Existence of self-similar solutions to Smoluchowski's coagulation equation

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Abstract

The existence of self-similar solutions to Smoluchowski's coagulation equation is conjectured since several years by physicists and numerical simulations have confirmed the validity of this conjecture. Still, there was no existence result up to now, except for the constant and additive kernels for which explicit formulae are available. In this paper, the existence of self-similar solutions decaying rapidly at infinity is established for a wide class of homogeneous coagulation kernels.

Key words. Smoluchowski's coagulation equation, self-similar solutions, mass conservation.

1 Introduction

The Smoluchowski coagulation equation provides a mean-field description of a system of an infinite number of particles growing by successive mergers, each particle being fully identified by its mass ranging in the set of positive real numbers. In fact, the only mechanism taken into account in this model is the coalescence of two particles to form a larger one. Denoting by $c(t,x) \geq 0$ the concentration of particles of mass $x \in (0,\infty)$ at time $t \geq 0$, the dynamics of c is given by [8, 23]

$$\partial_t c(t,x) = L_c(c(t,.))(x), \quad (t,x) \in (0,\infty) \times (0,\infty), \tag{1.1}$$

where the coagulation reaction term L_c is defined by

$$L_c(c)(x) = \frac{1}{2} \int_0^x K(y,x-y)\ c(y)\ c(x-y)\ dy - c(x) \int_0^\infty K(x,y)\ c(y)\ dy \tag{1.2}$$

for $x \in (0,\infty)$. In (1.2), $K(x,y)$ models the likelihood that two particles with respective masses x and y merge into a single one (with mass $x+y$) and the coagulation kernel K is a non-negative and symmetric function from $(0,\infty)^2$ into $[0,\infty)$. The first term of the right-hand side of (1.2) describes the formation of particles of mass x resulting from the coalescence of two particles with respective masses y and $x-y$, $y \in (0,x)$. The second term accounts for the disappearance of particles of mass x by coalescence with other particles.

Observe that, since the coalescence of two particles with respective masses x and y leads to the formation of a particle with mass $x+y$, the total mass of the whole system of particles is expected

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to remain constant throughout time evolution. In other words, a solution c to (1.1) is expected to satisfy

$$\int_0^\infty x\ c(t,x)\ dx = \int_0^\infty x\ c(0,x)\ dx \quad \text{for} \quad t \ge 0\,. \tag{1.3}$$

It is however well-known by now that this property fails to be true for coagulation kernels which increase sufficiently rapidly for large values of x and y. More precisely, some mass can be lost in finite time, a phenomenon known as *gelation*, and the occurrence of gelation has been extensively studied recently, either by physicists (see [7, 18] and the references therein) and by mathematicians (see [9, 12] and the references therein). For the coagulation kernels to be considered in this paper, the occurrence of the gelation phenomenon in finite time is excluded and solutions to (1.1) do satisfy (1.3).

A central issue is then to identify the large time behaviour of solutions. For *homogeneous* coagulation kernels K, that is, K satisfies

$$K(ux,uy) = u^\lambda\ K(x,y)\,, \qquad (u,x,y) \in (0,\infty)^3\,, \tag{1.4}$$

for some parameter $\lambda \in (-\infty,1]$, it is conjectured by physicists that a solution c to (1.1) approaches a self-similar profile for large masses and large times. More precisely, the so-called *dynamical scaling hypothesis* predicts that

$$c(t,x) \sim c_S(t,x) = \frac{1}{s(t)^2}\ \psi\left(\frac{x}{s(t)}\right) \tag{1.5}$$

after a sufficiently large time, and c_S is a *self-similar* solution to (1.1) with a finite mass

$$\int_0^\infty x\ c_S(t,x)\ dx = \int_0^\infty x\ \psi(x)\ dx < \infty\,, \tag{1.6}$$

see [1, 7, 15, 18] and the references therein. Here the particle mean mass $s(t)$ and the profile ψ are to be determined and depend on the coagulation kernel K but not on specific properties of the initial datum $c(0,.)$. Formal computations have been performed by physicists to identify $s(t)$ and the behaviour of ψ for small and large x [5, 7, 18, 19], while several numerical simulations seem to support the validity of (1.5) [10, 11, 14, 17, 20]. Still, nothing much is known from the rigorous point of view. In fact, the first difficulty encountered is the existence of the scaling profile ψ which satisfies a nonlinear integro-differential equation. It is the purpose of this work to prove the existence of at least one scaling profile ψ for three classes of coagulation kernels K, namely

$$K_1(x,y) = (x^\alpha + y^\alpha)(x^{-\beta} + y^{-\beta}), \quad \alpha \in [0,1),\ \beta \in (0,\infty),\ \lambda = \alpha - \beta \in (-\infty,1), \tag{1.7}$$

$$K_2(x,y) = (x^\alpha + y^\alpha)^\beta, \quad \alpha \in [0,\infty),\ \beta \in (0,\infty),\ \lambda = \alpha\beta \in [0,1), \tag{1.8}$$

$$K_3(x,y) = x^\alpha y^\beta + x^\beta y^\alpha, \quad \alpha \in (0,1),\ \beta \in (0,1),\ \lambda = \alpha + \beta \in [0,1), \tag{1.9}$$

which include several kernels considered in the literature. To our knowledge, no previous existence result was available, except for the constant kernel $K(x,y) = 1$ and the additive kernel $K(x,y) = x + y$. For these two kernels, explicit formulae for ψ are available, see, e.g., [7, 18]. It has actually been shown recently that, for the constant and additive kernels, there is a one-parameter family of self-similar solutions to (1.1) (not necessarily of the form (1.5), see [3, 22]). Nevertheless only one of them has a fast decay for large x and is expected to describe the large time behaviour of solutions to (1.1) with, say, compactly supported initial data. The self-similar solution to (1.1) we construct in this paper for K_1, K_2 and K_3 also has a fast decay at infinity. Still, it is likely that other self-similar solutions with "fat" tails also exist in that case.

Remark 1.1 *It actually seems that the proof of the existence of a scaling profile developed in this paper could be extended to other homogeneous coagulation kernels K, and in particular to $K(x,y) = (xy)^{-\alpha}$, $\alpha > 0$.*

Before stating our result, let us point out that the existence of a self-similar solution to (1.1) opens the way to the study of the validity of the dynamical scaling hypothesis (1.5). Still, a convergence proof seems to require also the uniqueness of such a solution (in a suitable class) and is still an open problem. For the constant and/or additive coagulation kernels, the validity of (1.5) has been investigated in [1, 4, 6, 13, 16, 18, 21, 22].

¿From now on, we assume that the coagulation kernel K is given by (1.7), (1.8) or (1.9). Observe that K satisfies (1.4) and that our assumptions imply that $\lambda < 1$. Inserting the self-similar ansatz (1.5) in (1.1) and using (1.4), we are led to find two positive real numbers (w, ϱ) and a non-negative function $\psi \in L^1(0,\infty; xdx)$ such that

$$w\partial_x(x^2\psi(x)) + xL_c(\psi)(x) = 0, \quad x \in (0,\infty), \tag{1.10}$$

$$\int_0^\infty x\ \psi(x)\ dx = \varrho\,, \tag{1.11}$$

see [7, 18]. We notice that, if ψ solves (1.10), (1.11) for the parameters (w, ϱ), then $\psi_{a,b}(x) = a\psi(bx)$ also solves (1.10), (1.11) but for the parameters $(awb^{-1-\lambda}, a\varrho b^{-2})$. Since $\lambda < 1$, the choice

$$a = \left(\frac{\varrho^{1+\lambda}}{(1-\lambda)^2 w^2}\right)^{1/(1-\lambda)}, \qquad b = \left(\frac{\varrho}{(1-\lambda)w}\right)^{1/(1-\lambda)} \tag{1.12}$$

allows us to consider (1.10), (1.11) with $w = 1/(1-\lambda)$ and $\varrho = 1$ without loss of generality. Setting next $s(t) = t^{1/(1-\lambda)}$, it readily follows from (1.10), (1.11) that $c_S(t,x) = s(t)^{-2}\psi(xs(t)^{-1})$ is a self-similar solution to (1.1) (in a weak sense) with mass 1.

To be more precise, we first define the notions of weak solution to (1.1) and (1.10) we will use in this paper. The following definition relies on the (formal) observation that, for $c : (0,\infty) \mapsto \mathbb{R}$ and for any test function $\phi : (0,\infty) \mapsto \mathbb{R}$ sufficiently smooth, we have

$$\begin{aligned}\int_0^\infty \phi(x) x L_c(c)(x) dx &= \int_0^\infty \int_0^\infty xK(x,y)[\phi(x+y) - \phi(x)]c(x)c(y)dydx \\ &= \int_0^\infty \phi'(z) \int_0^z \int_{z-x}^\infty xK(x,y)c(x)c(y)dydxdz.\end{aligned}$$

Definition 1.2 *Assume that K is given by (1.7), (1.8) or (1.9) and set $\gamma = 1/(1-\lambda)$.*
(i) A non-negative function $\psi \in L^1(0,\infty; xdx)$ is said to be a weak solution to

$$\gamma\partial_x(x^2\psi(x)) + xL_c(\psi)(x) = 0, \quad x \in (0,\infty), \tag{1.13}$$

if $\psi \in L^1(0,\infty; x^2dx)$ and is such that $(x,y) \longmapsto xyK(x,y)\psi(x)\psi(y) \in L^1\left((0,\infty)^2\right)$ and

$$\gamma z^2 \psi(z) = \int_0^z \int_{z-x}^\infty K(x,y)x\psi(x)\psi(y)dydx \quad \text{for a.e.} \quad z \in (0,\infty), \tag{1.14}$$

the right-hand side of (1.14) being finite for almost every $z \in (0,\infty)$.
(ii) A non-negative function $c \in L^\infty(0,\infty; L^1(0,\infty; xdx))$ is said to be a weak solution to (1.1) if, for every $t_2 > t_1 > 0$ and $\phi \in C_b^1([0,\infty))$, the function $(t,x,y) \longmapsto xyK(x,y)c(t,x)c(t,y)$ belongs to $L^1\left((t_1,t_2)\times(0,\infty)^2\right)$ and

$$\begin{aligned}&\int_0^\infty x\phi(x)\ \ (c(t_2,x) - c(t_1,x))\ \ dx \\ = &\int_{t_1}^{t_2} \int_0^\infty \int_0^\infty xK(x,y)[\phi(x+y) - \phi(x)]c(t,x)c(t,y)dydxdt.\end{aligned}$$

Note that, if ψ is a weak solution to (1.13) in the sense of Definition 1.2 (i), it satisfies

$$\gamma \int_0^\infty x^2 \psi(x)\phi'(x)dx = \int_0^\infty \int_0^\infty xK(x,y)[\phi(x+y)-\phi(x)]\psi(x)\psi(y)dydx \tag{1.15}$$

for any $\phi \in C_b^1([0,\infty))$.

The main result of this paper is the following.

Theorem 1.3 *Assume that K is given by (1.7), (1.8) or (1.9) and set $\gamma = 1/(1-\lambda)$.*
(i) There exists a weak solution ψ to (1.13) such that $\int_0^\infty x\psi(x)dx = 1$.
(ii) Introducing $c_S(t,x) = t^{-2\gamma}\psi(xt^{-\gamma})$ for $t>0$ and $x \in (0,\infty)$, c_S is a (self-similar) weak solution to (1.1), and $\int_0^\infty xc_S(t,x)dx = 1$ for each $t>0$.

We will only prove the assertion (i) of Theorem 1.3, since (ii) follows by a straightforward calculation. Furthermore, the weak solution ψ to (1.13) constructed in Theorem 1.3 enjoys the following properties.

Theorem 1.4 *Assume that K is given by (1.7), (1.8) or (1.9) and consider the weak solution ψ to (1.13) constructed in Theorem 1.3.*
(i) There exists a continuous and positive function $g \in C((0,\infty))$ such that $\psi(x) \geq g(x) > 0$ for each $x \in (0,\infty)$.
(ii) The following estimates hold:

$$\psi \in L^1(0,\infty; x^\sigma dx) \;\; \text{for every } \sigma \in \mathbb{R} \quad \text{if} \quad K = K_1\,, \tag{1.16}$$

$$\exists \varepsilon > 0, \int_0^\infty e^{\varepsilon x^{-\beta}} x^{-\beta}\psi(x)dx < \infty \;\; \text{if} \quad K = K_1\,, \tag{1.17}$$

$$\psi \in L^1(0,\infty; x^\sigma dx) \;\; \text{for every } \sigma \geq \lambda \quad \text{if} \quad K = K_2\,, \tag{1.18}$$

$$\psi \in L^1(0,\infty; x^\sigma dx) \;\; \text{for every } \sigma > \lambda \quad \text{if} \quad K = K_3\,. \tag{1.19}$$

Remark 1.5 *As already mentioned, some conjectures about the behaviour of ψ for small and large x have been proposed by physicists on the ground on formal computations [7, 18, 19]. On the one hand, it is conjectured that ψ decays at least exponentially for large x which is in perfect agreement with (1.16), (1.18) and (1.19). On the other hand, the small-x behaviour is conjectured to depend heavily on the kernel K. For K_1, ψ is expected to behave as $x^{-\tau}\, \exp\{-C\, x^{-\beta}\}$ (for some $\tau > 0$), which is in compliance with (1.16), (1.17). For K_2 and K_3, the conjectured behaviour of ψ is that $\psi(x) \sim C\, x^{-\tau}$ near $x = 0$ with $\tau < 1+\lambda$ for K_2 and $\tau = 1+\lambda$ for K_3, and (1.18) and (1.19) perfectly agree with this conjecture. Thus, the leading order in (1.17) and (1.19) seems to be optimal.*

Remark 1.6 *Owing to the expected singularity of ψ near $x=0$ for K given by (1.8) or (1.9), some of the terms in $L_c(\psi)$ are not well-defined and it is thus not likely that the weak formulation (1.14) can be improved. In addition, the properties of ψ obtained in Theorem 1.4 do not seem to be sufficient to prove that the right-hand side of (1.14) is continuous with respect to $z \in (0,\infty)$ (because of the singularity of ψ for $(x,y) \sim (z,z-x) \sim (z,0)$).*

The remainder of the paper is devoted to the proof of Theorems 1.3 and 1.4, which relies on a suitable discretization of (1.13), along with a compactness method. It turns out that it is more convenient to work with $Q(x) = x\,\psi(x)$. With this notation, the weak formulation (1.15) becomes

$$\gamma \int_0^\infty x\phi'(x)Q(x)dx = \int_0^\infty \int_0^\infty \frac{K(x,y)}{y}[\phi(x+y)-\phi(x)]Q(x)Q(y)dydx \tag{1.20}$$

for every $\phi \in C_b^1([0,\infty))$. We introduce and study a discrete approximation to (1.20) in Section 2. Moment and integrability estimates are the subject of the next two sections, Sections 3 and 4, respectively. The proof of Theorem 1.3, together with that of (1.16), (1.18) and (1.19), is performed in Section 5 while the last section of the paper is devoted to the proof of (1.17) and the strict positivity of ψ.

Throughout the paper we use the following notation: if μ is a measure on a set E and ϕ is a function from E in $\mathbb{R}$, we write $\langle \mu, \phi\rangle = \langle \mu(dx), \phi(x)\rangle = \int_E \phi(x)\ \mu(dx)$. For $k \in \mathbb{N}$, the space of bounded and C^k-smooth functions from $[0,\infty)$ in $\mathbb{R}$ which have bounded derivatives up to the order k is denoted by $C_b^k([0,\infty))$. Also, if $x \in \mathbb{R}$ and $y \in \mathbb{R}$, we put $x \wedge y = \min\{x, y\}$.

2 A discrete approximation

Let $N \geq 1$ be a positive integer and consider two families of non-negative real numbers $(v_i)_{1\leq i\leq N+1}$ and $(a_{i,j})_{1\leq i,j\leq N}$ such that

$$v_1 = v_{N+1} = 0 \quad \text{and} \quad a_{i,j} = a_{j,i} \quad \text{for} \quad 1 \leq i, j \leq N\,. \tag{2.1}$$

We define the function $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ by $F = (F_i)_{1\leq i\leq N}$ and

$$F_i(q) = v_{i+1}\ q_{i+1} - v_i\ q_i + \sum_{j=1}^{i-1} \frac{a_{i-j,j}}{j}\ q_{i-j}\ q_j - \sum_{j=1}^{N-i} \frac{a_{i,j}}{j}\ q_i\ q_j \tag{2.2}$$

for $1 \leq i \leq N$ and $q = (q_i)_{1\leq i\leq N} \in \mathbb{R}^N$. We first prove the existence of a zero of F.

Proposition 2.1 *Under the assumptions (2.1), there exists $q \in [0,\infty)^N$ such that*

$$\sum_{i=1}^{N} q_i = 1\,, \tag{2.3}$$

and $F(q) = 0$, that is,

$$v_{i+1}\ q_{i+1} - v_i\ q_i + \sum_{j=1}^{i-1} \frac{a_{i-j,j}}{j}\ q_{i-j}\ q_j - \sum_{j=1}^{N-i} \frac{a_{i,j}}{j}\ q_i\ q_j = 0 \quad \textit{for} \quad 1 \leq i \leq N\,. \tag{2.4}$$

Proof. Clearly F is a locally Lipschitz continuous function and (2.1) implies that

$$\sum_{i=1}^{N} F_i(q) = 0 \quad \text{for each} \quad q \in \mathbb{R}^N\,. \tag{2.5}$$

The Cauchy-Lipschitz theorem then ensures that, for each $q \in \mathbb{R}^N$, there is a unique maximal solution $\varphi(.,q) = (\varphi_i(.,q))_{1\leq i\leq N} \in C^1([0,t^+(q));\mathbb{R}^N)$ to the ordinary differential equation

$$\frac{d}{dt}\varphi(t,q) = F(\varphi(t,q))\,, \qquad \varphi(0,q) = q\,, \tag{2.6}$$

with the alternative:

$$\text{either} \quad t^+(q) = \infty \quad \text{or} \quad \left\{ t^+(q) < \infty \quad \text{and} \quad \lim_{t\to t^+(q)} \sum_{i=1}^{N} |\varphi_i(t,q)| = \infty \right\}. \tag{2.7}$$

Moreover, it follows from (2.5) that

$$\sum_{i=1}^N \varphi_i(t,q) = \sum_{i=1}^N q_i\,, \qquad t\in[0,t^+(q))\,, \quad q\in\mathbb{R}^N\,. \tag{2.8}$$

We now consider $q\in[0,\infty)^N$. Then, $q+t\ F(q)\in[0,\infty)^N$ for t small enough, namely

$$t\ \left(\sup_{1\le i\le N}\{v_i\} + \sup_{1\le i,j\le N}\{a_{i,j}\}\ \sum_{i=1}^N |q_i| \right) < 1\,.$$

Therefore, dist $\left(q+tF(q),[0,\infty)^N\right)=0$ for t small enough and the *subtangent condition*

$$\liminf_{t\to 0} \frac{1}{t}\ \text{dist}\left(q+tF(q),[0,\infty)^N\right)=0$$

is fulfilled for each $q\in[0,\infty)^N$. We then infer from [2, Theorem 16.5] that $[0,\infty)^N$ is positively invariant for the semiflow φ, that is, $\varphi(t,q)\in[0,\infty)^N$ for each $t\in[0,t^+(q))$ and $q\in[0,\infty)^N$. This property and (2.8) readily imply that

$$0\le \sum_{i=1}^N |\varphi_i(t,q)| = \sum_{i=1}^N \varphi_i(t,q) = \sum_{i=1}^N q_i < \infty$$

for $t\in[0,t^+(q))$ and $q\in[0,\infty)^N$. Recalling (2.7), we thus conclude that $t^+(q)=\infty$ for each $q\in[0,\infty)^N$.
We finally introduce the non-empty convex and compact subset $\mathcal{C}$ of $\mathbb{R}^N$ defined by

$$\mathcal{C} = \left\{ q=(q_i)_{1\le i\le N}\in[0,\infty)^N\,,\ \sum_{i=1}^N q_i = 1 \right\}\,.$$

Owing to the previous analysis and (2.8), $\mathcal{C}$ is positively invariant for the semiflow φ and we may apply [2, Proposition 22.13] to conclude that F has at least one zero in $\mathcal{C}$. □

We now use Proposition 2.1 to obtain a solution to a discrete approximation of (1.13). Let $n\ge 1$ be a positive integer and put $N=n^2$, $v_{N+1}^n=0$, and

$$v_i^n = \gamma\ \frac{i-1}{n}\,, \quad a_{i,j}^n = K\left(\frac{i}{n},\frac{j}{n}\right)\,, \quad 1\le i,j\le n^2\,.$$

The families (v_i^n) and $(a_{i,j}^n)$ fulfil (2.1) and we infer from Proposition 2.1 that there exists $q^n = (q_i^n)\in[0,\infty)^{n^2}$ such that

$$\sum_{i=1}^{n^2} q_i^n = 1\,, \tag{2.9}$$

and $F(q^n)=0$, that is,

$$\begin{aligned}\sum_{j=1}^{i-1} \frac{1}{j}\ K\left(\frac{i-j}{n},\frac{j}{n}\right)\ q_{i-j}^n\ q_j^n - \sum_{j=1}^{n^2-i} \frac{1}{j}\ K\left(\frac{i}{n},\frac{j}{n}\right)\ q_i^n\ q_j^n \\ = -\frac{\gamma}{n}\ \left[i\ \mathbb{1}_{\{1\le i\le n^2-1\}}\ q_{i+1}^n - (i-1)\ q_i^n \right]\end{aligned} \tag{2.10}$$

for $1 \leq i \leq n^2$. We then define the probability measure $Q^n(dx)$ on $(0,\infty)$ by

$$Q^n(dx) = \sum_{i=1}^{n^2} q_i^n \ \delta_{i/n}(dx)\,, \tag{2.11}$$

and first show that $Q^n(dx)$ is indeed a solution to an approximation of the weak formulation (1.20) of (1.13).

Lemma 2.2 *Let K be a symmetric and non-negative function from $(0,\infty)^2$ in $[0,\infty)$ and $n \geq 1$. There exists a probability measure $Q^n(dx)$ on $(0,\infty)$ such that, for any measurable function $\phi : (0,\infty) \mapsto \mathbb{R}$,*

$$\begin{aligned} &\left\langle Q^n(dx)Q^n(dy), \frac{K_n(x,y)}{y} \left[\phi(x+y) - \phi(x)\right] \right\rangle \\ &= \gamma \ \left\langle Q^n(dx), n\left(x - n^{-1}\right) \left[\phi(x) - \phi(x-n^{-1})\right] \mathbb{1}_{\{x \geq 2/n\}} \right\rangle\,, \end{aligned} \tag{2.12}$$

with $K_n(x,y) = K(x,y) \ \mathbb{1}_{\{x+y \leq n\}}$.

Proof. It suffices to consider $Q^n(dx)$ defined previously by (2.11). We multiply the i-th equation of (2.10) by $n\phi(i/n)$ and sum up the resulting identities to obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{n^2} \phi(i/n) \left[-\sum_{j=1}^{n^2-i} q_i^n q_j^n \frac{K(i/n,j/n)}{j/n} + \sum_{j=1}^{i-1} q_j^n q_{i-j}^n \frac{K(j/n,(i-j)/n)}{(i-j)/n} \right] \\ &\quad -\gamma \sum_{i=1}^{n^2} (i-1) q_i^n \phi(i/n) + \gamma \sum_{i=1}^{n^2-1} i q_{i+1}^n \phi(i/n) \\ &= \sum_{k=1}^{n^2} \sum_{l=1}^{n^2} q_k^n \ q_l^n \ \left[-\frac{K_n(k/n,l/n)}{l/n} \phi(k/n) + \frac{K_n(k/n,l/n)}{l/n} \phi((k+l)/n) \right] \\ &\quad +\gamma \sum_{k=2}^{n^2} q_k^n \ \left[-(k-1)\phi(k/n) + (k-1)\phi((k-1)/n) \right] \\ &= \left\langle Q^n(dx)Q^n(dy), \frac{K_n(x,y)}{y} \left[\phi(x+y) - \phi(x)\right] \right\rangle \\ &\quad -\gamma \left\langle Q^n(dx), n\left(x - n^{-1}\right) \left[\phi(x) - \phi(x-n^{-1})\right] \mathbb{1}_{\{x \geq 2/n\}} \right\rangle\,, \end{aligned}$$

whence (2.12). □

3 Moment estimates

We next prove some key moment estimates.

Lemma 3.1 *Assume that the coagulation kernel K is given by (1.7), (1.8) or (1.9). The sequence of probability measures $(Q^n(dx))$ constructed in Lemma 2.2 satisfies*

$$\sup_{n \geq 1} \langle Q^n(dx), x^\sigma \rangle < \infty \tag{3.1}$$

for every

$$\begin{array}{lll} \sigma\in\mathbb{R} & \textit{if} & K=K_1\,, \\ \sigma\in[\lambda-1,\infty) & \textit{if} & K=K_2\,, \\ \sigma\in(\lambda-1,\infty) & \textit{if} & K=K_3\,. \end{array}$$

Proof. We first note that, since the measure $Q^n(dx)$ has its support included in $[1/n,n]$, it is clear that $\langle Q_n(dx),x^\sigma\rangle$ is well-defined and finite for each n and $\sigma\in\mathbb{R}$. We then set

$$\begin{aligned} A_{n,\sigma} &= \left\langle Q^n(dx)Q^n(dy),\frac{K_n(x,y)}{y}\left[(x+y)^\sigma-x^\sigma\right]\right\rangle\,, \\ B_{n,\sigma} &= \gamma\,\left\langle Q^n(dx),n\left(x-n^{-1}\right)\left[x^\sigma-(x-n^{-1})^\sigma\right]1\!\!1_{\{x\geq 2/n\}}\right\rangle \end{aligned}$$

for $\sigma\in\mathbb{R}$ and notice that $A_{n,\sigma}=B_{n,\sigma}$ by (2.12).

Step 1. We first show that (3.1) holds true for K_1 and K_2 when $\sigma=\lambda-1$. On the one hand, since $K(x,y)\geq y^\lambda$, we have

$$\begin{aligned} |A_{n,\lambda-1}| &= \left\langle Q^n(dx)Q^n(dy),\frac{K_n(x,y)}{y}\,\left[1/x^{1-\lambda}-1/(x+y)^{1-\lambda}\right]\right\rangle \\ &\geq \left\langle Q^n(dx)Q^n(dy),(1/y^{1-\lambda})1\!\!1_{\{x+y\leq n\}}1\!\!1_{\{y\geq x\}}(1/x^{1-\lambda})[1-1/2^{1-\lambda}]\right\rangle \\ &\geq \varepsilon\left\langle Q^n(dx)Q^n(dy),\frac{1\!\!1_{\{y\geq x\}}}{(xy)^{1-\lambda}}\right\rangle-\varepsilon\left\langle Q^n(dx)Q^n(dy),\frac{1\!\!1_{\{x+y>n\}}}{(xy)^{1-\lambda}}\right\rangle. \end{aligned}$$

with $\varepsilon=1-1/2^{1-\lambda}>0$. An easy symmetry argument allows us to deduce that

$$\begin{aligned} |A_{n,\lambda-1}| &\geq \frac{\varepsilon}{2}\left\langle Q^n(dx),1/x^{1-\lambda}\right\rangle^2-2\varepsilon\left\langle Q^n(dx)Q^n(dy),\frac{1\!\!1_{\{x\geq n/2\}}}{(xy)^{1-\lambda}}\right\rangle \\ &\geq \frac{\varepsilon}{2}\left\langle Q^n(dx),1/x^{1-\lambda}\right\rangle^2-\frac{2^{2-\lambda}\varepsilon}{n^{1-\lambda}}\left\langle Q^n(dx),1/x^{1-\lambda}\right\rangle. \end{aligned}$$

On the other hand, an easy computation shows that

$$\begin{aligned} |B_{n,\lambda-1}| &\leq \gamma\left\langle Q^n(dx),(x-n^{-1})\frac{1-\lambda}{(x-n^{-1})^{2-\lambda}}1\!\!1_{\{x\geq 2/n\}}\right\rangle \\ &\leq \left\langle Q^n(dx),\frac{1\!\!1_{\{x\geq 2/n\}}}{(x-n^{-1})^{1-\lambda}}\right\rangle \\ &\leq 2^{1-\lambda}\left\langle Q^n(dx),1/x^{1-\lambda}\right\rangle, \end{aligned}$$

where the last inequality follows from the fact that $1/(x-n^{-1})\leq 2/x$ for every $x\geq 2/n$. We deduce from the bounds on $|A_{n,\lambda-1}|$ and $|B_{n,\lambda-1}|$ that there is a constant C which does not depend on $n\geq 1$ such that

$$\left\langle Q^n(dx),1/x^{1-\lambda}\right\rangle^2\leq C\left\langle Q^n(dx),1/x^{1-\lambda}\right\rangle$$

for each $n\geq 1$. Since $\left\langle Q^n(dx),1/x^{1-\lambda}\right\rangle<\infty$ for each n, we conclude that (3.1) holds true for K_1 and K_2 when $\sigma=\lambda-1$.

Step 2. We next prove that (3.1) holds true for K_2 and K_3 when $\sigma\geq 0$. Let $p\geq 1$ be a positive integer. Since $x\mapsto x^p$ is convex, we have

$$x^p-(x-1/n)^p\geq\frac{p}{n}\,(x-1/n)^{p-1}\quad\text{and}\quad(x+y)^p-x^p\leq p\;y\;(x+y)^{p-1}\,.$$

Consequently,

$$p\gamma\ \left\langle Q^n(dx),(x-1/n)^p \mathbb{1}_{\{x\ge 2/n\}}\right\rangle \le B_{n,p} = A_{n,p} \le p\ \left\langle Q^n(dx)Q^n(dy), K_n(x,y)\ (x+y)^{p-1}\right\rangle\ ,$$

and, since $(x+y)^p \le C\ (x^p+y^p)$ and $\langle Q^n(dx),1\rangle = 1$, we end up with

$$\langle Q^n(dx), x^p\rangle \le C\ \left(\frac{1}{n^p} + \left\langle Q^n(dx)Q^n(dy), K_n(x,y)\ (x+y)^{p-1}\right\rangle\right)\ . \tag{3.2}$$

Now, since $\lambda \in [0,1)$, it is straightforward to check that, for K_2 and K_3 and $\varepsilon > 0$, there is $C(\varepsilon,p) > 0$ such that $K(x,y)\ (x+y)^{p-1} \le \varepsilon\ (x^p+y^p) + C(\varepsilon,p)$. Using this inequality with $\varepsilon = 1/(4C)$ and the fact that $\langle Q^n(dx),1\rangle = 1$, we conclude that

$$\begin{aligned}\langle Q^n(dx), x^p\rangle &\le C\ \left(\frac{1}{n^p} + 2\varepsilon\ \langle Q^n(dx)Q^n(dy), x^p\rangle + C(\varepsilon,p)\right)\\ &\le \frac{1}{2}\ \langle Q^n(dx), x^p\rangle + C\ \left(\frac{1}{n^p} + C(\varepsilon,p)\right)\ .\end{aligned}$$

Consequently, (3.1) holds true for K_2 and K_3 when σ is a non-negative integer and thus for each $\sigma \ge 0$ by interpolation.

Step 3. We now check that (3.1) is true for K_1 when $\sigma \ge 0$. Let p be a positive integer such that $p > 1+\beta$. Recalling (3.2) and using a symmetry argument and the inequality $(x+y)^{p-1} \le C\ \left(x^{p-1}+y^{p-1}\right)$, we obtain

$$\langle Q^n(dx), x^p\rangle \le C\ \left(\frac{1}{n^p} + \left\langle Q^n(dx)Q^n(dy), K_n(x,y)\ x^{p-1}\right\rangle\right)\ ,$$

whence

$$\begin{aligned}\langle Q^n(dx), x^p\rangle \ &\le\ \frac{C}{n^p} + C\ \left\langle Q^n(dx), x^{p+\lambda-1}\right\rangle + C\ \left\langle Q^n(dx), x^{p+\alpha-1}\right\rangle\ \left\langle Q^n(dx), x^{-\beta}\right\rangle\\ &\quad + C\ \left\langle Q^n(dx), x^{p-1}\right\rangle\ \left\langle Q^n(dx), x^{\lambda}\right\rangle + C\ \left\langle Q^n(dx), x^{p-1-\beta}\right\rangle\ \left\langle Q^n(dx), x^{\alpha}\right\rangle\ .\end{aligned}$$

Now, since $p > 1+\beta \ge 1-\lambda$, one also has $0 < \lambda+p-1 < p$, $0 < \alpha+p-1 < p$ and $0 < p-1-\beta < p$. Since $Q^n(dx)$ is a probability measure, the Jensen inequality yields

$$\left\langle Q^n(dx), x^{\lambda+p-1}\right\rangle \le \langle Q^n(dx), x^p\rangle^{(p+\lambda-1)/p}\ , \qquad \left\langle Q^n(dx), x^{p-1}\right\rangle \le \langle Q^n(dx), x^p\rangle^{(p-1)/p}\ ,$$

$$\left\langle Q^n(dx), x^{\alpha+p-1}\right\rangle \le \langle Q^n(dx), x^p\rangle^{(p+\alpha-1)/p}\ , \qquad \left\langle Q^n(dx), x^{p-1-\beta}\right\rangle \le \langle Q^n(dx), x^p\rangle^{(p-1-\beta)/p}\ ,$$

$$\langle Q^n(dx), x^{\alpha}\rangle \le \langle Q^n(dx), x^p\rangle^{\alpha/p}\ .$$

Also, since $\beta \le 1-\lambda$, we deduce from Step 1 and the Jensen inequality that

$$\left\langle Q^n(dx), x^{-\beta}\right\rangle \le C$$

for some constant C independent of n. Consequently, by the Young inequality,

$$\begin{aligned}\langle Q^n(dx), x^p\rangle &\le C\left(1 + \langle Q^n(dx), x^p\rangle^{(p+\lambda-1)/p} + \langle Q^n(dx), x^p\rangle^{(p+\alpha-1)/p}\right)\\ &\quad + C\ \langle Q^n(dx), x^p\rangle^{(p-1)/p}\ \left\langle Q^n(dx), x^{\lambda}\right\rangle\\ &\le \frac{1}{4}\ \langle Q^n(dx), x^p\rangle + C\ \left(1 + \left\langle Q^n(dx), x^{\lambda}\right\rangle^p\right)\ .\end{aligned}$$

Now, either $\lambda \le 0$ and we infer from the Jensen inequality and Step 1 that

$$\left\langle Q^n(dx), x^\lambda \right\rangle^p \le \left\langle Q^n(dx), x^{\lambda-1} \right\rangle^{(p|\lambda|)/(1-\lambda)} \le C\,,$$

or $\lambda \in (0,1)$ and the Jensen and Young inequalities imply that

$$\left\langle Q^n(dx), x^\lambda \right\rangle^p \le \langle Q^n(dx), x^p \rangle^\lambda \le \frac{1}{4C}\ \langle Q^n(dx), x^p \rangle + C\,.$$

In both cases, we finally arrive at

$$\langle Q^n(dx), x^p \rangle \le \frac{1}{2}\ \langle Q^n(dx), x^p \rangle + C\,,$$

from which Step 3 follows at once.

Step 4. Up to now, we have proved that (3.1) is true for K_1 when $\sigma \ge \lambda - 1$. We now complete the proof of (3.1) for K_1. We consider $\delta > 0$ and notice that

$$\begin{aligned}
|A_{n,-(1+\delta)}| &= \left\langle Q^n(dx)Q^n(dy), \frac{K_n(x,y)}{y} \left[\frac{1}{x^{1+\delta}} - \frac{1}{(x+y)^{1+\delta}} \right] \right\rangle \\
&= \left\langle Q^n(dx)Q^n(dy), \frac{K_n(x,y)}{xy} \left[\frac{1}{x^{\delta}} - \frac{x}{(x+y)^{1+\delta}} \right] \right\rangle \\
&= \left\langle Q^n(dx)Q^n(dy), \frac{K_n(x,y)}{2xy} \left[\frac{1}{x^{\delta}} - \frac{x}{(x+y)^{1+\delta}} + \frac{1}{y^\delta} - \frac{y}{(x+y)^{1+\delta}} \right] \right\rangle \\
&= \left\langle Q^n(dx)Q^n(dy), \frac{K_n(x,y)}{2xy} \left[\frac{1}{x^{\delta}} - \frac{1}{2(x+y)^{\delta}} + \frac{1}{y^\delta} - \frac{1}{2(x+y)^{\delta}} \right] \right\rangle \\
&\ge \left\langle Q^n(dx)Q^n(dy), \frac{K_n(x,y)}{4xy} \left[\frac{1}{x^{\delta}} + \frac{1}{y^\delta} \right] \right\rangle \\
&= \frac{1}{2}\ \left\langle Q^n(dx)Q^n(dy), \frac{K_n(x,y)}{x^{1+\delta}y} \right\rangle \\
&\ge \frac{1}{2}\ \left\langle Q^n(dx)Q^n(dy), \frac{1\!\!1_{\{x+y\le n\}}}{x^{1+\delta+\beta}\ y^{1-\alpha}} \right\rangle \\
&\ge \frac{1}{2}\ \left\langle Q^n(dx)Q^n(dy), x^{-(1+\delta+\beta)}\ y^{\alpha-1} \right\rangle - \frac{1}{2}\ \left\langle Q^n(dx)Q^n(dy), \frac{1\!\!1_{\{x+y>n\}}}{x^{1+\delta+\beta}\ y^{1-\alpha}} \right\rangle \\
&\ge \frac{1}{2}\ \left\langle Q^n(dx), x^{-(1+\delta+\beta)} \right\rangle\ \left\langle Q^n(dx), x^{\alpha-1} \right\rangle \\
&\quad - \frac{2^{\delta+\beta}}{n^{1+\delta+\beta}}\ \left\langle Q^n(dx), x^{\alpha-1} \right\rangle - \frac{2^{-\alpha}}{n^{1-\alpha}}\ \left\langle Q^n(dx), x^{-(1+\delta+\beta)} \right\rangle \\
&\ge \left(\frac{1}{2}\ \left\langle Q^n(dx), x^{\alpha-1} \right\rangle - \frac{2^{-\alpha}}{n^{1-\alpha}} \right) \left\langle Q^n(dx), x^{-(1+\delta+\beta)} \right\rangle - \frac{2^{\delta+\beta}}{n^{1+\delta+\beta}}\ \left\langle Q^n(dx), x^{\alpha-1} \right\rangle
\end{aligned}$$

On the one hand, the Jensen inequality and Step 1 imply that

$$\left\langle Q^n(dx), x^{\alpha-1} \right\rangle \le \left\langle Q^n(dx), x^{\lambda-1} \right\rangle^{(1-\alpha)/(1-\lambda)} \le C\,,$$

since $0 < 1-\alpha < 1-\lambda$. On the other hand, we infer from Step 3 that

$$\begin{aligned}
\left\langle Q^n(dx), x^{\alpha-1} \right\rangle & \ge \left\langle Q^n(dx), \frac{1\!\!1_{\{x\le R\}}}{x^{1-\alpha}} \right\rangle \\
& \ge \frac{1}{R^{1-\alpha}} \left(\langle Q^n(dx), 1 \rangle - \frac{1}{R} \left\langle Q^n(dx), x \; 1\!\!1_{\{x\ge R\}} \right\rangle \right) \\
& \ge \frac{1}{R^{1-\alpha}} \left(1 - \frac{C}{R} \right) \\
& \ge 2\,\varepsilon
\end{aligned}$$

for some $\varepsilon > 0$ sufficiently small after choosing $R = 2/C$. Gathering the above three estimates we end up with

$$|A_{n,-(1+\delta)}| \ge \left(\varepsilon - \frac{2^{-\alpha}}{n^{1-\alpha}} \right) \left\langle Q^n(dx), x^{-(1+\delta+\beta)} \right\rangle - C \ge \frac{\varepsilon}{2} \left\langle Q^n(dx), x^{-(1+\delta+\beta)} \right\rangle - C$$

for n large enough, since $\alpha \in [0,1)$. Therefore,

$$\begin{aligned}
\frac{\varepsilon}{2} \left\langle Q^n(dx), x^{-(1+\delta+\beta)} \right\rangle & \le |A_{n,-(1+\delta)}| + C \\
& \le |B_{n,-(1+\delta)}| + C \\
& \le 2^{1+\delta} \gamma(1+\delta) \left\langle Q_n(dx), x^{-(1+\delta)} \right\rangle + C\,,
\end{aligned}$$

where we have proceeded as in Step 1 to obtain the last inequality. Recalling that $\beta > 0$, we have $0 < 1+\delta < 1+\delta+\beta$ and the Jensen and Young inequalities yield

$$\begin{aligned}
\frac{\varepsilon}{2} \left\langle Q^n(dx), x^{-(1+\delta+\beta)} \right\rangle & \le C \left(1 + \left\langle Q^n(dx), x^{-(1+\delta+\beta)} \right\rangle^{(1+\delta)/(1+\delta+\beta)} \right) \\
& \le \frac{\varepsilon}{4} \left\langle Q^n(dx), x^{-(1+\delta+\beta)} \right\rangle + C\,.
\end{aligned}$$

Thus, (3.1) is also valid for $\sigma < -1$ and an interpolation argument completes the proof for K_1.

Step 5. It remains to check that (3.1) holds true for K_3 when $\sigma \in (\lambda - 1, 0)$. We first remark that $K(x,y) \ge (xy)^{\lambda/2}$. Arguing as in Step 1, we find that

$$\begin{aligned}
|A_{n,\sigma}| & \ge \left\langle Q^n(dx)Q^n(dy), \frac{(xy)^{\lambda/2}}{y} \left[x^\sigma - (x+y)^\sigma \right] 1\!\!1_{\{x+y\le n\}} \right\rangle \\
|B_{n,\sigma}| & \le C \left\langle Q^n(dx), x^\sigma \right\rangle
\end{aligned}$$

We next define $\tau = 2/(1-\lambda+\sigma) > 0$ and $a_i = i^{-\tau}$ for $i \ge 1$. Then, for $n \ge 2$,

$$\begin{aligned}
|A_{n,\sigma}| & \ge |\sigma| \left\langle Q^n(dx)Q^n(dy), \frac{(xy)^{\lambda/2}}{(x+y)^{1-\sigma}} 1\!\!1_{\{x+y\le n\}} \right\rangle \\
& \ge |\sigma| \left\langle Q^n(dx)Q^n(dy), \frac{(xy)^{\lambda/2}}{(x+y)^{1-\sigma}} 1\!\!1_{\{x\in[0,1]\}} 1\!\!1_{\{y\in[0,1]\}} \right\rangle \\
& \ge |\sigma| \sum_{i\ge1} (2a_i)^{\sigma-1} \left\langle Q^n(dx)Q^n(dy), (xy)^{\lambda/2} 1\!\!1_{\{x\in(a_{i+1},a_i]\}} 1\!\!1_{\{y\in(a_{i+1},a_i]\}} \right\rangle \\
& \ge \varepsilon \sum_{i\ge1} (a_i)^{\sigma-1} \left\langle Q^n(dx), x^{\lambda/2} 1\!\!1_{\{x\in(a_{i+1},a_i]\}} \right\rangle^2 \qquad (3.3)
\end{aligned}$$

for some constant $\varepsilon > 0$. We next use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
& \left\langle Q^n(dx), \mathbb{1}_{\{x\in[0,1]\}}\ x^\sigma \right\rangle = \sum_{i=1}^\infty \left\langle Q^n(dx), x^{\sigma-(\lambda/2)} x^{\lambda/2} \mathbb{1}_{\{x\in(a_{i+1},a_i]\}} \right\rangle \\
\leq & \sum_{i=1}^\infty \frac{a_i^{(1-\sigma)/2}}{a_{i+1}^{-\sigma+(\lambda/2)}}\ a_i^{(\sigma-1)/2} \left\langle Q^n(dx), x^{\lambda/2} \mathbb{1}_{\{x\in(a_{i+1},a_i]\}} \right\rangle \\
\leq & \left(\sum_{i=1}^\infty \frac{a_i^{1-\sigma}}{a_{i+1}^{\lambda-2\sigma}} \right)^{1/2} \left(\sum_{i=1}^\infty (a_i)^{\sigma-1} \left\langle Q^n(dx), x^{\lambda/2} \mathbb{1}_{\{x\in(a_{i+1},a_i]\}} \right\rangle^2 \right)^{1/2} \\
\leq & C \left(\sum_{i=1}^\infty (a_i)^{\sigma-1} \left\langle Q^n(dx), x^{\lambda/2} \mathbb{1}_{\{x\in(a_{i+1},a_i]\}} \right\rangle^2 \right)^{1/2}, \qquad (3.4)
\end{aligned}$$

the last inequality resulting from the fact that

$$\frac{a_i^{1-\sigma}}{a_{i+1}^{\lambda-2\sigma}} = (i+1)^{\tau(\lambda-2\sigma)} i^{-\tau(1-\sigma)} \leq 2^{\tau(\lambda-2\sigma)} i^{-\tau(1-\lambda+\sigma)} = 2^{\tau(\lambda-2\sigma)} i^{-2}.$$

Gathering (3.3) and (3.4), we deduce that

$$|A_{n,\sigma}| \geq \varepsilon\ \left\langle Q^n(dx), \mathbb{1}_{\{x\in[0,1]\}}\ x^\sigma \right\rangle^2$$

for some $\varepsilon > 0$. Finally, since $|A_{n,\sigma}| = |B_{n,\sigma}|$ and $Q^n(dx)$ is a probability measure, the above bounds entail that

$$\langle Q^n(dx), x^\sigma \rangle^2 \leq 2 \left\langle Q^n(dx), x^\sigma \mathbb{1}_{\{x\in[0,1]\}} \right\rangle^2 + 2 \leq C \langle Q^n(dx), x^\sigma \rangle + 2,$$

from which the expected result readily follows by the Young inequality. □

4 Integrability estimates

Having studied the behaviour of $Q^n(dx)$ for small and large x in the previous section, we now turn to "integrability" properties of $Q^n(dx)$. Since $Q^n(dx)$ is a measure, such properties can obviously not be enjoyed by $Q^n(dx)$ but rather by

$$Z^n(x) = \gamma n\ \left\langle Q^n(dy), \left(y - \frac{1}{n} \right)\ \mathbb{1}_{\{x\leq y\leq x+1/n\}} \right\rangle, \quad x \in (0,\infty). \qquad (4.1)$$

In this direction, we have the following result:

Lemma 4.1 *(i) If K is given by (1.7), then Z^n belongs to $L^\infty(0,\infty)$ and*

$$\sup_{n\geq 1} \|Z^n\|_{L^\infty} < \infty. \qquad (4.2)$$

(ii) If K is given by (1.8) or (1.9) and $p \in (1, 1/\lambda)$, then $x \mapsto x^{\lambda-1}\ Z^n(x)$ belongs to $L^p(0,\infty)$ and

$$\sup_{n\geq 1} \int_0^\infty x^{p(\lambda-1)}\ Z^n(x)^p\ dx < \infty. \qquad (4.3)$$

Proof. (i) We take $\vartheta \in C_0^\infty((0,\infty))$ and choose

$$\phi(x) = \int_0^x \vartheta(y)\ dy\,, \quad x \in [0,\infty)\,,$$

in (2.12). Setting

$$A_n = \left\langle Q^n(dx)Q^n(dy), \frac{K_n(x,y)}{y}\ [\phi(x+y) - \phi(x)] \right\rangle\,,$$

we infer from the Fubini theorem and (2.12) that

$$\int_0^\infty \vartheta(x)\ Z^n(x)\ dx = A_n\,. \tag{4.4}$$

Using once more the Fubini theorem, we may estimate A_n as follows:

$$\begin{aligned}
|A_n| &\le \left| \left\langle Q^n(dx)Q^n(dy), \frac{K(x,y)}{y}\ \int_x^{x+y} \vartheta(z)\ dz \right\rangle \right| \\
&= \left| \int_0^\infty \vartheta(z)\ \left\langle Q^n(dx)Q^n(dy), \frac{K(x,y)}{y}\ 1\!\!1_{\{x\le z\}}\ 1\!\!1_{\{z-x\le y\}} \right\rangle\ dz \right| \\
&\le \int_0^\infty |\vartheta(z)|\ \left\langle Q^n(dx)Q^n(dy), \frac{K(x,y)}{y} \right\rangle\ dz \\
&\le \|\vartheta\|_{L^1}\ \left(\langle Q^n(dx), x^\lambda \rangle\ \langle Q^n(dx), x^{-1} \rangle + \langle Q^n(dx), x^\alpha \rangle\ \left\langle Q^n(dx), x^{-(\beta+1)} \right\rangle \right) \\
&\quad + \|\vartheta\|_{L^1}\ \left(\langle Q^n(dx), x^{-\beta} \rangle\ \langle Q^n(dx), x^{\alpha-1} \rangle + \langle Q^n(dx), 1 \rangle\ \langle Q^n(dx), x^{\lambda-1} \rangle \right) \\
&\le C\ \|\vartheta\|_{L^1}
\end{aligned}$$

by Lemma 3.1. Recalling (4.4) we conclude that, for some constant $C > 0$ not depending on n nor on ϑ,

$$\left| \int_0^\infty \vartheta(x)\ Z^n(x)\ dx \right| \le C\ \|\vartheta\|_{L^1}$$

for every $\vartheta \in C_0^\infty((0,\infty))$. A density argument next ensures that the previous estimate is actually true for every $\vartheta \in L^1(0,\infty)$. The dual space of $L^1(0,\infty)$ being $L^\infty(0,\infty)$, the first assertion of Lemma 4.1 readily follows.

(ii) We take $\vartheta \in C_0^\infty((0,\infty))$ and choose

$$\phi(x) = \int_0^x \vartheta(y)\ y^{\lambda-1}\ dy\,, \quad x \in [0,\infty)\,,$$

in (2.12). Setting

$$A_n = \left\langle Q^n(dx)Q^n(dy), \frac{K_n(x,y)}{y}\ [\phi(x+y) - \phi(x)] \right\rangle\,,$$

we infer from the Fubini theorem and (2.12) that

$$\int_0^\infty \vartheta(x)\ x^{\lambda-1}\ Z^n(x)\ dx = A_n\,. \tag{4.5}$$

To estimate A_n, we first remark that K_2 and K_3 satisfy, for some constant $C > 0$,

$$K(x,y) \le C(x^\mu\ y^\nu + x^\nu\ y^\mu)\,, \quad 0 \le \nu \le \mu < 1\,, \quad \lambda = \mu + \nu\,,$$

with $(\mu,\nu)=(\alpha\beta,0)$ for $K=K_2$ and $(\mu,\nu)=(\max\{\alpha,\beta\},\min\{\alpha,\beta\})$ for $K=K_3$. We next fix $p\in(1,1/\lambda)$. Then there exists $\varepsilon>\nu$ small enough such that

$$(\lambda+\varepsilon-\nu)\,p<1\,. \tag{4.6}$$

Introducing $\sigma=1-\lambda-\varepsilon<1-\lambda$, we infer from (4.6) that

$$(2-\lambda-\sigma-\nu)\,p>1 \quad\text{and}\quad 2-\lambda-\sigma-\nu-\frac{1}{p}<1-\lambda\,. \tag{4.7}$$

Now, since $\sigma<1-\lambda\le 1-\nu$, we infer from the Fubini theorem that

$$\begin{aligned}
|A_n| &\le \left\langle Q^n(dx)Q^n(dy), \frac{K(x,y)}{y}\int_x^{x+y}|\vartheta(z)|\ z^{\lambda-1}\ dz\right\rangle\\
&\le \int_0^\infty \frac{|\vartheta(z)|}{z^{1-\lambda}}\left\langle Q^n(dx)Q^n(dy), \frac{K(x,y)}{y}\ \mathbb{1}_{\{x\le z\}}\ \mathbb{1}_{\{z-x\le y\}}\right\rangle\ dz\\
&\le C\int_0^\infty \frac{|\vartheta(z)|}{z^{1-\lambda}}\left\langle Q^n(dx)Q^n(dy), \frac{(x^\mu\ y^\nu+x^\nu\ y^\mu)}{y}\ \mathbb{1}_{\{x\le z\}}\ \mathbb{1}_{\{z-x\le y\}}\right\rangle\ dz\\
&\le C\int_0^\infty \frac{|\vartheta(z)|}{z^{1-\lambda}}\left\langle Q^n(dx)Q^n(dy), \frac{1}{(z-x)^{1-\sigma-\nu}}\ \left(\frac{x^\mu}{y^\sigma}+\frac{x^\nu}{y^{\sigma+\nu-\mu}}\right)\ \mathbb{1}_{\{x\le z\}}\ \mathbb{1}_{\{z-x\le y\}}\right\rangle\ dz\\
&\le C\int_0^\infty \frac{|\vartheta(z)|}{z^{1-\lambda}}\left\langle Q^n(dx), \frac{x^\mu}{(z-x)^{1-\sigma-\nu}}\ \mathbb{1}_{\{x\le z\}}\right\rangle\left\langle Q^n(dy), y^{-\sigma}\right\rangle\ dz\\
&\quad + C\int_0^\infty \frac{|\vartheta(z)|}{z^{1-\lambda}}\left\langle Q^n(dx), \frac{x^\nu}{(z-x)^{1-\sigma-\nu}}\ \mathbb{1}_{\{x\le z\}}\right\rangle\left\langle Q^n(dy), y^{\mu-\nu-\sigma}\right\rangle\ dz\,.
\end{aligned}$$

Since $\sigma+\nu-\mu\le\sigma<1-\lambda$, we infer from Lemma 3.1 and the Fubini theorem that, setting $p'=p/(p-1)$,

$$\begin{aligned}
|A_n| &\le C\int_0^\infty \frac{|\vartheta(z)|}{z^{1-\lambda}}\left\langle Q^n(dx), \frac{x^\mu+x^\nu}{(z-x)^{1-\sigma-\nu}}\ \mathbb{1}_{\{x\le z\}}\right\rangle\ dz\\
&\le C\left\langle Q^n(dx), (x^\mu+x^\nu)\int_x^\infty \frac{|\vartheta(z)|}{(z-x)^{1-\sigma-\nu}\ z^{1-\lambda}}\ dz\right\rangle\\
&\le C\ \|\vartheta\|_{L^{p'}}\left\langle Q^n(dx), (x^\mu+x^\nu)\ \left(\int_x^\infty (z-x)^{-p(1-\sigma-\nu)}\ z^{-p(1-\lambda)}\ dz\right)^{1/p}\right\rangle\\
&\le C\ \|\vartheta\|_{L^{p'}}\left\langle Q^n(dx), \frac{x^\mu+x^\nu}{x^{2-\lambda-\sigma-\nu-1/p}}\right\rangle\ \left(\int_1^\infty u^{-p(1-\lambda)}\ (u-1)^{-p(1-\sigma-\nu)}\ du\right)^{1/p}\\
&\le C\ \|\vartheta\|_{L^{p'}}\,,
\end{aligned}$$

where the last inequality follows from Lemma 3.1, (4.6) and (4.7). Combining this estimate with (4.5) yields

$$\left|\int_0^\infty \vartheta(x)\ x^{\lambda-1}\ Z^n(x)\ dx\right|\le C\ \|\vartheta\|_{L^{p'}}$$

for every $\vartheta\in C_0^\infty((0,\infty))$. A density argument next ensures that the previous estimate is actually true for every $\vartheta\in L^{p'}(0,\infty)$. Since the dual space of $L^{p'}(0,\infty)$ is $L^p(0,\infty)$, this completes the proof of Lemma 4.1. □

5 Proof of Theorem 1.3

We are now in a position to complete the proof of Theorem 1.3, (1.16), (1.18) and (1.19).
Owing to (2.9) and Lemma 3.1, $(Q^n(dx))$ is a tight sequence of probability measures on $[0,\infty)$. Consequently, there are a subsequence $(Q^{n_k}(dx))$ of $(Q^n(dx))$ and a probability measure $Q(dx)$ on $[0,\infty)$ such that $Q^{n_k}(dx)$ converges narrowly towards $Q(dx)$, that is,

$$\lim_{k\to\infty} \langle Q^{n_k}(dx),\phi\rangle = \langle Q(dx),\phi\rangle \tag{5.1}$$

for any $\phi\in C_b([0,\infty))$. A straightforward consequence of Lemma 3.1 and (5.1) is that

$$\langle Q(dx), x^\sigma\rangle < \infty \tag{5.2}$$

for

$$\begin{array}{lll} \sigma\in\mathbb{R} & \text{if} & K=K_1\,, \\ \sigma\in[\lambda-1,\infty) & \text{if} & K=K_2\,, \\ \sigma\in(\lambda-1,\infty) & \text{if} & K=K_3\,. \end{array}$$

In addition, thanks to the uniform bounds on some negative moments of $Q^n(dx)$ given by Lemma 3.1, we deduce that $Q(\{0\})=0$, so that $Q(dx)$ is a probability measure on $(0,\infty)$.
We next check that $Q(dx)$ is absolutely continuous with respect to the Lebesgue measure on $(0,\infty)$. We first consider the case when K is given by (1.7). By Lemma 4.1, the sequence (Z^n) is bounded in $L^\infty(0,\infty)$ and we may thus assume that there is $Z\in L^\infty(0,\infty)$ such that (Z^{n_k}) converges weakly towards Z in $L^\infty(0,\infty)$. However, it is straightforward to deduce from (4.1) and (5.1) that (Z^{n_k}) converges towards $\gamma x Q(dx)$ in $\mathcal{D}'(0,\infty)$. Consequently, $\gamma x Q(dx) = Z(x)$ and, since $Q(dx)$ does not charge $\{0\}$ and satisfies (5.2), we conclude that $Q(dx)=Q(x)dx$ with $Q\in L^1(0,\infty)$. We next turn to the case where K is given by (1.8) or (1.9) and fix $p\in(1,1/\lambda)$. We infer from Lemma 4.1 that $(x^{\lambda-1}\,Z^n)$ is bounded in $L^p(0,\infty)$, so that we may assume that there is $\tilde{Z}\in L^p(0,\infty)$ such that $(x^{\lambda-1}Z^{n_k})$ converges weakly towards $\tilde{Z}$ in $L^p(0,\infty)$. But $(x^{\lambda-1}Z^{n_k})$ also converges towards $\gamma\, x^\lambda\, Q(dx)$ in $\mathcal{D}'(0,\infty)$ by (4.1) and (5.1). We then argue as before to conclude that $Q(dx)=Q(x)dx$ with $Q\in L^1(0,\infty)$.
We now prove that $\psi(x)=Q(x)/x$ is a weak solution to (1.13) and first check that (1.20) is satisfied by Q for every ϕ in $C_b^2([0,\infty))$ (the extension to C_b^1 functions being straightforward). Owing to (5.2), we have

$$\langle Q(dx)Q(dy), K(x,y)\rangle < \infty\,.$$

Since (2.12) holds with $n=n_k$ for each k, we only have to prove that

$$(i)\ \lim_{k\to\infty} A_k = A, \quad (ii)\ \lim_{k\to\infty} B_k = B, \tag{5.3}$$

where

$$\begin{aligned} A_k &= \left\langle Q^{n_k}(dx)Q^{n_k}(dy), \frac{K(x,y)\mathbb{1}_{\{x+y\le n_k\}}}{y}\,[\phi(x+y)-\phi(x)]\right\rangle, \\ A &= \left\langle Q(dx)Q(dy), \frac{K(x,y)}{y}\,[\phi(x+y)-\phi(x)]\right\rangle \\ B_k &= \left\langle Q^{n_k}(dx), n_k\left(x-n_k^{-1}\right)\left[\phi(x)-\phi(x-n_k^{-1})\right]\right\rangle, \\ B &= \langle Q(dx), x\phi'(x)\rangle\,. \end{aligned}$$

We first prove (5.3) (ii). Since $x\phi'(x)$ is continuous on $[0,\infty)$, it is clear from (5.1) that $(\langle Q^{n_k}(dx), x\phi'(x)\rangle)$ converges towards $\langle Q(dx), x\phi'(x)\rangle$ as $k\to\infty$. The claim (ii) then follows after noticing that

$$\begin{aligned} & \left\langle Q^{n_k}(dx), \left|x\phi'(x) - n_k\left(x-n_k^{-1}\right)\left[\phi(x)-\phi(x-n_k^{-1})\right]\right|\right\rangle \\ \le\ & \frac{\|\phi'\|_{L^\infty}}{n_k} + \frac{\|\phi''\|_{L^\infty}}{n_k}\ \langle Q^{n_k}(dx), x\rangle \\ \le\ & C\ \frac{\|\phi\|_{C_b^2}}{n_k} \underset{k\to\infty}{\longrightarrow} 0 \end{aligned}$$

by Lemma 3.1.
We next prove (5.3) (i). On the one hand, it is not difficult to deduce from (5.1) and (5.2) that

$$\lim_{k\to\infty}\left\langle Q^{n_k}(dx)Q^{n_k}(dy), \frac{K(x,y)}{y}\left[\phi(x+y)-\phi(x)\right]\right\rangle = B\,.$$

On the other hand,

$$\begin{aligned} & \left\langle Q^{n_k}(dx)Q^{n_k}(dy), \frac{K(x,y)\mathbb{1}_{\{x+y>n_k\}}}{y}\left|\phi(x+y)-\phi(x)\right|\right\rangle \\ \le\ & 2||\phi'||_{L^\infty}\left\langle Q^{n_k}(dx)Q^{n_k}(dy), K(x,y)\mathbb{1}_{\{x>n_k/2\}}\right\rangle \\ \le\ & \frac{4}{n_k}||\phi'||_{L^\infty}\left\langle Q^{n_k}(dx)Q^{n_k}(dy), xK(x,y)\right\rangle . \end{aligned}$$

The uniform moment estimates given in Lemma 3.1 allow us to conclude that (i) holds true. Therefore, $Q(dx) = Q(x)dx$ satisfies (1.20).
Finally, owing to the moment estimates (5.2), we may proceed as in Section 4 to show that

$$\int_0^z \int_{z-x}^\infty \frac{K(x,y)}{y}\ Q(y)\ Q(x)\ dydx < \infty$$

for each $z\in(0,\infty)$. For $\vartheta\in C_b([0,\infty))$, we then take

$$\phi(x) = \int_0^x \vartheta(y)\ dy\,, \qquad x\in[0,\infty)\,,$$

in (1.20) and use the Fubini theorem to deduce that

$$\gamma\int_0^\infty z\ Q(z)\ \vartheta(z)\ dz = \int_0^\infty \vartheta(z)\ \int_0^z\int_{z-x}^\infty \frac{K(x,y)}{y}\ Q(y)\ Q(x)\ dydx\ dz\,,$$

whence

$$\gamma z Q(z) = \int_0^z\int_{z-x}^\infty \frac{K(x,y)}{y} Q(y)Q(x)dydx \quad \text{ for a.e. } z\in(0,\infty).$$

Consequently, we realize that $\psi(z) = Q(z)/z$ satisfies all the properties claimed in Theorem 1.3, and the moment estimates (1.16), (1.18) or (1.19) as well.

6 Proof of Theorem 1.4

We first study the positivity of ψ.

Proposition 6.1 *Consider the function ψ constructed in Theorem 1.3. There is a continuous and positive function $g\in C((0,\infty))$ such that $\psi(x)\ge g(x)>0$ for any $x\in(0,\infty)$.*

Proof. Owing to (1.14) and the moment estimates (1.16), (1.18) or (1.19), it is sufficient to establish that ψ is positive a.e. in $(0,\infty)$. Indeed, we have $K(x,y)>0$ on $(0,\infty)^2$ and

$$\psi(z) \ge g(z) := \frac{1}{\gamma z^2} \int_0^z \int_{z-x}^\infty K(x,y) x (y\wedge 1) \psi(y) \psi(x) dydx\,, \quad z>0\,,$$

by (1.14), where g is clearly a continuous and positive function on $(0,\infty)$.
Step 1. We claim that ψ does not vanish in a neighbourhood of $x=0$. Indeed, assume for contradiction that supp $\psi \subset (\delta,\infty)$ for some $\delta\in(0,1)$. Setting

$$R(z) = \int_0^z x\psi(x)\ dx\,, \quad z\in[0,\infty)\,,$$

we infer from (1.14) and (1.16), (1.18) or (1.19) that, for $z>\delta$,

$$\begin{aligned} \gamma\ R'(z) &\le \frac{1}{z} \int_0^z \int_\delta^\infty x K(x,y) \psi(y)\ \psi(x)\ dydx \\ &\le \delta^{-2} \int_\delta^z \int_\delta^\infty C(\delta)\ (1+x+y)\ xy\psi(y)\ \psi(x)\ dydx \\ &\le C(\delta)\ (1+z)\ R(z)\,. \end{aligned}$$

To obtain the previous estimate, we have used that $K(x,y)\le C(\delta)\ (1+x+y)$ when $(x,y)\in[\delta,\infty)^2$ for the three classes of coagulation kernels (1.7), (1.8) and (1.9), and the moment estimates (1.16), (1.18) or (1.19). Consequently, for each $z>\delta$,

$$R(z) \le R(\delta)\ \exp\left\{ C(\delta)\ \int_\delta^z (1+y)\ dy \right\} = 0\,,$$

which contradicts the fact that $R(z)\to 1$ as $z\to\infty$.
Step 2. Assume now for contradiction that there is $z\in(0,\infty)$ such that $\psi(z)=0$. By (1.14) we have

$$\int_0^z \int_{z-x}^\infty x K(x,y)\ \psi(y)\ \psi(x)\ dydx = 0\,,$$

which in turn implies that

$$\int_0^{z/2} \int_{z/2}^z x K(x,y)\ \psi(y)\ \psi(x)\ dydx = 0\,.$$

Since $K(x,y)>0$ in $(0,\infty)^2$, we deduce that

$$\left(\int_0^{z/2} \psi(x) dx \right) \times \left(\int_{z/2}^z \psi(x) dx \right) = \int_0^{z/2} \int_{z/2}^z \psi(y)\ \psi(x)\ dydx = 0\,,$$

whence

$$\int_{z/2}^z \psi(y)\ dy = 0$$

by Step 1. Consequently, $\psi=0$ a.e. in $(z/2,z)$ and we may iterate the process to conclude that $\psi=0$ a.e. in $(0,z)$, which contradicts Step 1. □

We conclude the paper with the proof of (1.17).

Proposition 6.2 *Assume that $K = K_1$ and consider the function ψ constructed in Theorem 1.3. Then there is a constant $\varepsilon > 0$ such that*

$$\int_0^\infty e^{\varepsilon x^{-\beta}} x^{-\beta} \psi(x) dx < \infty\,. \tag{6.4}$$

Proof. For $a \in (0,1)$, $\varepsilon > 0$ and $x \in (0,\infty)$, we put $\vartheta_a(x) = \exp\left\{\varepsilon\ (x+a)^{-\beta}\right\}$ and $\phi(x) = \vartheta_a(x)/x$. Though $\phi \notin C_b^1([0,\infty))$, the moment estimates (1.16) allow us to use (1.15) for this choice of test function. On the one hand, we may proceed as in the proof of Step 4 of Lemma 3.1 to obtain that

$$\begin{aligned}
A &:= \int_0^\infty \int_0^\infty xK(x,y)\ [\phi(x) - \phi(x+y)]\ \psi(x)\ \psi(y)\ dydx \\
&\geq \int_0^\infty \int_0^\infty \frac{K(x,y)}{2}\ [\vartheta_a(x) + \vartheta_a(y) - \vartheta_a(x+y)]\ \psi(x)\ \psi(y)\ dydx \\
&\geq \int_0^\infty \int_0^\infty \frac{K(x,y)}{4}\ [\vartheta_a(x) + \vartheta_a(y)]\ \psi(x)\ \psi(y)\ dydx \\
&\geq \int_0^\infty \int_0^\infty \frac{K(x,y)}{2}\ \vartheta_a(x)\ \psi(x)\ \psi(y)\ dydx \\
&\geq \frac{1}{2} \left(\int_0^\infty x^\alpha\ \psi(x)\ dx\right) \left(\int_0^\infty \frac{\vartheta_a(x)}{x^\beta}\ \psi(x)\ dx\right)\,.
\end{aligned}$$

On the other hand, since $\int_0^\infty x\psi(x)dx = 1$, we have

$$\begin{aligned}
\gamma^{-1}\ B &:= -\int_0^\infty x^2\ \phi'(x)\ \psi(x)\ dx \\
&= \int_0^\infty \left(1 + \frac{\varepsilon\ \beta\ x}{(x+a)^{1+\beta}}\right)\ \vartheta_a(x)\ \psi(x)\ dx \\
&\leq \int_0^\infty \left(1 + \frac{\varepsilon\ \beta}{x^\beta}\right)\ \vartheta_a(x)\ \psi(x)\ dx \\
&\leq \int_0^{\varepsilon^{1/\beta}} \frac{\varepsilon}{x^\beta}\ \vartheta_a(x)\ \psi(x)\ dx + \frac{1}{\varepsilon^{1/\beta}} \int_{\varepsilon^{1/\beta}}^\infty x\vartheta_a(x)\ \psi(x)\ dx \\
&\quad + \varepsilon\ \beta \int_0^\infty \frac{\vartheta_a(x)}{x^\beta}\ \psi(x)\ dx \\
&\leq \varepsilon\ (1+\beta) \int_0^\infty \frac{\vartheta_a(x)}{x^\beta}\ \psi(x)\ dx + \frac{e}{\varepsilon^{1/\beta}}\,.
\end{aligned}$$

Since $2A = 2B$ by (1.15), we end up with

$$\left(\int_0^\infty x^\alpha\ \psi(x)\ dx - 2\ \varepsilon\ (1+\beta)\ \gamma\right) \int_0^\infty \frac{\vartheta_a(x)}{x^\beta}\ \psi(x)\ dx \leq \frac{2\ \gamma\ e}{\varepsilon^{1/\beta}}\,.$$

Choosing

$$\varepsilon = \frac{1}{4\ \gamma\ (1+\beta)} \int_0^\infty x^\alpha\ \psi(x)\ dx > 0$$

which is positive since $\int_0^\infty x\psi(x)dx = 1$, we obtain that

$$\int_0^\infty \frac{\vartheta_a(x)}{x^\beta}\ \psi(x)\ dx \leq \frac{4\ \gamma\ e}{\varepsilon^{1/\beta}} \left(\int_0^\infty x^\alpha \psi(x) dx\right)^{-1}\,. \tag{6.5}$$

Since the right-hand side of the above estimate does not depend on $a \in (0,1)$, we let $a \to 0$ in (6.5) and use the Fatou lemma to complete the proof. □

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