 0 \le \beta \le 1$ are considered in [21] and [8, Theorem 2.9], respectively. Similar results for the coagulation-fragmentation equations may be found in [5, 13]. A basic feature of the previous uniqueness results is that they require the finiteness of two moments of the solution (at least for almost every t), namely $\langle c, \varphi \rangle$ and $\langle c, \varphi^2 \rangle$. Observe that these two moments indeed provide different information on c for small and/or large x, except when $\varphi \equiv 1$, that is, for bounded coagulation kernels. In that particular case, the well-posedness of (1.1) in the space of non-negative and bounded measures is rather easy to establish and is included in [19, Theorem 2.1]. But, for unbounded coagulation kernels, the previous uniqueness results require the finiteness of two different moments.
A noticeable exception is to be found in [18] where the uniqueness of measure-valued solutions is established for the kernels $a(x,y) = 2$, $x+y$ and xy under the sole assumption that the initial data has a finite moment of order 0, 1 and 2, respectively. In other words, uniqueness holds in the class of measures having a finite moment of order the degree of homogeneity of the coagulation kernel and our purpose is to show that such a property is enjoyed for a wider class of *homogeneous* coagulation kernels, that is, satisfying

$$a(ux,uy) = u^\lambda \; a(x,y)\,, \qquad (u,x,y) \in (0,\infty)^3\,, \tag{1.2}$$

for some parameter $\lambda \in (-\infty,2] \setminus \{0\}$. We mention at this point that the homogeneity assumption on the coagulation kernel seems to be rather natural, since several coagulation kernels derived from physical arguments enjoy this property (see, e.g., [1, Table 1] where eight among the nine coagulation kernels listed there and taken from the physical literature are homogeneous).

Consider thus an homogeneous coagulation kernel a with degree $\lambda \in (-\infty,2] \setminus \{0\}$. Our aim in this paper is to prove that, in many cases, there exists a unique measure-valued solution to (1.1) as soon as the moment of order λ of the initial condition is finite and as long as the solution has also a finite moment of order λ. It is indeed well-known that, if $\lambda \in (1,2]$, solutions to (1.1) cannot enjoy this property for all times. When $\lambda \neq 1$, we also allow the initial datum to have an infinite total mass, that is, an infinite first moment. Let us already point out that the main novelty of our results is the uniqueness statement in Theorem 2.2 below. Unlike the uniqueness proof performed in [18] which relies on the Laplace transform, our proof is based on the use of a specific Wasserstein-type distance between solutions, which depends on the homogeneity parameter λ. We show that such a distance between two solutions satisfies a Gronwall inequality, from which uniqueness readily follows.
The choice of this distance is actually strongly motivated by the study of stochastic coalescents performed in [9, Theorem 2.2], though a distance of this kind has also been used in [12, Theorem 2.1] to establish the uniqueness of solutions to the Becker-Döring equations and in [14, Proposition 12] to investigate the large time behaviour of solutions to (1.1) when $a \equiv 1$ (we also refer to [10] for additional information on the connection between the approach in [9] and the present paper). Furthermore, though particularly well-suited for homogeneous coagulation kernels, our method also applies to non-homogeneous coagulation kernels. We also use this approach to prove our existence result but it could also probably be obtained by other classical arguments (see, e.g., [6, Theorem 2.8] where the existence of measure-valued solutions to (1.1) is shown in a different functional framework, namely the initial datum is a non-negative Radon measure with finite first

moment and $a(x,y) = x^\alpha y^\beta + x^\beta y^\alpha$, the parameters α and β ranging in $[-1,1]$ and satisfying $\alpha+\beta \in [0,1)$ and $\alpha > \beta - 1$).

We state our result in Section 2, together with the definition of measure-valued solutions to (1.1), and also give some examples of coagulation kernels to which our results apply. The proofs are then performed in Section 3. In the last section, a similar method allows us to show a contraction property and a comparison principle for the primitives of the solutions to (1.1) for the constant coagulation kernel.

2 Well-posedness

We first give some notations and definitions. Since we will deal with measure-valued solutions, the underlying functional setting is the set $\mathcal{M}^+$ of non-negative Radon measures on $(0,\infty)$. For $\lambda \in \mathbb{R}$ and $c \in \mathcal{M}^+$, we set

$$M_\lambda(c) := \int_0^\infty x^\lambda c(dx), \quad \mathcal{M}_\lambda^+ = \left\{ c \in \mathcal{M}^+,\ M_\lambda(c) < \infty \right\}. \tag{2.1}$$

Next, for $\lambda \in (-\infty, 2] \setminus \{0\}$, we introduce the space $\mathcal{H}_\lambda$ of test functions which will be needed to define measure-valued solutions to (1.1):

$$\begin{array}{lll} \text{if} & \lambda < 0, & \mathcal{H}_\lambda := \left\{ \phi \in \mathcal{C}([0,\infty)) \text{ such that } \sup_{x>0} x^{-\lambda} |\phi(x)| < \infty \right\}, \\ \text{if} & \lambda \in (0,1], & \mathcal{H}_\lambda := \left\{ \phi \in \mathcal{C}([0,\infty)) \text{ such that } \phi(0) = 0 \text{ and } \sup_{x \ne y} \dfrac{|\phi(x) - \phi(y)|}{|x-y|^\lambda} < \infty \right\}, \\ \text{if} & \lambda \in (1,2], & \mathcal{H}_\lambda := \left\{ \phi \in \mathcal{C}([0,\infty)) \text{ such that } \phi(0) = 0 \text{ and } \sup_{x \ne y} \dfrac{|\phi(x) - \phi(y)|}{|x^\lambda - y^\lambda|} < \infty \right\}. \end{array}$$

Notice that the set $\mathcal{C}_c^1((0,\infty))$ of $\mathcal{C}^1$-smooth functions with compact support in $(0,\infty)$ is included in $\mathcal{H}_\lambda$ for $\lambda \in (-\infty,2] \setminus \{0\}$. We may now define the notion of weak solution to (1.1) we will use in the sequel.

Definition 2.1 *Consider $\lambda \in (-\infty,2] \setminus \{0\}$ and a coagulation kernel a satisfying*

$$\left. \begin{array}{lll} \text{if} & \lambda \in (-\infty,1], & a(x,y) \le \kappa_0 \ (x+y)^\lambda, \\ \text{if} & \lambda \in (1,2], & a(x,y) \le \kappa_0 \ \left(x \ y^{\lambda-1} + x^{\lambda-1} \ y \right), \end{array} \right\} \tag{2.2}$$

for $(x,y) \in (0,\infty)^2$ and some constant $\kappa_0 > 0$.
Let $T \in (0,\infty]$ and $c^{in} \in \mathcal{M}_\lambda^+$. A family $\{c_t\}_{t\in[0,T)} \subset \mathcal{M}^+$ is a (c^{in}, a, T, λ)-weak solution to (1.1) if $c_0 = c^{in}$,

$$t \longmapsto \int_0^\infty \phi(x) c_t(dx) \quad \textit{is differentiable on} \quad [0,T)$$

for each $\phi \in \mathcal{H}_\lambda$, and, for every $t \in [0,T)$ and $\phi \in \mathcal{H}_\lambda$,

$$\sup_{s\in[0,t]} M_\lambda(c_s) < \infty, \tag{2.3}$$

and

$$\frac{d}{dt} \int_0^\infty \phi(x) c_t(dx) = \frac{1}{2} \int_0^\infty \int_0^\infty a(x,y) (A\phi)(x,y) c_t(dx) c_t(dy), \tag{2.4}$$

where the function $(A\phi)$ *is defined by*

$$(A\phi)(x,y) := \phi(x+y) - \phi(x) - \phi(y)\,, \quad (x,y) \in (0,\infty)^2\,. \tag{2.5}$$

According to (2.2), (2.3) and Lemma 3.1 below, the integrals in (2.4) are absolutely convergent and bounded with respect to $t \in [0,s]$ for every $s < T$. The weak formulation (2.4) of the Smoluchowski equation (1.1) is standard, see, e.g., [19].

Our result then reads as follows.

Theorem 2.2 *Consider* $\lambda \in (-\infty,2] \setminus \{0\}$ *and* $c^{in} \in \mathcal{M}_\lambda^+$. *Assume that the coagulation kernel* a *belongs to* $W^{1,\infty}((\varepsilon,1/\varepsilon)^2)$ *for every* $\varepsilon \in (0,1)$ *and satisfies (2.2). Assume further that*

$$\left.\begin{array}{lll} \text{if} & \lambda \in (-\infty,0), & (x^\lambda + y^\lambda)|\partial_x a(x,y)| \le \kappa_1\ x^{\lambda-1}y^\lambda, \\ \text{if} & \lambda \in (0,1], & (x^\lambda \wedge y^\lambda)|\partial_x a(x,y)| \le \kappa_1\ x^{\lambda-1}y^\lambda, \\ \text{if} & \lambda \in (1,2], & (x\wedge y)\left(x^{\lambda-1} + y^{\lambda-1}\right)|\partial_x a(x,y)| \le \kappa_1\ x^{\lambda-1}y^\lambda, \end{array}\right\} \tag{2.6}$$

for all $(x,y) \in (0,\infty)^2$ *and some constant* $\kappa_1 > 0$.

(i) If $\lambda \in (-\infty,1]$, *there exists a unique* $(c^{in},a,\infty,\lambda)$*-weak solution* $\{c_t\}_{t\in[0,\infty)}$ *to (1.1).*

(ii) If $\lambda \in (1,2]$, *then there exists* $T_* \in (0,\infty]$ *such that there exists a unique* (c^{in},a,T_*,λ)*-weak solution* $\{c_t\}_{t\in[0,T_*)}$ *to (1.1), with the alternative* $T_* = \infty$ *or* $T_* < \infty$ *and* $\lim_{t\to T_*} M_\lambda(c_t) = \infty$. *Furthermore,* $T_* \ge T_0 := \left(\lambda M_\lambda(c^{in})\kappa_0\right)^{-1}$.

Here an below, we use the notation $x \wedge y := \min\{x,y\}$ and $x \vee y := \max\{x,y\}$ for $(x,y) \in [0,\infty)^2$. Notice that we cannot exclude that $T_* = \infty$ in *(ii)* since no lower bound is assumed for a. Nevertheless, under the additional assumption that $a(x,y) \ge \kappa_2 (xy)^{\lambda/2}$ for some $\kappa_2 > 0$, it follows from the analysis performed in [11, 7] that $T_* < \infty$.

Theorem 2.2 applies in particular to the following coagulation kernels (in all the examples below, we have excluded the case $\lambda = 0$ since it is included in [19, Theorem 2.1]).

$$\begin{array}{llcl}
\text{(k1)} & a(x,y) & = & (x^\alpha + y^\alpha)^\beta, \quad \alpha \in (0,\infty),\ \beta \in (-\infty,\infty),\ \lambda = \alpha\beta \in (-\infty,1]\setminus\{0\}, \\
\text{(k2)} & a(x,y) & = & x^\alpha y^\beta + x^\beta y^\alpha, \quad 0 \le \alpha \le \beta \le 1,\ \lambda = \alpha+\beta \in (0,2], \\
\text{(k3)} & a(x,y) & = & (xy)^\alpha (x+y)^{-\beta}, \quad \alpha > 0,\ \beta > 0,\ \lambda = 2\alpha - \beta \in (-\infty,2]\setminus\{0\}, \\
\text{(k4)} & a(x,y) & = & (x^\alpha + y^\alpha)^\beta |x^\gamma - y^\gamma|, \quad \alpha \in (0,\infty),\ \beta \in (0,\infty),\ \gamma \in (0,1], \lambda = \alpha\beta + \gamma \in (0,1], \\
\text{(k5)} & a(x,y) & = & |x-y|^\alpha (x+y)^{-\beta}, \quad \alpha \in [1,\infty),\ \beta \in (0,\infty),\ \lambda = \alpha - \beta \in (-\infty,1]\setminus\{0\}, \\
\text{(k6)} & a(x,y) & = & (x^{1/3} + y^{1/3})(xy)^{1/2}(x+y)^{-3/2}, \quad \lambda = -1/6, \\
\text{(k7)} & a(x,y) & = & (x+y)^\lambda e^{-\beta(x+y)^\alpha}, \quad \alpha \in (0,\infty),\ \beta \in (0,\infty),\ \lambda \in (-\infty,0), \\
\text{(k8)} & a(x,y) & = & (x+y)^\lambda e^{-\beta(x+y)^{-\alpha}}, \quad \alpha \in (0,\infty),\ \beta \in (0,\infty),\ \lambda \in (0,1), \\
\text{(k9)} & a(x,y) & = & \text{any linear combination of previously listed kernels with the same degree } \lambda.
\end{array}$$

The previous list includes in particular the kernel (k2) which is used as a model case in the mathematical literature, together with several physical homogeneous kernels collected in [4, Section 4.3] and [1, Table 1]: for instance, (k1) with $\alpha = 1/3$ and $\beta \in \{7/3, 3\}$, (k4) with $\alpha = 1/3$, $\beta = 2$ and $\gamma \in \{1/6, 1/3, 2/3\}$, (k5) with $\alpha = 2$ and $\beta = 1$ and (k6). Finally, (k7) and (k8) show that our result also applies to some non-homogeneous kernels.

Unfortunately, Smoluchowski's coagulation kernel $a(x,y) = (x^{1/3}+y^{1/3})(x^{-1/3}+y^{-1/3})$ (see [20]) or $a(x,y) = (x^{1/3}+y^{1/3})^2(x^{-1}+y^{-1})^{1/2}$ are not covered by our analysis.
As a final comment, we point out that the uniqueness results stated in [19, Theorem 2.1] and in Theorem 2.2 are in some sense complementary. Indeed, Theorem 2.2 requires weaker assumptions on the initial data but is only valid for sufficiently smooth coagulation kernels. On the other hand, there is no regularity assumption on the coagulation kernel in [19, Theorem 2.1] but stronger properties are required on the initial data.

3 Proofs

We first establish some properties of $(A\phi)$ for $\phi \in \mathcal{H}_\lambda$ which allow us to justify that the weak formulation (2.4) is meaningful.

Lemma 3.1 *Consider $\lambda \in (-\infty, 2] \setminus \{0\}$ and $\phi \in \mathcal{H}_\lambda$. Then there exists C_ϕ depending only on ϕ and λ such that*

$$(x+y)^\lambda |(A\phi)(x,y)| \le C_\phi\ (xy)^\lambda \quad \text{if} \quad \lambda \in (-\infty, 1]\,,$$

$$\left(xy^{\lambda-1} + x^{\lambda-1}y\right) |(A\phi)(x,y)| \le C_\phi (xy)^\lambda \quad \text{if} \quad \lambda \in (1,2]\,,$$

for all $(x,y) \in (0,\infty)^2$.

Proof. Assume first that $\lambda \in (-\infty, 0)$. Since $|\phi(x)| \le C x^\lambda$ for some constant $C>0$, we have

$$(x+y)^\lambda |(A\phi)(x,y)| \le C(x^\lambda \wedge y^\lambda)\left[(x+y)^\lambda + x^\lambda + y^\lambda\right] \le C x^\lambda y^\lambda.$$

When $\lambda \in (0,1]$, there is $C>0$ such that $|\phi(x)-\phi(y)| \le C|x-y|^\lambda$. Since $(x+y)^\lambda \le x^\lambda + y^\lambda$ and $\phi(0)=0$, we have

$$\begin{aligned}
(x+y)^\lambda |(A\phi)(x,y)| &\le (x^\lambda + y^\lambda)|(A\phi)(x,y)| \\
&\le x^\lambda \left[|\phi(x+y)-\phi(x)| + |\phi(y)-\phi(0)|\right] \\
&\quad + y^\lambda \left[|\phi(x+y)-\phi(y)| + |\phi(x)-\phi(0)|\right] \\
&\le C x^\lambda y^\lambda.
\end{aligned}$$

Finally, if $\lambda \in (1,2]$, we have $|\phi(x)-\phi(y)| \le C\left|x^\lambda - y^\lambda\right|$ for some $C>0$. Using that $\lambda - 2 \le 0 < \lambda - 1 \le 1$ and that $\phi(0)=0$, we realize that

$$\begin{aligned}
\left(xy^{\lambda-1}+x^{\lambda-1}y\right)|(A\phi)(x,y)| &\le \left(xy^{\lambda-1}+x^{\lambda-1}y\right)\ (|\phi(x+y)-\phi(x\vee y)| + |\phi(x\wedge y)-\phi(0)|) \\
&\le C\ \left(xy^{\lambda-1}+x^{\lambda-1}y\right)\ \left(\lambda\ (x+y)^{\lambda-1}\ (x\wedge y) + (x\wedge y)^\lambda\right) \\
&\le C\ \left(xy^{\lambda-1}+x^{\lambda-1}y\right)\ \left(x^{\lambda-1}+y^{\lambda-1}\right)\ (x\wedge y) \\
&\le C\ \left(x^\lambda y^{\lambda-1} + xy^\lambda y^{\lambda-2} + x^\lambda y x^{\lambda-2} + x^{\lambda-1}y^\lambda\right)\ (x\wedge y) \\
&\le C\ \left(x^\lambda y^{\lambda-1} + xy^\lambda (x\wedge y)^{\lambda-2} + x^\lambda y (x\wedge y)^{\lambda-2} + x^{\lambda-1}y^\lambda\right)\ (x\wedge y) \\
&\le C\ \left(x^\lambda y^\lambda + xy^\lambda (x\wedge y)^{\lambda-1} + x^\lambda y (x\wedge y)^{\lambda-1} + x^\lambda y^\lambda\right) \\
&\le C\ x^\lambda y^\lambda\,,
\end{aligned}$$

which completes the proof of Lemma 3.1. □

We now turn to the cornerstone of the uniqueness proof, for which we introduce a specific distance between solutions depending on λ. This distance involves the primitives of the solutions to (1.1) and we gather in the next lemma some notations and properties we will use later on.

Lemma 3.2 *For $c \in \mathcal{M}^+$ and $x \in (0,\infty)$, we put*

$$F^c(x) := \int_0^\infty \mathbf{1}_{(x,\infty)}(y) c(dy) \quad \text{and} \quad G^c(x) := \int_0^\infty \mathbf{1}_{(0,x]}(y) c(dy)\,.$$

(i) If $c \in \mathcal{M}_\lambda^+$ for some $\lambda \in (-\infty, 0)$, then

$$\int_0^\infty x^{\lambda-1} G^c(x) dx = M_\lambda(c)/|\lambda|\,, \quad \lim_{x\to 0} x^\lambda G^c(x) = \lim_{x\to\infty} x^\lambda G^c(x) = 0\,,$$

and $G^c \in L^\infty(0,R)$ for each $R > 0$.

(ii) If $c \in \mathcal{M}_\lambda^+$ for some $\lambda \in (0,2]$, then

$$\int_0^\infty x^{\lambda-1} F^c(x) dx = M_\lambda(c)/\lambda\,, \quad \lim_{x\to 0} x^\lambda F^c(x) = \lim_{x\to\infty} x^\lambda F^c(x) = 0\,,$$

and $F^c \in L^\infty(\varepsilon,\infty)$ for each $\varepsilon > 0$.

Proof. We check (i): by the Fubini theorem we have

$$\int_0^\infty x^{\lambda-1} G^c(x) dx = \int_0^\infty x^{\lambda-1} \int_0^\infty \mathbf{1}_{(0,x]}(y) c(dy) dx = \int_0^\infty c(dy) \int_y^\infty x^{\lambda-1} dx = M_\lambda(c)/(-\lambda)\,.$$

We next notice that, for $R > 0$ and $x \in (0,R)$, the negativity of λ implies that

$$G^c(x) \le G^c(R) \le \int_0^\infty \left(\frac{y}{R}\right)^\lambda \mathbf{1}_{(0,R]}(y) c(dy) \le R^{-\lambda} M_\lambda(c)\,,$$

so that $G^c \in L^\infty(0,R)$. Next, since $\lambda < 0$, we have

$$x^\lambda G^c(x) = x^\lambda \int_0^\infty \mathbf{1}_{(0,x]}(y) c(dy) \le \int_0^\infty \mathbf{1}_{(0,x]}(y) y^\lambda c(dy)\,,$$

and the Lebesgue dominated convergence theorem entails that $x^\lambda G^c(x) \longrightarrow 0$ as $x \to 0$. Similarly, for $R > 0$ and $x > R$, we have

$$\begin{aligned} x^\lambda G^c(x) &= x^\lambda \int_0^\infty \mathbf{1}_{(0,R]}(y) c(dy) + x^\lambda \int_0^\infty \mathbf{1}_{(R,x]}(y) c(dy) \\ &\le \left(\frac{x}{R}\right)^\lambda \int_0^\infty \mathbf{1}_{(0,R]}(y) y^\lambda c(dy) + \int_0^\infty \mathbf{1}_{(R,x]}(y) y^\lambda c(dy) \\ &\le \left(\frac{x}{R}\right)^\lambda M_\lambda(c) + \int_0^\infty \mathbf{1}_{(R,x]}(y) y^\lambda c(dy)\,, \end{aligned}$$

whence for all $R > 0$,

$$\limsup_{x\to\infty} x^\lambda G^c(x) \le \int_0^\infty \mathbf{1}_{(R,\infty)}(y) y^\lambda c(dy)\,,$$

since $\lambda < 0$. Since $c \in \mathcal{M}_\lambda^+$, we may then let $R \to \infty$ and complete the proof of (i). We then argue in a similar way to prove (ii). □

We may now state the fundamental inequality on which the uniqueness proof relies.

Proposition 3.3 *Consider $\lambda \in (-\infty,2]\backslash\{0\}$, $T \in (0,\infty]$ and a coagulation kernel a satisfying the assumptions of Theorem 2.2. Let c^{in} and d^{in} be two initial conditions in $\mathcal{M}_\lambda^+$ and denote by $\{c_t\}_{t\in[0,T)}$ a (c^{in},a,T,λ)-weak solution to (1.1) and by $\{d_t\}_{t\in[0,T)}$ a (d^{in},a,T,λ)-weak solution to (1.1). In addition,*

(i) if $\lambda \in (-\infty,0)$, we put $E(t,x) = G^{c_t}(x) - G^{d_t}(x)$, $\varrho(x) = x^{\lambda-1}$ and

$$\tilde{R}(t,x) = \int_x^\infty \varrho(z)\ \text{sign}\ (E(t,z))dz \quad \text{for} \quad (t,x) \in [0,T)\times(0,\infty)\,,$$

(ii) if $\lambda \in (0,2]$, we put $E(t,x) = F^{c_t}(x) - F^{d_t}(x)$, $\varrho(x) = x^{\lambda-1}$ and

$$\tilde{R}(t,x) = \int_0^x \varrho(z)\ \text{sign}\ (E(t,z))dz \quad \text{for} \quad (t,x) \in [0,T)\times(0,\infty)\,.$$

Then, for each $t \in [0,T)$, $\tilde{R}(t) \in \mathcal{H}_\lambda$ and

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty \varrho(x)\,|E(t,x)|\,dx \\
&\le \frac{1}{2}\int_0^\infty\int_0^\infty a(x,y)\big[\varrho(x+y)-\varrho(x)\big](c_t+d_t)(dy)\,|E(t,x)|\,dx \\
&\quad + \frac{1}{2}\int_0^\infty\int_0^\infty \partial_x a(x,y)\left[\tilde{R}(t,x+y)-\tilde{R}(t,x)-\tilde{R}(t,y)\right](c_t+d_t)(dy)E(t,x)dx. \qquad (3.1)
\end{aligned}$$

At a formal level, the inequality (3.1) is proved as follows: one first uses (2.4) to obtain an equation satisfied by $c_t - d_t$ and then multiply the resulting equation by $\tilde{R}$, the latter being possible since $\tilde{R}$ belongs to $\mathcal{H}_\lambda$. Performing suitable integrations by parts then yields (3.1). In fact, this formal computation would give an equality in (3.1) with $\big[\varrho(x+y)\ \text{sign}\ (E(t,x)E(t,x+y)) - \varrho(x)\big]$ instead of $\big[\varrho(x+y)-\varrho(x)\big]$ in the first integral of the right-hand side of (3.1). However, under our assumptions on $\{c_t\}_{t\in[0,T)}$ and $\{d_t\}_{t\in[0,T)}$, the convergence of the first integral of the right-hand side of (3.1) with this term is not clear for $\lambda \in (0,2]$ and we are only able to prove the inequality (3.1). It however suffices to prove Theorem 2.2. We also mention here that (3.1) is likely to be valid for any other choice of the function ϱ, provided the growth conditions on a, $\{c_t\}_{t\in[0,T)}$ and $\{d_t\}_{t\in[0,T)}$ are modified accordingly.

Before proceeding to the proof of Proposition 3.3, we state and prove two auxiliary results: the first one (Lemma 3.4) allows us to check that the integrals of the right-hand side of (3.1) are indeed convergent, while the second one (Lemma 3.5) is devoted to the time differentiability of E.

Lemma 3.4 *Under the notations and assumptions of Proposition 3.3, there is a positive constant $C > 0$ depending only on λ, κ_0 and κ_1 such that for $(t,x,y) \in [0,T)\times(0,\infty)^2$,*

$$\text{if} \quad \lambda \in (-\infty,0), \qquad a(x,y)(\varrho(x+y)+\varrho(x)) \le Cx^{\lambda-1}y^\lambda, \qquad (3.2)$$

$$\text{if} \quad \lambda \in (0,2], \qquad a(x,y)|\varrho(x+y)-\varrho(x)| \le Cx^{\lambda-1}y^\lambda, \qquad (3.3)$$

while

$$a(x,y)\left|\tilde{R}(t,x+y)-\tilde{R}(t,x)-\tilde{R}(t,y)\right| \le Cx^\lambda y^\lambda, \qquad (3.4)$$

$$\left|\partial_x a(x,y)\left[\tilde{R}(t,x+y)-\tilde{R}(t,x)-\tilde{R}(t,y)\right]\right| \le Cx^{\lambda-1}y^\lambda. \qquad (3.5)$$

Proof.
Case 1: $\lambda \in (-\infty, 0)$. By (2.2) we have

$$a(x,y)(\varrho(x+y)+\varrho(x)) \le \kappa_0 (x+y)^\lambda [(x+y)^{\lambda-1} + x^{\lambda-1}] \le 2\kappa_0 x^{\lambda-1} y^\lambda .$$

Next,

$$\left|\tilde{R}(t,x+y) - \tilde{R}(t,x) - \tilde{R}(t,y)\right| \le \int_x^{x+y} z^{\lambda-1} dz + \int_y^\infty z^{\lambda-1} dz \le \frac{x^\lambda + y^\lambda}{|\lambda|},$$

so that (3.4) and (3.5) follow from (2.2) and (2.6), respectively.
Case 2: $\lambda \in (0,1]$. On the one hand, the monotonicity of ϱ, the subadditivity of $x \mapsto x^\lambda$ and (2.2) ensure that

$$\begin{aligned} a(x,y)|\varrho(x+y)-\varrho(x)| &\le \kappa_0 \left(x^\lambda + y^\lambda\right) (\varrho(x) - \varrho(x+y)) \\ &\le \kappa_0 x^\lambda \varrho(x)^{1-\lambda} (\varrho(x)-\varrho(x+y))^\lambda + \kappa_0 y^\lambda \varrho(x) \\ &\le \kappa_0 x^{\lambda-(1-\lambda)^2} \left(\int_x^{x+y} (1-\lambda) z^{\lambda-2} dz\right)^\lambda + \kappa_0\ x^{\lambda-1} y^\lambda \\ &\le \kappa_0 x^{\lambda-(1-\lambda)^2} y^\lambda x^{\lambda(\lambda-2)} + \kappa_0\ x^{\lambda-1} y^\lambda \\ &\le 2\kappa_0\ x^{\lambda-1} y^\lambda . \end{aligned}$$

On the other hand, the subadditivity of $x \mapsto x^\lambda$ ensures that

$$\begin{aligned} \left|\tilde{R}(t,x+y) - \tilde{R}(t,x) - \tilde{R}(t,y)\right| &\le \int_{x\vee y}^{x+y} z^{\lambda-1} dz + \int_0^{x\wedge y} z^{\lambda-1} dz \\ &\le \lambda^{-1} \left((x+y)^\lambda - (x\vee y)^\lambda + (x\wedge y)^\lambda\right) \le 2\lambda^{-1} (x\wedge y)^\lambda, \end{aligned}$$

so that (3.4) and (3.5) follow from (2.2) and (2.6), respectively.
Case 3: $\lambda \in (1,2]$. We first infer from (2.2) and the subadditivity of ϱ that

$$\begin{aligned} a(x,y)|\varrho(x+y)-\varrho(x)| &\le \kappa_0 \left(xy^{\lambda-1} + x^{\lambda-1} y\right) (\varrho(x+y) - \varrho(x)) \\ &\le \kappa_0 x y^{\lambda-1} \int_x^{x+y} z^{\lambda-2}\ dz + \kappa_0 x^{\lambda-1} y \left(x^{\lambda-1} + y^{\lambda-1} - x^{\lambda-1}\right) \\ &\le \kappa_0 x y^{\lambda-1} y x^{\lambda-2} + \kappa_0\ x^{\lambda-1} y^\lambda \\ &\le 2\kappa_0\ x^{\lambda-1} y^\lambda . \end{aligned}$$

We next use once more the subadditivity of ϱ to deduce that

$$\begin{aligned} \left|\tilde{R}(t,x+y) - \tilde{R}(t,x) - \tilde{R}(t,y)\right| &\le \int_{x\vee y}^{x+y} z^{\lambda-1} dz + \int_0^{x\wedge y} z^{\lambda-1} dz \\ &\le (x+y)^{\lambda-1} (x+y-x\vee y) + (x\wedge y)^\lambda \\ &\le 2(x\wedge y)(x+y)^{\lambda-1} \le 2(x\wedge y)(x^{\lambda-1} + y^{\lambda-1}), \end{aligned}$$

and we complete the proof using (2.2) and (2.6). □

Lemma 3.5 *Consider* $\lambda \in (-\infty,2]\backslash\{0\}$, $T \in (0,\infty]$ *and a coagulation kernel a satisfying the assumptions of Theorem 2.2. Let* $c^{in} \in \mathcal{M}_\lambda^+$ *and denote by* $\{c_t\}_{t\in[0,T)}$ *a* (c^{in}, a, T, λ)*-weak solution to (1.1). Then,* *(i) if $\lambda \in (-\infty,0)$, $(t,x) \longmapsto \partial_t G^{c_t}(x)$ belongs to $L^\infty(0,s;L^1(0,\infty;x^{\lambda-1}dx))$ for each $s \in [0,T)$,*

(ii) if $\lambda \in (0,2]$, $(t,x) \longmapsto \partial_t F^{c_t}(x)$ belongs to $L^\infty(0,s;L^1(0,\infty;x^{\lambda-1}dx))$ for each $s \in [0,T)$.

Proof. We first consider the case $\lambda \in (-\infty,0)$. Let $\vartheta \in \mathcal{C}([0,\infty))$ with compact support in $(0,\infty)$ and put

$$\phi(x) = \int_x^\infty \vartheta(y)dy \quad \text{for} \quad x \in (0,\infty)\,.$$

Clearly, $\phi \in \mathcal{H}_\lambda$ and it follows from (2.4) and Lemma 3.2 that for all $t \in [0,T)$,

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty \vartheta(x)G^{c_t}(x)dx &= \frac{d}{dt}\int_0^\infty \phi(x)c_t(dx) \\
&= -\frac{1}{2}\int_0^\infty\int_0^\infty a(x,y)\left(\int_x^{x+y}\vartheta(z)dz + \int_y^\infty \vartheta(z)dz\right)c_t(dy)c_t(dx) \\
&= -\frac{1}{2}\int_0^\infty \vartheta(z)\int_0^z\int_{z-x}^\infty a(x,y)c_t(dy)c_t(dx)dz \\
&\quad -\frac{1}{2}\int_0^\infty \vartheta(z)\int_0^\infty\int_0^z a(x,y)c_t(dy)c_t(dx)dz\,,
\end{aligned}$$

whence

$$\partial_t G^{c_t}(z) = -\frac{1}{2}\int_0^z\int_{z-x}^\infty a(x,y)c_t(dy)c_t(dx) - \frac{1}{2}\int_0^\infty\int_0^z a(x,y)c_t(dy)c_t(dx) \tag{3.6}$$

for $(t,z) \in [0,T)\times(0,\infty)$. Owing to (2.2) and the Fubini theorem, we observe that, for each $t \in [0,T)$,

$$\begin{aligned}
&\int_0^\infty z^{\lambda-1}\left|\partial_t G^{c_t}(z)\right| dz \\
= &\ \frac{1}{2}\int_0^\infty z^{\lambda-1}\left(\int_0^z\int_{z-x}^\infty a(x,y)c_t(dy)c_t(dx) + \int_0^\infty\int_0^z a(x,y)c_t(dy)c_t(dx)\right)dz \\
\le &\ \frac{\kappa_0}{2}\int_0^\infty\int_0^\infty (x+y)^\lambda\left(\int_x^{x+y} z^{\lambda-1}dz + \int_y^\infty z^{\lambda-1}dz\right)c_t(dy)c_t(dx) \\
\le &\ \frac{\kappa_0}{2(-\lambda)}\int_0^\infty\int_0^\infty \left(x^\lambda \wedge y^\lambda\right)\left(x^\lambda + y^\lambda\right)c_t(dy)c_t(dx) \\
\le &\ \frac{\kappa_0}{(-\lambda)}M_\lambda(c_t)^2\,,
\end{aligned}$$

and the right-hand side of the above inequality is locally bounded in $[0,T)$ by (2.3). We have thus proved (i).

We next turn to the case $\lambda \in (0,2]$ and consider $\vartheta \in \mathcal{C}([0,\infty))$ with compact support in $(0,\infty)$. Putting

$$\phi(x) = \int_0^x \vartheta(y)dy \quad \text{for} \quad x \in (0,\infty)\,,$$

which clearly belongs to $\mathcal{H}_\lambda$, we infer from (2.4) and Lemma 3.2 that

$$
\begin{aligned}
\frac{d}{dt}\int_0^\infty \vartheta(x)F^{c_t}(x)dx &= \frac{d}{dt}\int_0^\infty \phi(x)c_t(dx) \\
&= \frac{1}{2}\int_0^\infty\int_0^\infty a(x,y)\left(\int_x^{x+y}\vartheta(z)dz - \int_0^y \vartheta(z)dz\right)c_t(dx)c_t(dy) \\
&= \frac{1}{2}\int_0^\infty \vartheta(z)\int_0^z\int_{z-x}^\infty a(x,y)c_t(dy)c_t(dx)dz \\
&\quad -\frac{1}{2}\int_0^\infty \vartheta(z)\int_0^\infty\int_z^\infty a(x,y)c_t(dy)c_t(dx)dz \\
&= \frac{1}{2}\int_0^\infty \vartheta(z)\int_0^z\int_0^z \mathbf{1}_{[z,\infty)}(x+y)a(x,y)c_t(dy)c_t(dx)dz \\
&\quad -\frac{1}{2}\int_0^\infty \vartheta(z)\int_z^\infty\int_z^\infty a(x,y)c_t(dy)c_t(dx)dz\,,
\end{aligned}
$$

whence

$$
\partial_t F^{c_t}(z) = \frac{1}{2}\int_0^z\int_0^z \mathbf{1}_{[z,\infty)}(x+y)a(x,y)c_t(dy)c_t(dx) - \frac{1}{2}\int_z^\infty\int_z^\infty a(x,y)c_t(dy)c_t(dx) \qquad (3.7)
$$

for $(t,z)\in[0,T)\times(0,\infty)$. On the one hand, if $\lambda\in(0,1]$, it follows from (2.2) and the Fubini theorem that, for each $t\in[0,T)$,

$$
\begin{aligned}
&\int_0^\infty z^{\lambda-1}\left|\partial_t F^{c_t}(z)\right| dz \\
\le\ & \frac{1}{2}\int_0^\infty z^{\lambda-1}\left(\int_0^z\int_0^z \mathbf{1}_{[z,\infty)}(x+y)a(x,y)c_t(dy)c_t(dx) + \int_z^\infty\int_z^\infty a(x,y)c_t(dy)c_t(dx)\right) dz \\
\le\ & \frac{\kappa_0}{2}\int_0^\infty z^{\lambda-1}\left(\int_0^z\int_0^z \mathbf{1}_{[z,\infty)}(x+y)(x^\lambda+y^\lambda)c_t(dy)c_t(dx) + \int_z^\infty\int_z^\infty (x^\lambda+y^\lambda)c_t(dy)c_t(dx)\right) dz \\
\le\ & \kappa_0\int_0^\infty z^{\lambda-1}\left(\int_0^z\int_0^z \mathbf{1}_{[z,\infty)}(x+y)x^\lambda c_t(dy)c_t(dx) + \int_z^\infty\int_z^\infty x^\lambda c_t(dy)c_t(dx)\right) dz \\
\le\ & \kappa_0\int_0^\infty\int_0^\infty x^\lambda\left(\int_{x\vee y}^{x+y} z^{\lambda-1}dz + \int_0^{x\wedge y} z^{\lambda-1}dz\right)c_t(dy)c_t(dx) \\
\le\ & \frac{2\kappa_0}{\lambda}\int_0^\infty\int_0^\infty x^\lambda (x\wedge y)^\lambda c_t(dy)c_t(dx) \\
\le\ & \frac{2\kappa_0}{\lambda} M_\lambda(c_t)^2\,.
\end{aligned}
$$

Since the right-hand side of the above inequality is locally bounded on $[0,T)$ by (2.3), we obtain the expected result for $\lambda\in(0,1]$. On the other hand, if $\lambda\in(1,2]$, we infer from (2.2) and the

Fubini theorem that

$$
\begin{aligned}
& \int_0^\infty z^{\lambda-1} \left|\partial_t F^{c_t}(z)\right| dz \\
\leq\ & \frac{1}{2} \int_0^\infty z^{\lambda-1} \left(\int_0^z \int_0^z \mathbf{1}_{[z,\infty)}(x+y) a(x,y) c_t(dy) c_t(dx) + \int_z^\infty \int_z^\infty a(x,y) c_t(dy) c_t(dx) \right)\ dz \\
\leq\ & \kappa_0 \int_0^\infty z^{\lambda-1} \left(\int_0^z \int_0^z \mathbf{1}_{[z,\infty)}(x+y) x y^{\lambda-1} c_t(dy) c_t(dx) + \int_z^\infty \int_z^\infty x y^{\lambda-1} c_t(dy) c_t(dx) \right)\ dz \\
\leq\ & \kappa_0 \int_0^\infty \int_0^\infty x y^{\lambda-1} \left(\int_{x\vee y}^{x+y} z^{\lambda-1} dz + \int_0^{x\wedge y} z^{\lambda-1} dz \right) c_t(dy) c_t(dx) \\
\leq\ & \kappa_0 \int_0^\infty \int_0^\infty x y^{\lambda-1} \left((x+y)^{\lambda-1} (x\wedge y) + (x\wedge y)^\lambda \right) c_t(dy) c_t(dx) \\
\leq\ & 2\kappa_0 \int_0^\infty \int_0^\infty x y^{\lambda-1} (x\wedge y) \left(x^{\lambda-1} + y^{\lambda-1} \right) c_t(dy) c_t(dx) \\
\leq\ & 4\kappa_0 M_\lambda(c_t)^2\,,
\end{aligned}
$$

since $xy^{\lambda-1}(x\wedge y)y^{\lambda-1} \leq x^\lambda y^\lambda$ for $(x,y) \in (0,\infty)^2$ (recall that $\lambda \in (1,2]$). Consequently, the property (ii) also holds true for $\lambda \in (1,2]$. □

Proof of Proposition 3.3. Let $t \in [0,T)$. We first note that, since $s \longmapsto M_\lambda(c_s)$ and $s \longmapsto M_\lambda(d_s)$ are in $L^\infty(0,t)$ by (2.3), it follows from Lemma 3.2 and Lemma 3.4 that the three integrals in (3.1) are absolutely convergent. Also, $\tilde{R}(t)$ belongs to $\mathcal{H}_\lambda$ for each $t \in [0,T)$ by (3.4).

Case 1: $\lambda \in (0,2]$. Let $t \in [0,T)$. By Lemma 3.2 and Lemma 3.5, $E \in W^{1,\infty}(0,s;L^1(0,\infty;x^{\lambda-1}dx))$ for every $s \in (0,T)$, so that

$$
\begin{aligned}
\frac{d}{dt} \int_0^\infty x^{\lambda-1} |E(t,x)| dx &= \int_0^\infty x^{\lambda-1} \text{ sign } (E(t,x)) \partial_t E(t,x) dx \\
&= \int_0^\infty \partial_x \tilde{R}(t,x) \left(\partial_t F^{c_t}(x) - \partial_t F^{d_t}(x) \right) dx\,.
\end{aligned}
$$

Owing to (2.3), Lemma 3.2 and (3.4), we may use (3.7) and the Fubini theorem to obtain

$$
\begin{aligned}
& \frac{d}{dt} \int_0^\infty x^{\lambda-1} |E(t,x)| dx \\
=\ & \frac{1}{2} \int_0^\infty \partial_x \tilde{R}(t,z) \int_0^z \int_0^z \mathbf{1}_{[z,\infty)}(x+y) a(x,y) (c_t(dy)c_t(dx) - d_t(dy)d_t(dx)) dz \\
& -\frac{1}{2} \int_0^\infty \partial_x \tilde{R}(t,z) \int_z^\infty \int_z^\infty a(x,y) (c_t(dy)c_t(dx) - d_t(dy)d_t(dx)) dz \\
=\ & \frac{1}{2} \int_0^\infty \int_0^\infty a(x,y) \left[\int_{x\vee y}^{x+y} \partial_x \tilde{R}(t,z) dz - \int_0^{x\wedge y} \partial_x \tilde{R}(t,z) dz \right] (c_t(dy)c_t(dx) - d_t(dy)d_t(dx)) \\
=\ & \frac{1}{2} \int_0^\infty \int_0^\infty a(x,y) (A\tilde{R}(t))(x,y) \left[(c_t(dy) - d_t(dy)) c_t(dx) + d_t(dy)(c_t(dx) - d_t(dx)) \right] \\
=\ & \frac{1}{2} \int_0^\infty \int_0^\infty a(x,y) (A\tilde{R}(t))(x,y) (c_t + d_t)(dy) (c_t - d_t)(dx)\,,
\end{aligned}
$$

where we have used a symmetry argument to deduce the last equality. Introducing

$$
I(t,x) := \int_0^\infty a(x,y) (A\tilde{R}(t))(x,y) (c_t+d_t)(dy)\,, \quad x \in (0,\infty)\,,
$$

the previous identity reads

$$\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx = \frac{1}{2}\int_0^\infty I(t,x)(c_t-d_t)(dx)\,, \tag{3.8}$$

while it follows from (3.4) that

$$|I(t,x)| \le Cx^\lambda M_\lambda(c_t+d_t)\,, \quad x\in(0,\infty)\,,\ t\in[0,T). \tag{3.9}$$

The next step is to perform an integration by part in the right-hand side of (3.8). It is however not clear whether I is differentiable with respect to x. We thus fix $\varepsilon\in(0,1)$ and put

$$I_\varepsilon(t,x) := \int_\varepsilon^{1/\varepsilon} a(x,y)(A\tilde{R}(t))(x,y)(c_t+d_t)(dy)\,, \quad x\in(0,\infty)\,.$$

Since we have assumed that a belongs to $W^{1,\infty}((\alpha,1/\alpha)^2)$ for every $\alpha\in(0,1)$, we easily deduce from the estimates $|\tilde{R}(t,x)|\le\lambda^{-1}x^\lambda$ and $|\partial_x\tilde{R}(t,x)|\le x^{\lambda-1}$ that $I_\varepsilon(t,.)$ belongs to $W^{1,\infty}(\alpha,1/\alpha)$ for each $\alpha\in(0,1)$ with

$$\begin{aligned}\partial_x I_\varepsilon(t,x) &= \int_\varepsilon^{1/\varepsilon} \partial_x a(x,y)(A\tilde{R}(t))(x,y)(c_t+d_t)(dy)\\ &\quad + \int_\varepsilon^{1/\varepsilon} a(x,y)\left(\partial_x\tilde{R}(t,x+y)-\partial_x\tilde{R}(t,x)\right)(c_t+d_t)(dy)\,.\end{aligned} \tag{3.10}$$

Recalling (3.8) and using an integration by parts, we end up with

$$\begin{aligned}\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx &= \frac{1}{2}\int_0^\infty (I-I_\varepsilon)(t,x)(c_t-d_t)(dx) - \frac{1}{2}\left[I_\varepsilon(t,x)E(t,x)\right]_{x=0}^{x=\infty}\\ &\quad + \frac{1}{2}\int_0^\infty \partial_x I_\varepsilon(t,x)E(t,x)dx\,.\end{aligned} \tag{3.11}$$

We first infer from (3.4) that

$$\begin{aligned}&\left|\int_0^\infty (I-I_\varepsilon)(t,x)(c_t-d_t)(dx)\right|\\ \le\ & \int_0^\infty |(I-I_\varepsilon)(t,x)|(c_t+d_t)(dx)\\ \le\ & C\int_0^\infty \left(\int_0^\varepsilon (xy)^\lambda(c_t+d_t)(dy) + \int_{1/\varepsilon}^\infty (xy)^\lambda(c_t+d_t)(dy)\right)(c_t+d_t)(dx)\\ \le\ & C\ M_\lambda(c_t+d_t)\left(\int_0^\varepsilon y^\lambda(c_t+d_t)(dy) + \int_{1/\varepsilon}^\infty y^\lambda(c_t+d_t)(dy)\right)\,,\end{aligned}$$

whence

$$\lim_{\varepsilon\to0}\int_0^\infty (I-I_\varepsilon)(t,x)(c_t-d_t)(dx) = 0 \tag{3.12}$$

by (2.3). It next follows from (2.3) and (3.9) that

$$|I_\varepsilon(t,x)E(t,x)| \le CM_\lambda(c_t+d_t)x^\lambda\left(F^{c_t}(x)+F^{d_t}(x)\right)\,,$$

from which we readily conclude by Lemma 3.2 that

$$\lim_{x\to0} I_\varepsilon(t,x)E(t,x) = \lim_{x\to\infty} I_\varepsilon(t,x)E(t,x) = 0\,. \tag{3.13}$$

Finally, (2.3), Lemma 3.2 and (3.5) imply that

$$\begin{aligned}
& \lim_{\varepsilon\to 0} \int_0^\infty \int_\varepsilon^{1/\varepsilon} \partial_x a(x,y)(A\tilde{R}(t))(x,y)(c_t+d_t)(dy)E(t,x)dx \\
= & \int_0^\infty \int_0^\infty \partial_x a(x,y)(A\tilde{R}(t))(x,y)(c_t+d_t)(dy)E(t,x)dx\,, \qquad (3.14)
\end{aligned}$$

while

$$\begin{aligned}
& \limsup_{\varepsilon\to 0} \int_0^\infty \int_\varepsilon^{1/\varepsilon} a(x,y)\left(\partial_x\tilde{R}(t,x+y)-\partial_x\tilde{R}(t,x)\right)(c_t+d_t)(dy)E(t,x)dx \\
= & \limsup_{\varepsilon\to 0} \int_0^\infty \int_\varepsilon^{1/\varepsilon} a(x,y)\left(\varrho(x+y)\ \mathrm{sign}\ (E(t,x+y)E(t,x))-\varrho(x)\right)(c_t+d_t)(dy)|E(t,x)|dx \\
\le & \limsup_{\varepsilon\to 0} \int_0^\infty \int_\varepsilon^{1/\varepsilon} a(x,y)\left(\varrho(x+y)-\varrho(x)\right)(c_t+d_t)(dy)|E(t,x)|dx \\
= & \int_0^\infty \int_0^\infty a(x,y)\left(\varrho(x+y)-\varrho(x)\right)(c_t+d_t)(dy)|E(t,x)|dx\,, \qquad (3.15)
\end{aligned}$$

where we have used (3.3) to obtain the last formula. Owing to (3.11) and (3.13), we have

$$\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx = \frac{1}{2}\int_0^\infty (I-I_\varepsilon)(t,x)(c_t-d_t)(dx) + \frac{1}{2}\int_0^\infty \partial_x I_\varepsilon(t,x)E(t,x)dx$$

for each $\varepsilon\in(0,1)$. We may then let $\varepsilon\to 0$ in the above identity and use (3.12), (3.10), (3.14) and (3.15) to conclude that (3.1) holds true.

Case 2: $\lambda\in(-\infty,0)$. Let $t\in[0,T)$. Owing to Lemma 3.2 and Lemma 3.5, we may argue as in the previous case (using (3.6) instead of (3.7)) to obtain that

$$\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx = -\frac{1}{2}\int_0^\infty I(t,x)(c_t-d_t)(dx)\,, \qquad (3.16)$$

where

$$I(t,x) := \int_0^\infty a(x,y)(A\tilde{R}(t))(x,y)(c_t+d_t)(dy)\,, \quad x\in(0,\infty)\,.$$

Since a belongs to $W^{1,\infty}((\alpha,1/\alpha)^2)$ for every $\alpha\in(0,1)$, it readily follows from (3.2) and (3.5) that $I(t,.)$ belongs to $W^{1,\infty}(\alpha,1/\alpha)$ for each $\alpha\in(0,1)$ with

$$\begin{aligned}
\partial_x I(t,x) = & \int_0^\infty \partial_x a(x,y)(A\tilde{R}(t))(x,y)(c_t+d_t)(dy) \\
& + \int_0^\infty a(x,y)\left(\partial_x\tilde{R}(t,x+y)-\partial_x\tilde{R}(t,x)\right)(c_t+d_t)(dy)\,. \qquad (3.17)
\end{aligned}$$

Notice that this case is easier than the previous one $\lambda\in(0,2]$ because we may use (3.2) instead of (3.3). Furthermore,

$$|I(t,x)|\le Cx^\lambda M_\lambda(c_t+d_t)\,, \quad x\in(0,\infty)\,,$$

by (3.4). This last property and Lemma 3.2 then ensure that

$$\lim_{x\to 0} I(t,x)E(t,x) = \lim_{x\to\infty} I(t,x)E(t,x) = 0\,.$$

Consequently, we may perform an integration by parts in the right-hand side of (3.16) and use (3.17) to obtain

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx &= -\frac{1}{2}\left[I(t,x)E(t,x)\right]_{x=0}^{x=\infty} + \frac{1}{2}\int_0^\infty \partial_x I(t,x)E(t,x)dx \\
&= \frac{1}{2}\int_0^\infty\int_0^\infty \partial_x a(x,y)(A\tilde{R}(t))(x,y)(c_t+d_t)(dy)E(t,x)dx \\
&\quad +\frac{1}{2}\int_0^\infty\int_0^\infty a(x,y)\left(\partial_x\tilde{R}(t,x+y)-\partial_x\tilde{R}(t,x)\right)(c_t+d_t)(dy)E(t,x)dx \\
&= \frac{1}{2}\int_0^\infty\int_0^\infty \partial_x a(x,y)\left[\tilde{R}(t,x+y)-\tilde{R}(t,x)-\tilde{R}(t,y)\right](c_t+d_t)(dy)E(t,x)dx \\
&\quad +\frac{1}{2}\int_0^\infty\int_0^\infty a(x,y)\left[\varrho(x+y)\ \text{sign}\ (E(t,x+y)E(t,x))-\varrho(x)\right](c_t+d_t)(dy)\,|E(t,x)|\,dx \\
&\le \frac{1}{2}\int_0^\infty\int_0^\infty \partial_x a(x,y)\left[\tilde{R}(t,x+y)-\tilde{R}(t,x)-\tilde{R}(t,y)\right](c_t+d_t)(dy)E(t,x)dx \\
&\quad +\frac{1}{2}\int_0^\infty\int_0^\infty a(x,y)\left[\varrho(x+y)-\varrho(x)\right](c_t+d_t)(dy)\,|E(t,x)|\,dx.
\end{aligned}$$

which completes the proof of Proposition 3.3. □

Observe that when $\lambda\in(-\infty,1]\setminus\{0\}$, ϱ is a non-increasing function and the first term of the right-hand side of (3.1) is nonpositive.

Corollary 3.6 *Under the notations and assumptions of Proposition 3.3, there is a positive constant C depending only on λ, κ_0 and κ_1 such that, for each $t\in[0,T)$,*

$$\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx \le CM_\lambda(c_t+d_t)\int_0^\infty x^{\lambda-1}|E(t,x)|dx. \tag{3.18}$$

Proof. When $\lambda\in(-\infty,1]\setminus\{0\}$, Corollary 3.6 readily follows from (3.1), (3.5), and the nonpositivity of the first term of the right-hand side of (3.1) already mentioned. When $\lambda\in(1,2]$, Corollary 3.6 is a straightforward consequence of (3.1), (3.3) and (3.5). □

We are now in a position to complete the proof of Theorem 2.2.

Proof of Theorem 2.2: uniqueness. Owing to (2.3), Lemma 3.2 and Corollary 3.6, the uniqueness assertion of Theorem 2.2 readily follows from the Gronwall Lemma. □

Proof of Theorem 2.2: existence. For $n\ge 1$, we consider $c^{in,n}(dx)=\mathbf{1}_{[1/n,n]}c^{in}(dx)$, which belongs to $\mathcal{M}_\delta^+$ for any $\delta\in(-\infty,\infty)$ since $c^{in}\in\mathcal{M}_\lambda^+$. We also notice that $(G^{c^{in,n}})$ converges towards $G^{c^{in}}$ in $L^1(0,\infty;x^{\lambda-1}dx)$ as $n\to\infty$ if $\lambda\in(-\infty,0)$ while $(F^{c^{in,n}})$ converges towards $F^{c^{in}}$ in $L^1(0,\infty;x^{\lambda-1}dx)$ as $n\to\infty$ if $\lambda\in(0,2]$.

We now split the proof into three cases, according to the values of λ.

Case 1: $\lambda\in(-\infty,0)$. Introducing $\varphi_\lambda(x)=\sqrt{\kappa_0}x^{\lambda/2}$ for $x\in(0,\infty)$, we infer from (2.2) that $a(x,y)\le\varphi_\lambda(x)\varphi_\lambda(y)$ for every $(x,y)\in(0,\infty)^2$. Since φ_λ is subadditive, we are in a position to apply [19, Theorem 2.1] to deduce that, for each $n\ge1$, there exists a $(c^{in,n},a,\infty,\lambda)$-weak solution $\{c_t^n\}_{t\ge0}$ to (1.1).

Since $\lambda < 0$, $(x+y)^\lambda - x^\lambda - y^\lambda \le 0$ for all $(x,y) \in (0,\infty)^2$ and it readily follows from (2.4) with $\phi(x) = x^\lambda$ that

$$M_\lambda(c_t^n) \le M_\lambda(c^{in,n}) \le M_\lambda(c^{in})\,, \quad t \ge 0\,, n \ge 1\,. \tag{3.19}$$

We then easily deduce from Corollary 3.6 and (3.19) that there is a constant $C > 0$ such that

$$\int_0^\infty x^{\lambda-1} \left| G^{c_t^m}(x) - G^{c_t^n}(x) \right| dx \le e^{Ct} \int_0^\infty x^{\lambda-1} \left| G^{c^{in,m}}(x) - G^{c^{in,n}}(x) \right| dx$$

for $t \ge 0$, $n \ge 1$ and $m \ge 1$. Recalling that $t \mapsto G^{c_t^n}$ belongs to $\mathcal{C}([0,\infty); L^1(0,\infty; x^{\lambda-1}dx))$ for each $n \ge 1$ by Lemma 3.2 and Lemma 3.5, we conclude that $\left(t \mapsto G^{c_t^n}\right)_{n\ge1}$ is a Cauchy sequence in $\mathcal{C}([0,\infty); L^1(0,\infty; x^{\lambda-1}dx))$ and there is $g \in \mathcal{C}([0,\infty); L^1(0,\infty; x^{\lambda-1}dx))$ such that

$$\lim_{n\to\infty} \sup_{s\in[0,t]} \int_0^\infty x^{\lambda-1} \left| G^{c_s^n}(x) - g(s,x) \right| dx = 0 \quad \text{for each} \quad t \in [0,\infty)\,. \tag{3.20}$$

As a first consequence of (3.20), we obtain that $x \mapsto g(t,x)$ is non-decreasing and non-negative for each $t \in [0,\infty)$. In addition, since $g \in \mathcal{C}([0,\infty); L^1(0,\infty; x^{\lambda-1}dx))$, we have

$$\lim_{\varepsilon\to0} \sup_{s\in[0,t]} \left[\int_0^\varepsilon x^{\lambda-1} g(s,x)dx + \int_{1/\varepsilon}^\infty x^{\lambda-1} g(s,x)dx \right] = 0 \tag{3.21}$$

for each $t \in (0,\infty)$.

We next show that the sequence $(c_s^n)_{n\ge1}$ is tight in $\mathcal{M}_\lambda^+$, uniformly with respect to $s \in [0,t]$. For that purpose, we consider $\varepsilon \in (0,1/4)$ and notice that, since $x \mapsto G^{c_s^n}(x)$ is non-decreasing and $\lambda < 0$, it follows from Lemma 3.2 that

$$\begin{aligned}
& \int_0^\varepsilon x^\lambda c_s^n(dx) + \int_{1/\varepsilon}^\infty x^\lambda c_s^n(dx) \\
= \ & |\lambda| \int_0^\varepsilon x^{\lambda-1} G^{c_s^n}(x)dx + \varepsilon^\lambda G^{c_s^n}(\varepsilon) + |\lambda| \int_{1/\varepsilon}^\infty x^{\lambda-1} \left(G^{c_s^n}(x) - G^{c_s^n}(1/\varepsilon) \right) dx \\
\le \ & |\lambda| \int_0^\varepsilon x^{\lambda-1} G^{c_s^n}(x)dx + \varepsilon^{\lambda-1} \int_\varepsilon^{2\varepsilon} G^{c_s^n}(x)dx + |\lambda| \int_{1/\varepsilon}^\infty x^{\lambda-1} G^{c_s^n}(x)dx \\
\le \ & |\lambda| \int_0^\varepsilon x^{\lambda-1} G^{c_s^n}(x)dx + 2^{1-\lambda} \int_\varepsilon^{2\varepsilon} x^{\lambda-1} G^{c_s^n}(x)dx + |\lambda| \int_{1/\varepsilon}^\infty x^{\lambda-1} G^{c_s^n}(x)dx \\
\le \ & (|\lambda| + 2^{1-\lambda}) \int_0^{2\varepsilon} x^{\lambda-1} G^{c_s^n}(x)dx + |\lambda| \int_{1/\varepsilon}^\infty x^{\lambda-1} G^{c_s^n}(x)dx\,.
\end{aligned}$$

The Lebesgue dominated convergence theorem, (3.20) and (3.21) then entail that

$$\lim_{\varepsilon\to0} \sup_{n\ge1} \sup_{s\in[0,t]} \left(\int_0^\varepsilon x^\lambda c_s^n(dx) + \int_{1/\varepsilon}^\infty x^\lambda c_s^n(dx) \right) = 0\,, \tag{3.22}$$

for every $t \in [0,\infty)$, whence the claimed tightness of $(c_t^n)_{n\ge1}$ in $\mathcal{M}_\lambda^+$. Consequently, denoting by $c_t(dx) := \partial_x g(t,x)$ the first derivative with respect to x of g in the sense of distributions for $t \in (0,\infty)$, we deduce from (3.19), (3.20) and (3.22) that $c_t(dx) \in \mathcal{M}_\lambda^+$ with $M_\lambda(c_t) \le M_\lambda(c^{in})$.

Consider now $\phi \in C_c^1((0,\infty))$ and recall that $|\phi'(x)| \le Cx^{\lambda-1}$ for some constant C. On the one hand, the time continuity of g implies that

$$t \longmapsto \int_0^\infty \phi(x)c_t(dx) = -\int_0^\infty \phi'(x)g(t,x)dx$$

is continuous on $[0,\infty)$. On the other hand, the convergence (3.20) ensures that

$$\lim_{n\to\infty}\sup_{s\in[0,t]}\left|\int_0^\infty \phi(x)(c_s^n-c_s)(dx)\right| = \lim_{n\to\infty}\sup_{s\in[0,t]}\left|\int_0^\infty \phi'(x)(G^{c_s^n}-G^{c_s})(x)dx\right|$$
$$\le \lim_{n\to\infty}\sup_{s\in[0,t]} C\int_0^\infty x^{\lambda-1}|G^{c_s^n}(x)-g(s,x)|dx = 0 \tag{3.23}$$

for every $t\in[0,\infty)$. We then infer from (3.22), (3.23), Lemma 3.1 and a density argument that, for every $\phi\in\mathcal{H}_\lambda$, the map $t\mapsto\int_0^\infty\phi(x)c_t(dx)$ is continuous and

$$\lim_{n\to\infty}\sup_{s\in[0,t]}\left|\int_0^\infty \phi(x)(c_s^n-c_s)(dx)\right| = 0,$$
$$\lim_{n\to\infty}\sup_{s\in[0,t]}\left|\int_0^\infty\int_0^\infty a(x,y)(A\phi)(x,y)(c_s^n(dx)c_s^n(dy)-c_s(dx)d_s(dy))\right| = 0. \tag{3.24}$$

We may thus pass to the limit as $n\to\infty$ in the integrated form of (2.4) for $\{c_t^n\}_{t\ge0}$ and deduce that, for all $t\ge0$ and $\phi\in\mathcal{H}_\lambda$, we have

$$\int_0^\infty \phi(x)c_t(dx) = \int_0^\infty \phi(x)c^{in}(dx) + \int_0^t\int_0^\infty\int_0^\infty a(x,y)(A\phi)(x,y)c_s(dy)c_s(dx)ds. \tag{3.25}$$

Classical arguments then allow us to differentiate (3.25) with respect to time and conclude that $\{c_t\}_{t\in[0,\infty)}$ is a $(c^{in},a,\infty,\lambda)$-weak solution to (1.1).

Case 2: $\lambda\in(0,1]$. Introducing $\varphi_\lambda(x)=\sqrt{\kappa_0}(1+x)^\lambda$ for $x\in(0,\infty)$, we infer from (2.2) that $a(x,y)\le\varphi_\lambda(x)\varphi_\lambda(y)$ for every $(x,y)\in(0,\infty)^2$. Since φ_λ is subadditive, we are in a position to apply [19, Theorem 2.1] to deduce that, for each $n\ge1$, there exists a $(c^{in,n},a,\infty,\lambda)$-weak solution $\{c_t^n\}_{t\ge0}$ to (1.1).
In that case, we also have $(x+y)^\lambda-x^\lambda-y^\lambda\le0$ for all $(x,y)\in(0,\infty)^2$, so that (3.19) still holds true. We then proceed as in the previous case to deduce from Corollary 3.6 and (3.19) that $\left(F^{c_t^n}\right)_{n\ge1}$ is a Cauchy sequence in $\mathcal{C}([0,\infty);L^1(0,\infty;x^{\lambda-1}dx))$ and there is $f\in\mathcal{C}([0,\infty);L^1(0,\infty;x^{\lambda-1}dx))$ such that $\left(F^{c_t^n}\right)_{n\ge1}$ converges towards f in $\mathcal{C}([0,\infty;L^1(0,\infty;x^{\lambda-1}dx))$. Arguing in a similar way as in the previous case, this convergence implies the tightness of $(c_t^n)_{n\ge1}$ in $\mathcal{M}_\lambda^+$ (in the sense that (3.22) still holds) with the help of the following inequality

$$\int_0^\varepsilon x^\lambda c_t^n(dx) + \int_{1/\varepsilon}^\infty x^\lambda c_t^n(dx) \le C\int_0^\varepsilon x^{\lambda-1}F^{c_t^n}(x)dx + C\int_{1/(2\varepsilon)}^\infty x^{\lambda-1}F^{c_t^n}(x)dx\,, \tag{3.26}$$

which is valid for $\varepsilon\in(0,1/4)$ and $\lambda\in(0,2)$. We conclude as in the previous case that, setting $c_t(dx):=-\partial_x f(t,x)$ (in the sense of distributions), $\{c_t\}_{t\in[0,\infty)}$ is a $(c^{in},a,\infty,\lambda)$-weak solution to (1.1).

Case 3: $\lambda\in(1,2]$. In that case, [19, Theorem 2.1] only provides a local (in time) existence result, a fact which is strongly related to the gelation phenomenon, see, e.g., [7, 11] and the references therein. In addition, the dependence of the existence time on the initial data is not suitable for our purpose. We thus return to a more classical approach and proceed in three steps: we first truncate the coagulation kernel and prove a local existence result for initial data with fast decay at infinity by a compactness method. Nevertheless, there is a positive lower bound for the existence time which depends only on the moment of order λ of the initial data. The case of general initial data in $\mathcal{M}_\lambda^+$ is then handled as for $\lambda\in[0,1]$ for some non-optimal existence time. We finally extend the solution to a maximal existence time.

Step 1. We prove that, for

$$c^{in} \in \mathcal{M}_1^+ \cap \mathcal{M}_3^+ \,, \tag{3.27}$$

there exists a $(c^{in}, a, T_0, \lambda)$-weak solution to (1.1) with $T_0 := (\lambda \kappa_0 M_\lambda(c^{in}))^{-1}$.
For $n \geq 1$, we define a coagulation kernel a_n by $a_n(x,y) = a(x,y) \wedge n$ for $(x,y) \in (0,\infty)^2$. Notice that (2.2) and (2.6) warrant that

$$a_n(x,y) \leq \kappa_0 \, \left(x\, y^{\lambda-1} + x^{\lambda-1}\, y\right) \quad \text{and} \quad (x \wedge y)\left(x^{\lambda-1} + y^{\lambda-1}\right) |\partial_x a_n(x,y)| \leq \kappa_1 \, x^{\lambda-1} y^\lambda \,, \tag{3.28}$$

for $(x,y) \in (0,\infty)^2$ and $n \geq 1$. We next put $c^{in,n}(dx) = \mathbf{1}_{[1/n,n]} c^{in}(dx)$ which belongs to $\mathcal{M}_\delta^+$ for any $\delta \in (-\infty,\infty)$ and deduce from [19, Theorem 2.1] that, for each $n \geq 1$, there exists a $(c^{in,n}, a_n, \infty, \lambda)$-weak solution $\{c_t^n\}_{t \geq 0}$ to (1.1) which satisfies

$$M_1(c_t^n) = M_1(c^{in,n}) \leq M_1(c^{in}) \,, \quad t \in [0,\infty) \,. \tag{3.29}$$

We next take $\phi(x) = x^\lambda$, $x \in (0,\infty)$, in (2.4) for $\{c_t^n\}_{t\geq 0}$ and use (3.28) to obtain

$$\begin{aligned}
\frac{d}{dt} \int_0^\infty x^\lambda c_t^n(dx) &\leq \frac{\kappa_0}{2} \int_0^\infty \int_0^\infty \left[(x+y)^\lambda - x^\lambda - y^\lambda\right] \left(xy^{\lambda-1} + yx^{\lambda-1}\right) c_t^n(dx) c_t^n(dy) \\
&\leq \kappa_0 \int_0^\infty \int_0^\infty xy^{\lambda-1} \left[(x+y)^\lambda - x^\lambda - y^\lambda\right] c_t^n(dx) c_t^n(dy) \,.
\end{aligned}$$

Since

$$\left[(x+y)^\lambda - x^\lambda - y^\lambda\right] = \lambda(\lambda-1) \int_0^x \int_0^y (u+v)^{\lambda-2} dv du \leq \lambda(\lambda-1) y \int_0^x u^{\lambda-2} du \leq \lambda x^{\lambda-1} y \,,$$

we end up with

$$\frac{d}{dt} M_\lambda(c_t^n) \leq \lambda \; \kappa_0 \; M_\lambda(c_t^n)^2 \,.$$

Therefore, recalling that $T_0 = (\lambda M_\lambda(c^{in})\kappa_0)^{-1}$, we have

$$M_\lambda(c_t^n) \leq \frac{M_\lambda(c^{in,n})}{1 - \lambda M_\lambda(c^{in,n})\kappa_0 t} \leq \frac{M_\lambda(c^{in})}{1 - \lambda M_\lambda(c^{in})\kappa_0 t} \,, \quad t \in [0, T_0) \,. \tag{3.30}$$

Similarly, we take $\phi(x) = x^3$, $x \in (0,\infty)$, in (2.4) for $\{c_t^n\}_{t\geq 0}$ and use (3.28) and the inequality

$$\begin{aligned}
\left[(x+y)^3 - x^3 - y^3\right] xy^{\lambda-1} &= 3\left[x^2 y^{\lambda+1} + x^3 y^\lambda\right] \\
&= 3\left[\left(x^\lambda y^3\right)^{1/(3-\lambda)} \; \left(x^3 y^\lambda\right)^{(2-\lambda)/(3-\lambda)} + x^3 y^\lambda\right] \\
&\leq C\left[x^\lambda y^3 + x^3 y^\lambda\right] \,,
\end{aligned}$$

to obtain that

$$\frac{d}{dt} M_3(c_t^n) \leq C \; M_\lambda(c_t^n) \; M_3(c_t^n) \,,$$

whence

$$M_3(c_t^n) \leq M_3(c^{in}) \; \exp\left\{C \int_0^t M_\lambda(c_s^n) \; ds\right\} \,, \quad t \in [0,\infty) \,. \tag{3.31}$$

We now fix $t \in (0, T_0)$. Owing to (3.29), (3.30) and (3.31), there is a positive constant $C_1(t)$ depending only on λ, κ_0, c^{in} and t such that

$$M_1(c_s^n) + M_\lambda(c_s^n) + M_3(c_s^n) \leq C_1(t) \,, \quad s \in [0,t] \,. \tag{3.32}$$

In addition, we infer from (2.4) for c^n, (3.28), Lemma 3.1 and (3.32) that

$$s \longmapsto \int_0^\infty \phi(x) c_s^n(dx) \quad \text{is bounded in} \quad W^{1,\infty}(0,t) \quad \text{for every} \quad \phi \in \mathcal{H}_\lambda\,. \tag{3.33}$$

As $\lambda \in (1,3)$, (3.32), (3.33), the Arzelà-Ascoli theorem and a separability argument ensure that there are $c = \{c_s\}_{s\in[0,t]}$ in $L^\infty(0,t;\mathcal{M}_\lambda^+)$ and a subsequence of (c^n) (not relabeled) such that

$$\lim_{n\to\infty} \sup_{s\in[0,t]} \left| \int_0^\infty \phi(x)(c_s^n - c_s)(dx) \right| = 0 \quad \text{for every} \quad \phi \in \mathcal{H}_\lambda\,. \tag{3.34}$$

Thanks to the convergence (3.34), classical arguments allow us to deduce that $\{c_s\}_{s\in[0,t)}$ is a (c^{in}, a, t, λ)-weak solution to (1.1). In addition, we already know that such a solution is unique and we can extend it to a $(c^{in}, a, T_0, \lambda)$-weak solution to (1.1) since t is arbitrary in $[0, T_0)$.
We have thus constructed a unique $(c^{in}, a, T_0, \lambda)$-weak solution to (1.1) for an initial datum $c^{in} \in \mathcal{M}_\lambda^+$ which satisfies the additional property (3.27). We emphasize at this point that the existence time T_0 only depends on λ, κ_0 and $M_\lambda(c^{in})$ and that it follows from (3.30) and (3.34) that

$$M_\lambda(c_t) \le \frac{M_\lambda(c^{in})}{1 - \lambda M_\lambda(c^{in})\kappa_0 t}\,, \quad t \in [0, T_0)\,. \tag{3.35}$$

Step 2. We now prove that, for any $c^{in} \in \mathcal{M}_\lambda^+$, there exists a $(c^{in}, a, T_0, \lambda)$-weak solution to (1.1) with $T_0 := (\lambda\kappa_0 M_\lambda(c^{in}))^{-1}$.
For $n \ge 1$, we put $c^{in,n}(dx) := \mathbf{1}_{[1/n,n]} c^{in}(dx)$. Clearly $c^{in,n}$ belongs to $\mathcal{M}_1^+ \cap \mathcal{M}_3^+$ and we infer from the previous step that there exists a unique $(c^{in,n}, a, T_0^n, \lambda)$-weak solution $\{c_t^n\}_{t\in[0,T_0^n)}$ to (1.1) with $T_0^n := (\lambda M_\lambda(c^{in,n})\kappa_0)^{-1}$. Noticing that $M_\lambda(c^{in,n}) \le M_\lambda(c^{in})$, we realize that $T_0 \le T_0^n$, while (3.35) reads

$$M_\lambda(c_t^n) \le \frac{M_\lambda(c^{in,n})}{1 - \lambda M_\lambda(c^{in,n})\kappa_0 t} \le \frac{M_\lambda(c^{in})}{1 - \lambda M_\lambda(c^{in})\kappa_0 t}\,, \quad t \in [0, T_0).$$

Since (3.26) also holds true for $\lambda \in (1,2]$, we next argue as for the case $\lambda \in (0,1]$ to conclude that there is a unique $(c^{in}, a, T_0, \lambda)$-weak solution to (1.1).

Step 3. Consider $c^{in} \in \mathcal{M}_\lambda^+$ and put

$$T_* := \sup\left\{T \ge 0 \quad \text{such that there exists a} \quad (c^{in}, a, T, \lambda) - \text{weak solution to (1.1)}\right\} \in [0, \infty]\,.$$

Owing to the previous step, we already know that $T_* > 0$ with $T_* \ge T_0 := (\lambda M_\lambda(c^{in})\kappa_0)^{-1}$. Now, either $T_* = \infty$ and the proof is finished, or $T_* < \infty$. In the latter case, it remains to prove that $M_\lambda(c_t) \to \infty$ as $t \to T_*$. Indeed, assume for contradiction that there are a sequence $(t_n)_{n\ge1}$ in $(0, T_*)$ and $m \in [0,\infty)$ such that $t_n \to T_*$ and $M_\lambda(c_{t_n}) \to m$ as $n \to \infty$. We put $\varepsilon = (4\lambda m\kappa_0)^{-1}$ and choose $N \ge 1$ such that

$$T_* - \varepsilon < t_N < T_* \quad \text{and} \quad M_\lambda(c_{t_N}) \le 2m\,.$$

Setting $\tilde{c}^{in} := c_{t_N}$, we infer from the previous step that there exists a $(\tilde{c}^{in}, a, \tilde{T}_0, \lambda)$-weak solution $\{\tilde{c}_t\}_{t\in[0,\tilde{T}_0)}$ to (1.1) with $\tilde{T}_0 = (\lambda M_\lambda(\tilde{c}^{in})\kappa_0)^{-1} \ge (2\lambda m\kappa_0)^{-1} = 2\varepsilon$. Therefore, $t_N + \tilde{T}_0 \ge t_N + 2\varepsilon \ge T_* + \varepsilon$. We next point out that our uniqueness result ensures that $c_{t_N+t} = \tilde{c}_t$ for all $t \in [0, T_* - t_N)$. Putting $d_t = c_t$ for all $t \in [0, T_*)$ and $d_t = \tilde{c}_{t-t_N}$ for $t \in [T_*, T_* + \varepsilon) \subset [T_*, t_N + \tilde{T}_0)$, we deduce that $\{d_t\}_{t\in[0,T_*+\varepsilon)}$ is a $(\tilde{c}^{in}, a, T_* + \varepsilon, \lambda)$-weak solution to (1.1), which contradicts the definition of T_* and thus completes the proof. □

4 A comparison principle for the constant kernel $a \equiv 2$

The aim of this last section is to point out that, for the constant coagulation kernel $a \equiv 2$, the approach used to prove Theorem 2.2 and more precisely the analogue to Proposition 3.3 reveals a comparison principle and a contraction property which seem to have been unnoticed before. We first recall that for such a kernel, we know from [19, Theorem 2.1] that there exists a unique $(c^{in}, a, \infty, 0)$-weak solution to (1.1) as soon as $c^{in} \in \mathcal{M}_0^+$.

Proposition 4.1 *Assume that $a \equiv 2$ and consider two initial conditions c^{in} and d^{in} in $\mathcal{M}_0^+$. Denote by $\{c_t\}_{t\ge 0}$ the unique $(c^{in}, a, \infty, 0)$-weak solution to (1.1) and by $\{d_t\}_{t\ge 0}$ the unique $(d^{in}, a, \infty, 0)$-weak solution to (1.1).*

(i) If c^{in} and d^{in} belong to $\mathcal{M}_0^+ \cap \mathcal{M}_1^+$, we have for all $t \ge 0$,

$$\int_0^\infty F^{c_t}(x)dx = \int_0^\infty F^{c^{in}}(x)dx\,, \quad \int_0^\infty F^{d_t}(x)dx = \int_0^\infty F^{d^{in}}(x)dx\,, \tag{4.1}$$

and

$$\int_0^\infty \left|F^{c_t}(x) - F^{d_t}(x)\right| dx \le \int_0^\infty \left|F^{c^{in}}(x) - F^{d^{in}}(x)\right| dx. \tag{4.2}$$

(ii) Assume now that c^{in} and d^{in} only belong to $\mathcal{M}_0^+$. If $F^{c^{in}}(x) \le F^{d^{in}}(x)$ for all $x \in (0,\infty)$, then we have $F^{c_t}(x) \le F^{d_t}(x)$ for $(t,x) \in [0,\infty) \times (0,\infty)$.

Proof. We first prove (i) and assume that c^{in} and d^{in} belong to $\mathcal{M}_0^+ \cap \mathcal{M}_1^+$. In that case, c_t and d_t belong to $\mathcal{M}_1^+$ and satisfy $M_0(F^{c_t}) = M_1(c_t) = M_1(c^{in}) = M_0(F^{c^{in}})$ for each $t \ge 0$ (it suffices to apply (2.4) with the choice $\phi(x) = x$ and notice that $(A\phi) \equiv 0$, see [19]). The same property holds for $\{d_t\}_{t\ge 0}$. We next argue as in the proof of Proposition 3.3 with $\varrho \equiv 1$ to obtain that

$$\frac{d}{dt} \int_0^\infty \left|F^{c_t}(x) - F^{d_t}(x)\right| dx \le 0$$

since $\partial_x a \equiv 0$. We have thus proved (i). We now prove (ii) and proceed in two steps.
Step 1. We first assume that c^{in} and d^{in} belong to $\mathcal{M}_0^+ \cap \mathcal{M}_1^+$. A classical consequence of (4.1) and (4.2) (see [3]) is that

$$\int_0^\infty \left(F^{c_t}(x) - F^{d_t}(x)\right)_+ dx \le \int_0^\infty \left(F^{c^{in}}(x) - F^{d^{in}}(x)\right)_+ dx$$

for $t \ge 0$, where $r_+ := r \vee 0$ denotes the positive part of $r \in \mathbb{R}$. Indeed, since $r_+ = (|r| + r)/2$ for $r \in \mathbb{R}$, it follows from (4.1) and (4.2) that

$$\begin{aligned}
\int_0^\infty \left(F^{c_t}(x) - F^{d_t}(x)\right)_+ dx &= \frac{1}{2}\int_0^\infty \left|F^{c_t}(x) - F^{d_t}(x)\right| dx + \frac{1}{2}\int_0^\infty \left(F^{c_t}(x) - F^{d_t}(x)\right) dx \\
&\le \frac{1}{2}\int_0^\infty \left|F^{c^{in}}(x) - F^{d^{in}}(x)\right| dx + \frac{1}{2}\int_0^\infty \left(F^{c^{in}}(x) - F^{d^{in}}(x)\right) dx \\
&= \int_0^\infty \left(F^{c^{in}}(x) - F^{d^{in}}(x)\right)_+ dx\,,
\end{aligned}$$

from which the assertion (ii) for initial data in $\mathcal{M}_0^+ \cap \mathcal{M}_1^+$ readily follows.
Step 2. Assume now that c^{in} and d^{in} only belong to $\mathcal{M}_0^+$. For each $n \ge 1$, we consider the approximations $c^{in,n}$ and $d^{in,n}$ of c^{in} and d^{in} defined by $c^{in,n}(dx) := \mathbf{1}_{(0,n]} c^{in}(dx)$ and $d^{in,n}(dx) =$

$\mathbf{1}_{(0,n]}d^{in}(dx) + F^{d^{in}}(n)\delta_n$, where δ_n denotes the Dirac mass at $x = n$. Clearly, $c^{in,n}$ and $d^{in,n}$ belong to $\mathcal{M}_0^+ \cap \mathcal{M}_1^+$ and the sequences $(c^{in,n})_{n\geq 1}$ and $(d^{in,n})_{n\geq 1}$ converge narrowly in $\mathcal{M}_0^+$ to c^{in} and d^{in}, respectively, since $\lim_{n\to\infty} F^{d^{in}}(n) = 0$. Furthermore, for every $n \geq 1$ and $x \in (0,\infty)$,

$$F^{c^{in,n}}(x) = \left(F^{c^{in}}(x) - F^{c^{in}}(n)\right)_+ \leq \left(F^{d^{in}}(x) - F^{d^{in}}(n)\right)_+ + F^{d^{in}}(n)\mathbf{1}_{(0,n)}(x) = F^{d^{in,n}}(x).$$

Denoting by $\{c_t^n\}_{t\geq 0}$ the unique $(c^{in,n}, a, \infty, 0)$-weak solution to (1.1) and by $\{d_t^n\}_{t\geq 0}$ the unique $(d^{in,n}, a, \infty, 0)$-weak solution to (1.1), we deduce from the previous step that

$$F^{c_t^n}(x) \leq F^{d_t^n}(x) \quad \text{for} \quad (t,x) \in [0,\infty) \times (0,\infty) . \tag{4.3}$$

Arguing as in [19], we can prove that the tight convergence in $\mathcal{M}_0^+$ of $\left(c^{in,n}\right)_{n\geq 1}$ towards c^{in} and of $\left(d^{in,n}\right)_{n\geq 1}$ towards d^{in} imply that of $(c_t^n)_{n\geq 1}$ to c_t and $(d_t^n)_{n\geq 1}$ to d_t for each $t \geq 0$. We now fix $t > 0$ and denote by $\mathcal{N}(t)$ the set of discontinuity points of either F^{c_t} or F^{d_t}, that is

$$\mathcal{N}(t) := \left\{x \in (0,\infty) \quad \text{such that} \quad F^{c_t}(x-) \neq F^{c_t}(x+) \quad \text{or} \quad F^{d_t}(x-) \neq F^{d_t}(x+)\right\} .$$

Since F^{c_t} and F^{d_t} are both non-increasing functions on $(0,\infty)$, $\mathcal{N}(t)$ is a countable subset of $(0,\infty)$. For $x \in (0,\infty) \setminus \mathcal{N}(t)$, it follows from [2, Theorem 2.3] that

$$\lim_{n\to\infty} F^{c_t^n}(x) = F^{c_t}(x) \quad \text{and} \quad \lim_{n\to\infty} F^{d_t^n}(x) = F^{d_t}(x) .$$

Owing to (4.3), we thus conclude that $F^{c_t}(x) \leq F^{d_t}(x)$ for $x \in (0,\infty)\setminus\mathcal{N}(t)$. Finally, for $x \in \mathcal{N}(t)$, there is a non-increasing sequence $(x_k)_{k\geq 1}$ in $(0,\infty)\setminus\mathcal{N}(t)$ such that $x_k > x$ for $k \geq 1$ and $x_k \to x$ as $k \to \infty$. Since $F^{c_t}(x_k) \leq F^{d_t}(x_k)$ for $k \geq 1$ and F^{c_t} and F^{d_t} are both right continuous at x, the last assertion of Proposition 4.1 follows by letting $k \to \infty$ in the previous inequality. □

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