A microscopic probabilistic description of a locally regulated population and macroscopic approximations

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December 22, 2006

Abstract

We consider a discrete model describing a locally regulated spatial population with mortality selection. This model was studied in parallel by Bolker and Pacala, [2] and Dieckmann, Law and Murrell [9], [4], [10]. We first generalize this model by adding spatial dependence. Then we give a path-wise description in terms of Poisson point measures. We show that different re-normalizations may lead to different macroscopic approximations of this model. The first approximation is deterministic and gives a rigorous sense to the *number density*. The second one is a superprocess already studied by Etheridge [6]. Finally, we study in specific cases the long time behavior of the system and of its deterministic approximation.

Key words: Interacting measure-valued processes, Regulated population, Deterministic macroscopic approximation, Nonlinear superprocess, Equilibrium.

MSC 2000: 60J80, 60K35.

1 Introduction

We consider a spatial ecological system consisting of motionless individuals (such as *plants*). Individuals are characterized by their location. We assume that each plant produces seeds, at a given rate. When a seed is born, it immediately disperses from its *mother* and becomes a mature plant. We also assume that plants are subjected to *mortality selection*. That is, each plant dies at a rate depending on the local population density. All these events occur randomly, in continuous time. This model was introduced by Bolker-Pacala [2] and Dieckmann-Law-Murrell [9]. To study the system, Bolker-Pacala derived approximations for the time evolution of the moments (mean and spatial covariance) of the population distribution. In the present paper, we aim to
(i) give a rigorous definition of the underlying *microscopic* stochastic process,
(ii) rewrite rigorously the moment equations of [2],
(iii) derive some tracktable macroscopic approximations,
(iv) study the long time behavior of the stochastic process and/or its approximations.
Unfortunately, we obtained only some partial results concering point (iv).

In Section 2, we describe the BPDL (Bolker-Pacala-Dieckmann-Law) process in details. In fact, we generalize slightly the model by adding a spatial dependence in all the rates. Then we give a path-wise representation of the system in terms of Poisson point measures. We also produce a

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numerical algorithm to simulate the BPDL process.
Section 3 is devoted to existence and uniqueness. We also show some martingale properties of the BPDL process.
In Section 4, we re-find the mean equation that Bolker-Pacala [2] intuitively obtained. We also give a rigorous sense to the covariance terms formally defined in [2] or [9], [4], [10].
Section 5 is concerned with macroscopic approximations of the BPDL process. We first show that, conveniently re-normalized, the BPDL process converges to the solution of a deterministic integro-differential equation. We propose this as a rigorous interpretation of the *density number*, often introduced by biologists with a proper definition. We also show that with another re-normalization, the BPDL process converges to the superprocess version of the BPDL model, introduced and studied by Etheridge [6].
We give some partial results about extinction and survival for the BPDL process in Section 6.
In Section 7, we study the convergence to equilibrium of the deterministic approximation. We obtain only some partial results. We next show that in the *detailed balance case* to be specified later on, there exists a non-trival steady state for the BPDL process. We conclude the paper with some simulations.

2 The model

Let us first describe the model in details.

2.1 Definition of the parameters and heuristics

The plants are supposed to be motionless and characterized by their spatial location. We assume that the spatial domain is the closure $\bar{\mathcal{X}}$ of an open connected subset $\mathcal{X}$ of $\mathbb{R}^d$, for some $d \geq 1$. We will denote by $M_F(\bar{\mathcal{X}})$ (resp. $\mathcal{P}(\bar{\mathcal{X}})$) the set of finite non-negative measures (resp. probability measures) on $\bar{\mathcal{X}}$. Let also $\mathcal{M}$ be the subset of $M_F(\bar{\mathcal{X}})$ consisting of all finite point measures:

$$\mathcal{M} = \left\{ \sum_{i=1}^{n} \delta_{x_i},\ n \geq 0, x_1, ..., x_n \in \bar{\mathcal{X}} \right\}. \tag{2.1}$$

Here and below, δ_x denotes the Dirac mass at x. For any $m = \sum_{i=1}^{n} \delta_{x_i} \in \mathcal{M}$, any measurable function f on $\bar{\mathcal{X}}$, we set $\langle m, f \rangle = \int_{\bar{\mathcal{X}}} f dm = \sum_{i=1}^{n} f(x_i)$.

Notation 2.1 *For all x in $\bar{\mathcal{X}}$, we introduce the following quantities:*
(i) $\mu(x) \in [0, \infty)$ is the rate of "intrinsic" death of plants located at x,
(ii) $\gamma(x) \in [0, \infty)$ is the rate of seed production of plants located at x,
(iii) $D(x, dz)$ is the dispersion law of the seeds of plants located at x, it is assumed to satisfy, for each $x \in \bar{\mathcal{X}}$, $\int_{z\in\mathbb{R}^d,\ x+z\in\bar{\mathcal{X}}} D(x, dz) = 1$ and $\int_{z\in\mathbb{R}^d,\ x+z\notin\bar{\mathcal{X}}} D(x, dz) = 0$.
(iv) $\alpha(x) \in [0, \infty)$ is the rate of interaction of plants located at x,
and, for x, y in $\bar{\mathcal{X}}$,
(v) $U(x, y) = U(y, x) \in [0, \infty)$ is the competition kernel.

The competition kernel $U(x, y)$ describes the strength of competition between plants located at x and y. It thus can be thought to be of the form $U(x, y) = h(|x - y|)$, for some non-increasing function h from $\mathbb{R}_+$ into $\mathbb{R}_+$.
We aim to study the stochastic process ν_t, taking its values in $\mathcal{M}$, and describing the *distribution* of plants at time t. We will write:

$$\nu_t = \sum_{i=1}^{I(t)} \delta_{X_t^i}, \tag{2.2}$$

$I(t) \in \mathbb{N}$ standing for the number of plants alive at time t, and $X_t^1, ..., X_t^{I(t)}$ describing their locations (in $\bar{\mathcal{X}}$). The supposed dynamics for this population can be roughly summarized as:
(i) at time $t = 0$, we have a (possibly random) distribution $\nu_0 \in \mathcal{M}$,
(ii) each plant (located at some $x \in \bar{\mathcal{X}}$) has three independent exponential clocks: a *seed production* clock with parameter $\gamma(x)$, a *natural death* clock with parameter $\mu(x)$, and a *competition mortality* clock with parameter $\alpha(x) \sum_{i=1}^{I(t)} U(x, X_t^i)$,
(iii) if one of the two *death* clocks of a plant rings, then this plant disappears,
(iv) if the *seed production* clock of a plant (located at some $x \in \bar{\mathcal{X}}$) rings, then it produces a seed. This seed immediately becomes a mature plant. Its location is given by $y = x + z$, where z is chosen randomly according to the dispersion law $D(x, dz)$.

In [2], γ, μ, α, and D were assumed to be space-independent. Our generalization might allow one to take into account external effects such as landscape, resource distribution, etc.
Note also that assuming that all these clocks are exponentially distributed allows us to reset all the clocks to 0 at each time that one clock rings.
We whish to describe the system by the evolution in time of the empirical measure ν_t. More precisely, we are looking for a $\mathcal{M}$-valued Markov process $(\nu_t)_{t\geq 0}$ with infinitesimal generator L, defined for a large class of functions ϕ from $\mathcal{M}$ into $\mathbb{R}$, for all $\nu \in \mathcal{M}$, by

$$\begin{aligned} L\phi(\nu) &= \int_{\bar{\mathcal{X}}} \nu(dx) \int_{\mathbb{R}^d} D(x, dz) \left[\phi(\nu + \delta_{x+z}) - \phi(\nu)\right] \gamma(x) \\ &\quad + \int_{\bar{\mathcal{X}}} \nu(dx) \left[\phi(\nu - \delta_x) - \phi(\nu)\right] \left\{ \mu(x) + \alpha(x) \int_{\bar{\mathcal{X}}} \nu(dy) U(x, y) \right\}. \end{aligned} \tag{2.3}$$

The first term is linear (in ν) and describes the seed production and dispersal phenomenon. The second is nonlinear, and describes the death due to age or competition. This infinitesimal generator can be compared with formula (3) in Bolker-Pacala [2] p. 182.

2.2 Description in terms of Poisson measures

We will now give a path-wise description of the $\mathcal{M}$-valued stochastic process $(\nu_t)_{t\geq 0}$. To this end, we will use Poisson point measures. For the sake of simplicity, we assume that the spatial dependence of all the parameters is bounded in some sense.

Assumption (A): There exist some constants $\bar{\alpha}$, $\bar{\gamma}$ and $\bar{\mu}$ such that for all $x \in \bar{\mathcal{X}}$,

$$\alpha(x) \leq \bar{\alpha}, \quad \gamma(x) \leq \bar{\gamma}, \quad \mu(x) \leq \bar{\mu}. \tag{2.4}$$

There exist a constant $C > 0$ and a probability density $\tilde{D}$ on $\mathbb{R}^d$ such that for all $x \in \bar{\mathcal{X}}$,

$$D(x, dz) = D(x, z)dz \quad \text{with} \quad D(x, z) \leq C\tilde{D}(z). \tag{2.5}$$

The competition kernel U is bounded by some constant $\bar{U}$.

We also introduce the following notation.

Notation 2.2 *Let* $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. *Let* $H = (H^1, ..., H^k, ...) : \mathcal{M} \mapsto (\mathbb{R}^d)^{\mathbb{N}^*}$ *be defined by*

$$H\left(\sum_{i=1}^n \delta_{x_i}\right) = (x_{\sigma(1)}, ..., x_{\sigma(n)}, 0, ..., 0, ...), \tag{2.6}$$

where $x_{\sigma(1)} \preccurlyeq ... \preccurlyeq x_{\sigma(n)}$, *for some arbitrary order* $\preccurlyeq$ *on* $\mathbb{R}^d$ *(one may for example choose the lexicographic order).*

This function H will allow us to overcome the following (purely notational) problem: assume that a population of plants is described by a point measure $\nu \in \mathcal{M}$. Choosing a plant uniformly among all plants consists in choosing i uniformly in $\{1, ..., \langle \nu, 1\rangle\}$, and then in choosing the plant *number* i (from the arbitrary order point of view). The location of such a plant is thus $H^i(\nu)$.

Notation 2.3 *We consider the path space $\mathcal{T} \subset \mathbb{D}([0,\infty), M_F(\bar{\mathcal{X}}))$ defined by*

$$\mathcal{T} = \left\{ (\nu_t)_{t\geq 0} \;\Big/\; \begin{array}{l} \forall t \geq 0,\ \nu_t \in \mathcal{M},\ \text{and}\ \exists\, 0 = t_0 < t_1 < t_2 < ..., \\ \lim_n t_n = \infty \text{ and } \nu_t = \nu_{t_i}\ \forall t \in [t_i, t_{i+1}) \end{array} \right\}. \tag{2.7}$$

Note that for $(\nu_t)_{t\geq 0} \in \mathcal{T}$, and $t > 0$, we can define ν_{t-} in the following way: if $t \notin \cup_i\{t_i\}$, $\nu_{t-} = \nu_t$, while if $t = t_i$ for some $i \geq 1$, $\nu_{t-} = \nu_{t_{i-1}}$.

We now introduce the probabilistic objects we will need.

Definition 2.4 *Let $(\Omega, \mathcal{F}, P)$ be a (sufficiently large) probability space. On this space, we consider the following four independent random elements:*
(i) a $\mathcal{M}$-valued random variable ν_0 (the initial distribution),
(ii) a Poisson point measure $N(ds, di, dz, d\theta)$ on $[0,\infty) \times \mathbb{N}^* \times \mathbb{R}^d \times [0,1]$, with intensity measure $\bar{\gamma} ds \left(\sum_{k\geq 1} \delta_k(di)\right) \left(C\tilde{D}(z)dz\right) d\theta$ (the seed production Poisson measure),
(iii) a Poisson point measure $M(ds, di, d\theta)$ on $[0,\infty) \times \mathbb{N}^* \times [0,1]$, with intensity measure $\bar{\mu} ds \left(\sum_{k\geq 1} \delta_k(di)\right) d\theta$ (the "intrinsic" death Poisson measure),
(iv) a Poisson point measure $Q(ds, di, dj, d\theta, d\theta')$ on $[0,\infty) \times \mathbb{N}^* \times \mathbb{N}^* \times [0,1] \times [0,1]$, with intensity measure $\bar{U}\bar{\alpha} ds \left(\sum_{k\geq 1} \delta_k(di)\right) \left(\sum_{k\geq 1} \delta_k(dj)\right) d\theta d\theta'$ (the "competition" mortality Poisson measure).
We also consider the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by these processes.

We finally write the BPDL model in terms of these stochastic objects.

Definition 2.5 *Assume (A). A $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process $\nu = (\nu_t)_{t\geq 0}$ belonging a.s. to $\mathcal{T}$ will be called a BPDL process if a.s., for all $t \geq 0$,*

$$\begin{aligned} \nu_t \;=\;& \nu_0 + \int_0^t \int_{\mathbb{N}^*} \int_{\mathbb{R}^d} \int_0^1 \mathbf{1}_{\{i \leq \langle \nu_{s-}, 1\rangle\}} \delta_{(H^i(\nu_{s-})+z)} \mathbf{1}_{\left\{\theta \leq \frac{\gamma(H^i(\nu_{s-}))D(H^i(\nu_{s-}),z)}{\bar{\gamma} C \tilde{D}(z)}\right\}} N(ds, di, dz, d\theta) \\ & - \int_0^t \int_{\mathbb{N}^*} \int_0^1 \mathbf{1}_{\{i \leq \langle \nu_{s-}, 1\rangle\}} \delta_{H^i(\nu_{s-})} \mathbf{1}_{\left\{\theta \leq \frac{\mu(H^i(\nu_{s-}))}{\bar{\mu}}\right\}} M(ds, di, d\theta) \\ & - \int_0^t \int_{\mathbb{N}^*} \int_{\mathbb{N}^*} \int_0^1 \int_0^1 \mathbf{1}_{\{i \leq \langle \nu_{s-}, 1\rangle\}} \mathbf{1}_{\{j \leq \langle \nu_{s-}, 1\rangle\}} \delta_{H^i(\nu_{s-})} \mathbf{1}_{\left\{\theta' \leq \frac{U(H^i(\nu_{s-}), H^j(\nu_{s-}))}{\bar{U}}\right\}} \\ & \qquad \mathbf{1}_{\left\{\theta \leq \frac{\alpha(H^i(\nu_{s-}))}{\bar{\alpha}}\right\}} Q(ds, di, dj, d\theta, d\theta'). \end{aligned} \tag{2.8}$$

Although the formula looks complicated, the principle is very simple. The indicator functions involving θ and θ' are related to the *rates* and appear when the parameters depend on the space variable x. In the case where the rates are constant studied in [2], one may cancel all the integrals and indicator functions involving θ.
Let us now show that if ν solves of (2.8), then it follows the dynamics we are interested in.

Proposition 2.6 *Assume (A). Consider a solution $(\nu_t)_{t\geq 0}$ to equation (2.8). Then $(\nu_t)_{t\geq 0}$ is a Markov process. Its infinitesimal generator L is defined for all bounded and measurable maps $\phi : \mathcal{M} \mapsto \mathbb{R}$, all $\nu \in \mathcal{M}$, by (2.3). In particular, the law of $(\nu_t)_{t\geq 0}$ does not depend on the chosen order (see Notation 2.2).*

Proof The fact that $(\nu_t)_{t\geq 0}$ is a Markov process is classical. Let us now consider a function ϕ as in the statement. Recall that with our notation, $\nu_0 = \sum_{i=1}^{\langle \nu_0,1\rangle} \delta_{H^i(\nu_0)}$. Recall also that $L\phi(\nu_0) = \partial_t E\left[\phi(\nu_t)\right]_{t=0}$. A simple computation, using the fact that a.s., $\phi(\nu_t) = \phi(\nu_0) + \sum_{s\leq t}[\phi(\nu_{s-} + \{\nu_s - \nu_{s-}\}) - \phi(\nu_{s-})]$, shows that

$$\begin{aligned}
\phi(\nu_t) \quad = \quad & \phi(\nu_0) + \int_0^t \int_{\mathbb{N}^*} \int_{\mathbb{R}^d} \int_0^1 \left[\phi(\nu_{s-} + \delta_{(H^i(\nu_{s-})+z)}) - \phi(\nu_{s-})\right] \mathbf{1}_{\{i\leq \langle \nu_{s-},1\rangle\}} \\
& \qquad \mathbf{1}_{\left\{\theta \leq \frac{\gamma(H^i(\nu_{s-}))D(H^i(\nu_{s-}),z)}{\bar{\gamma} C \tilde{D}(z)}\right\}} N(ds,di,dz,d\theta) \\
& + \int_0^t \int_{\mathbb{N}^*} \int_0^1 \left[\phi(\nu_{s-} - \delta_{H^i(\nu_{s-})}) - \phi(\nu_{s-})\right] \mathbf{1}_{\{i\leq \langle \nu_{s-},1\rangle\}} \mathbf{1}_{\left\{\theta \leq \frac{\mu(H^i(\nu_{s-}))}{\bar{\mu}}\right\}} M(ds,di,d\theta) \\
& + \int_0^t \int_{\mathbb{N}^*} \int_{\mathbb{N}^*} \int_0^1 \int_0^1 \left[\phi(\nu_{s-} - \delta_{H^i(\nu_{s-})}) - \phi(\nu_{s-})\right] \mathbf{1}_{\{i\leq \langle \nu_{s-},1\rangle\}} \mathbf{1}_{\{j\leq \langle \nu_{s-},1\rangle\}} \\
& \qquad \mathbf{1}_{\left\{\theta' \leq \frac{U(H^i(\nu_{s-}),H^j(\nu_{s-}))}{\bar{U}}\right\}} \mathbf{1}_{\left\{\theta \leq \frac{\alpha(H^i(\nu_{s-}))}{\bar{\alpha}}\right\}} Q(ds,di,dj,d\theta,d\theta').
\end{aligned}$$

Taking expectations, we obtain

$$\begin{aligned}
& E[\phi(\nu_t)] = E[\phi(\nu_0)] \\
& + \int_0^t dsE\Big[\int_{\mathbb{R}^d} dz \bar{\gamma} C \tilde{D}(z) \sum_{i=1}^{\langle \nu_s,1\rangle} \frac{\gamma(H^i(\nu_{s-}))D(H^i(\nu_{s-}),z)}{\bar{\gamma} C \tilde{D}(z)} \left[\phi(\nu_{s-} + \delta_{(H^i(\nu_{s-})+z)}) - \phi(\nu_{s-})\right] \Big] \\
& + \int_0^t dsE\Big[\bar{\mu} \sum_{i=1}^{\langle \nu_s,1\rangle} \frac{\mu(H^i(\nu_{s-}))}{\bar{\mu}} \left[\phi(\nu_{s-} - \delta_{H^i(\nu_{s-})}) - \phi(\nu_{s-})\right] \Big] \\
& + \int_0^t dsE\Big[\bar{U}\bar{\alpha} \sum_{i=1}^{\langle \nu_s,1\rangle} \sum_{j=1}^{\langle \nu_s,1\rangle} \frac{U(H^i(\nu_{s-}),H^j(\nu_{s-}))}{\bar{U}} \frac{\alpha(H^i(\nu_{s-}))}{\bar{\alpha}} \left[\phi(\nu_{s-} - \delta_{H^i(\nu_{s-})}) - \phi(\nu_{s-})\right] \Big] \\
& = E[\phi(\nu_0)] + \int_0^t dsE\Big[\int_{\bar{\mathcal{X}}} \nu_s(dx) \int_{\mathbb{R}^d} dz \gamma(x) D(x,z) \left[\phi(\nu_s + \delta_{(x+z)}) - \phi(\nu_s)\right] \Big] \\
& + \int_0^t dsE\Big[\int_{\bar{\mathcal{X}}} \nu_s(dx) \left[\phi(\nu_s - \delta_x) - \phi(\nu_s)\right] \left\{\mu(x) + \alpha(x) \int_{\bar{\mathcal{X}}} \nu_s(dy) U(x,y)\right\} \Big].
\end{aligned}$$

Differentiating this expression at $t = 0$ leads to (2.3). □

2.3 About simulation

This path-wise definition of the BPDL process leads to the following simulation algorithm:
Step 0: Simulate the initial state ν_0, and set $T_0 = 0$.
Step 1: Compute the total *event* rate, given by $m(0) = m_1(0) + m_2(0) + m_3(0)$, with

$$m_1(0) = C\bar{\gamma} \langle \nu_0, 1\rangle \ , \ m_2(0) = \bar{\mu} \langle \nu_0, 1\rangle \ , \ m_3(0) = \bar{\alpha}\bar{U} \langle \nu_0, 1\rangle^2 . \tag{2.9}$$

Simulate S_1 exponentially distributed, with parameter $m(0)$, and set $T_1 = T_0 + S_1$. Set $\nu_t = \nu_0$ for all $t < T_1$. Choose whether to go to Step 1.1, 1.2, or 1.3 with probability $m_1(0)/m(0)$, $m_2(0)/m(0)$ and $m_3(0)/m(0)$.
Step 1.1: choose i uniformly in $\{1, ..., \langle \nu_0, 1\rangle\}$. Choose $z \in \mathbb{R}^d$ according to the law $\tilde{D}(z)dz$. With probability $1 - \frac{\gamma(H^i(\nu_0))D(H^i(\nu_0),z)}{\bar{\gamma} C \tilde{D}(z)}$, do nothing (i.e. set $\nu_{T_1} = \nu_0$). Else, add a new plant at the location $H^i(\nu_0) + z$ (i.e. set $\nu_{T_1} = \nu_0 + \delta_{(H^i(\nu_0)+z)}$).

Step 1.2: choose i uniformly in $\{1, ..., \langle \nu_0, 1\rangle\}$. With probability $1 - \frac{\mu(H^i(\nu_0))}{\bar{\mu}}$, do nothing (i.e. set $\nu_{T_1} = \nu_0$). Else, remove the i-th plant (i.e. set $\nu_{T_1} = \nu_0 - \delta_{H^i(\nu_0)}$).
Step 1.3: choose i and j uniformly in $\{1, ..., \langle \nu_0, 1\rangle\}^2$. With probability $1 - \frac{U(H^i(\nu_0)), H^j(\nu_0)))}{\bar{U}} \frac{\alpha(H^i(\nu_0))}{\bar{\alpha}}$, do nothing (i.e. set $\nu_{T_1} = \nu_0$). Else, remove the i-th plant (i.e. set $\nu_{T_1} = \nu_0 - \delta_{H^i(\nu_0)}$).
Step 2: Compute the total *event* rate, given by $m(T_1) = m_1(T_1) + m_2(T_1) + m_3(T_1)$, with

$$m_1(T_1) = C\bar{\gamma} \langle \nu_{T_1}, 1\rangle \ , \ m_2(T_1) = \bar{\mu} \langle \nu_{T_1}, 1\rangle \ , \ m_3(T_1) = \bar{\alpha}\bar{U} \langle \nu_{T_1}, 1\rangle^2 . \tag{2.10}$$

Simulate S_2 exponentially distributed, with parameter $m(T_1)$, and set $T_2 = T_1 + S_2$. Set $\nu_t = \nu_{T_1}$ for all $t \in [T_1, T_2[$, etc.

3 Existence and first properties

We now show existence, uniqueness, and some moment estimates for the BPDL process.

Theorem 3.1 *(i) Assume (A) and that $E(\langle \nu_0, 1\rangle) < \infty$. Then there exists a unique BPDL process $(\nu_t)_{t\geq 0}$ in the sense of Definition 2.5.*
(ii) If furthermore for some $p \geq 1$, $E(\langle \nu_0, 1\rangle^p) < \infty$, then for any $T < \infty$,

$$E\left(\sup_{t\in[0,T]} \langle \nu_t, 1\rangle^p\right) < \infty. \tag{3.1}$$

Proof We first prove (ii). Consider thus a BPDL process $(\nu_t)_{t\geq 0}$. We introduce for each n the stopping time $\tau_n = \inf\{t \geq 0,\ \langle \nu_t, 1\rangle \geq n\}$. Then a simple computation using (A) shows that, neglecting the non-positive death terms,

$$\begin{aligned}\sup_{s\in[0,t\wedge\tau_n]} \langle \nu_s, 1\rangle^p \leq \langle \nu_0, 1\rangle^p + \int_0^{t\wedge\tau_n} \int_{\mathbb{N}^*} \int_{\mathbb{R}^d} \int_0^1 [(\langle \nu_{s-}, 1\rangle + 1)^p - \langle \nu_{s-}, 1\rangle^p] \\ \mathbf{1}_{\{i\leq\langle \nu_{s-},1\rangle\}} \mathbf{1}_{\left\{\theta\leq \frac{\gamma(H^i(\nu_{s-}))D(H^i(\nu_{s-}),z)}{\bar{\gamma}C\tilde{D}(z)}\right\}} N(ds, di, dz, d\theta) \\ \leq \langle \nu_0, 1\rangle^p + C_p \int_0^{t\wedge\tau_n} \int_{\mathbb{N}^*} \int_{\mathbb{R}^d} \int_0^1 \left[1 + \langle \nu_{s-}, 1\rangle^{p-1}\right] \mathbf{1}_{\{i\leq\langle \nu_{s-},1\rangle\}} N(ds, di, dz, d\theta),\end{aligned} \tag{3.2}$$

for some constant C_p. Taking expectations, we thus obtain, the value of C_p changing from line to line,

$$\begin{aligned}E\left(\sup_{s\in[0,t\wedge\tau_n]} \langle \nu_s, 1\rangle^p\right) &\leq C_p + C_p E\left(\int_0^{t\wedge\tau_n} ds\bar{\gamma}C \int_{\mathbb{R}^d} dz\tilde{D}(z) [\langle \nu_{s-}, 1\rangle + \langle \nu_{s-}, 1\rangle^p]\right) \\ &\leq C_p + C_p E\left(\int_0^t ds [1 + \langle \nu_{s\wedge\tau_n}, 1\rangle^p]\right).\end{aligned} \tag{3.3}$$

The Gronwall Lemma allows us to conclude that for any $T < \infty$, there exists a constant $C_{p,T}$, not depending on n, such that

$$E\left(\sup_{t\in[0,T\wedge\tau_n]} \langle \nu_t, 1\rangle^p\right) \leq C_{p,T}. \tag{3.4}$$

First, we deduce that τ_n tends a.s. to infinity. Indeed, if not, one may find a $T < \infty$ such that $\epsilon_T = P(\sup_n \tau_n < T) > 0$. This would imply that for all n, $E\left(\sup_{t\in[0,T\wedge\tau_n]} \langle \nu_t, 1\rangle^p\right) \geq \epsilon_T n^p$, which contradicts (3.4).
We may let n go to infinity in (3.4) thanks to the Fatou Lemma. This leads to (3.1).

Point (i) is a consequence of point (ii). Indeed, one can for example build the solution $(\nu_t)_{t\geq 0}$ using the simulation algorithm previously described, choosing the rates and acceptance-rejection according to the Poisson measures N, M, and Q. One only has to check that the sequence of (effective or fictitious) jump instants T_n goes a.s. to infinity as n tends to infinity. But this follows from (3.1) with $p = 1$. Uniqueness also holds, since one has no choice in the construction. $\square$

We now prove that if there is at most one plant at each location at time $t = 0$, then this also holds for all $t \geq 0$.

Proposition 3.2 *Assume (A) and that $E(\langle \nu_0, 1\rangle) < \infty$. Assume also that a.s., $\sup_{x\in\bar{\mathcal{X}}} \nu_0(\{x\}) \leq 1$. Consider the BPDL process $(\nu_t)_{t\geq 0}$. Then for all $t \geq 0$, a.s.,*

$$\int_{\bar{\mathcal{X}}} \nu_t(dx)\nu_t(\{x\}) = \langle \nu_t, 1\rangle, \quad i.e. \quad \sup_{x\in\bar{\mathcal{X}}} \nu_t(\{x\}) \leq 1. \tag{3.5}$$

Proof Consider the non-negative function ϕ defined on $\mathcal{M}$ by $\phi(\nu) = \int_{\bar{\mathcal{X}}} \nu(dx)\nu(\{x\}) - \langle \nu, 1\rangle$. Then note that a.s., $\phi(\nu_0) = 0$, and that for any $\nu \in \mathcal{M}$, any $x \in \text{supp }\nu$, $\phi(\nu - \delta_x) - \phi(\nu) \leq 0$. Consider, for each $n \geq 1$, the stopping time $\tau_n = \inf\{t \geq 0,\ \langle \nu_t, 1\rangle \geq n\}$. A simple computation allows us to obtain, for all $t \geq 0$, all $n \geq 1$,

$$E\left[\phi(\nu_{t\wedge\tau_n})\right] \leq 0 + E\left[\int_0^{t\wedge\tau_n} ds \int_{\bar{\mathcal{X}}} \nu_s(dx) \int_{\mathbb{R}^d} D(x,dz)\gamma(x)\left\{\phi(\nu_s + \delta_{(x+z)}) - \phi(\nu_s)\right\}\right]. \tag{3.6}$$

One easily checks, using that ν is atomic, that the RHS term identically vanishes, since $D(x, dz)$ has a density. Hence, a.s., $\phi(\nu_{t\wedge\tau_n}) = 0$. Thanks to (3.1) with $p = 1$, τ_n a.s. grows to infinity with n, which concludes the proof. $\square$

We carry on with a property concerning the absolute continuity of the expectation of ν_t. For ν a random measure, we define the deterministic measure $E(\nu)$ by $\langle E(\nu), f\rangle = E(\langle \nu, f\rangle)$.

Proposition 3.3 *Assume (A), that $E[\langle \nu_0, 1\rangle] < \infty$ and that $E(\nu_0)$ admits a density $\tilde{n}_0$ with respect to the Lebesgue measure. Consider the Bolker-Pacala process $(\nu_t)_{t\geq 0}$. Then for all $t \geq 0$, $E(\nu_t)$ has a density $\tilde{n}_t$: for all measurable non-negative functions f on $\bar{\mathcal{X}}$, $E[\langle \nu_t, f\rangle] = \int_{\bar{\mathcal{X}}} dx f(x)\tilde{n}_t(x)$.*

Proof Consider a Borel set A of $\mathbb{R}^d$ with Lebesgue measure zero. Consider also, for each $n \geq 1$, the stopping time $\tau_n = \inf\{t \geq 0,\ \langle \nu_t, 1\rangle \geq n\}$. A simple computation allows us to obtain, for all $t \geq 0$, all $n \geq 1$,

$$\begin{aligned} E\left[\langle \nu_{t\wedge\tau_n}, \mathbf{1}_A\rangle\right] = E(\langle \nu_0, \mathbf{1}_A\rangle) + E\bigg(\int_0^{t\wedge\tau_n} ds \int_{\bar{\mathcal{X}}} \nu_s(dx)\gamma(x) \int_{\mathbb{R}^d} dz D(x,z)\mathbf{1}_A(x+z)\bigg) \\ -E\bigg(\int_0^{t\wedge\tau_n} ds \int_{\bar{\mathcal{X}}} \nu_s(dx)\mathbf{1}_A(x)\left(\mu(x) + \alpha(x)\int_{\bar{\mathcal{X}}} \nu_s(dy)U(x,y)\right)\bigg). \end{aligned} \tag{3.7}$$

By assumption, the first term on the RHS is zero. The second term is also zero, since for any $x \in \bar{\mathcal{X}}$, $\int_{\mathbb{R}^d} dz \mathbf{1}_A(x+z)D(x,z) = 0$. The third term is of course non-positive. Hence for each n, $E(\langle \nu_{t\wedge\tau_n}, \mathbf{1}_A\rangle)$ is non-positive and thus zero. Thanks to (3.1) with $p = 1$, τ_n a.s. grows to infinity with n, which concludes the proof. $\square$

We finally give some martingale properties of the process $(\nu_t)_{t\geq 0}$.

Proposition 3.4 *Assume (A), and that for some $p \geq 2$, $E\left[\langle \nu_0, 1\rangle^p\right] < \infty$. Consider the Bolker-Pacala process $(\nu_t)_{t\geq 0}$, and recall that L is defined by (2.3).*

(i) For all measurable functions ϕ from $\mathcal{M}$ into $\mathbb{R}$ such that for some constant C, for all $\nu \in \mathcal{M}$, $|\phi(\nu)| + |L\phi(\nu)| \leq C(1 + \langle \nu, 1 \rangle^p)$, the process

$$\phi(\nu_t) - \phi(\nu_0) - \int_0^t ds L\phi(\nu_s) \tag{3.8}$$

is a càdlàg L^1-$(\mathcal{F}_t)_{t\geq 0}$-martingale starting from 0.
(ii) Point (i) applies to any measurable ϕ satisfying $|\phi(\nu)| \leq C(1 + \langle \nu, 1 \rangle^{p-2})$.
(iii) Point (i) applies to any function $\phi(\nu) = \langle \nu, f \rangle^q$, with $0 \leq q \leq p-1$ and with f bounded and measurable on $\bar{\mathcal{X}}$.
(iv) For any f bounded and measurable on $\bar{\mathcal{X}}$, the process

$$\begin{aligned} M_t^f &= \langle \nu_t, f \rangle - \langle \nu_0, f \rangle - \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx)\gamma(x) \int_{\mathbb{R}^d} dz D(x,z) f(x+z) \\ &\quad + \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx) f(x) \Big[\mu(x) + \alpha(x) \int_{\bar{\mathcal{X}}} \nu_s(dy) U(x,y)\Big] \end{aligned} \tag{3.9}$$

is a càdlàg L^2-martingale starting from 0 with (predictable) quadratic variation

$$\langle M^f \rangle_t = \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx) \Big\{ \gamma(x) \int_{\mathbb{R}^d} dz f^2(x+z) D(x,z) + f^2(x) \Big[\mu(x) + \alpha(x) \int_{\bar{\mathcal{X}}} \nu_s(dy) U(x,y)\Big]\Big\}. \tag{3.10}$$

Proof First of all note that point (i) is immediate thanks to Proposition 2.6 and (3.1). Points (ii) and (iii) follow from straightforward computations using (2.3). To prove (iv), we first assume that $E\left[\langle \nu_0, 1 \rangle^3\right] < \infty$. We apply (i) with $\phi(\nu) = \langle \nu, f \rangle$. This yields that M^f is a martingale. To compute its bracket, we first apply (i) with $\phi(\nu) = \langle \nu, f \rangle^2$ and obtain that

$$\begin{aligned} &\langle \nu_t, f \rangle^2 - \langle \nu_0, f \rangle^2 - \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx)\gamma(x) \int_{\mathbb{R}^d} dz D(x,z) \Big[f^2(x+z) + 2f(x+z) \langle \nu_s, f \rangle \Big] \\ &\qquad - \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx) \Big\{ f^2(x) - 2f(x) \langle \nu_s, f \rangle \Big\} \Big[\mu(x) + \alpha(x) \int_{\bar{\mathcal{X}}} \nu_s(dy) U(x,y)\Big] \end{aligned} \tag{3.11}$$

is a martingale. Then we apply the Itô formula to compute $\langle \nu_t, f \rangle^2$ from (3.9). We deduce that

$$\begin{aligned} &\langle \nu_t, f \rangle^2 - \langle \nu_0, f \rangle^2 - \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx)\gamma(x) \int_{\mathbb{R}^d} dz D(x,z) 2f(x+z) \langle \nu_s, f \rangle \\ &+ \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx) 2f(x) \langle \nu_s, f \rangle \Big[\mu(x) + \alpha(x) \int_{\bar{\mathcal{X}}} \nu_s(dy) U(x,y)\Big] - \langle M^f \rangle_t \end{aligned} \tag{3.12}$$

is a martingale. Comparing (3.11) and (3.12) leads to (3.10). The extension to the case where only $E\left[\langle \nu_0, 1 \rangle^2\right] < \infty$ is is straightforward, since even in this case, $E[\langle M^f \rangle_t] < \infty$ thanks to (3.1) with $p = 2$. □

4 On the the BPDL Moment Equations

We now aim to give a sense to the mean moment equation given in [2] formula (6). Note that in the biology literature, one may be confused by the notation between the discrete measure ν_t, its expectation $E(\nu_t)$ (defined by $\langle E(\nu_t), f \rangle = E(\langle \nu_t, f \rangle))$, and a measure with density $n_t(x)$ of which the definition is not clear. Following [2], we assume in this section that

Assumption (B): The spatial domain is $\bar{\mathcal{X}} = \mathbb{R}^d$. All the parameters α, γ, μ, and D of the model are independent of x. Moreover the (bounded) competition kernel $U(x, y)$ has the form $U(x - y)$, and both dispersal and competition kernels are symmetric probability distribution functions, i.e. $D(z) = D(-z)$, $U(x - y) = U(y - x)$, and $\int_{\mathbb{R}^d} dz D(z) = \int_{\mathbb{R}^d} dz U(z) = 1$.

We moreover assume that $E(\langle \nu_0, 1\rangle^2) < \infty$ and that there is at most one plant at each location at time $t = 0$. So (3.1) with $p = 1$ holds and we can define for each time $t \in [0, T]$

$$n(t) = E(\langle \nu_t, 1\rangle). \tag{4.1}$$

Using Proposition 3.4-(iv) with $f = 1$ and taking expectations in (3.9), we obtain

$$E(\langle \nu_t, 1\rangle) = E(\langle \nu_0, 1\rangle) + \int_0^t ds(\gamma - \mu)E(\langle \nu_s, 1\rangle) - \alpha \int_0^t dsE\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_s(dx)\nu_s(dy)U(x-y)\Big). \tag{4.2}$$

Hence,

$$\begin{aligned} n(t) \quad &= \quad n(0) + (\gamma - \mu)\int_0^t dsn(s) - \alpha \int_0^t dsE\Big(\int_{\mathbb{R}^d} \nu_s(dx)U(0)\nu_s(\{x\})\Big) \\ &\quad -\alpha \int_0^t dsE\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_s(dx)\nu_s(dy)\mathbf{1}_{\{x\neq y\}}U(x-y)\Big). \end{aligned} \tag{4.3}$$

But thanks to Proposition 3.2, we know that for all $s \geq 0$, $\int_{\mathbb{R}^d} \nu_s(dx)U(0)\nu_s(\{x\}) = U(0)\langle \nu_s, 1\rangle$. We thus obtain

$$n(t) = n(0) + (\gamma - \mu - \alpha U(0))\int_0^t dsn(s) - \alpha \int_0^t dsE\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_s(dx)\nu_s(dy)\mathbf{1}_{\{x\neq y\}}U(x-y)\Big). \tag{4.4}$$

Let us now explain the *covariance term* of Bolker-Pacala. Writing

$$\begin{aligned} &\alpha E\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_s(dx)\nu_s(dy)\mathbf{1}_{\{x\neq y\}}U(x-y)\Big) \\ &= \alpha E\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_s(dx)\Big(\nu_s(dy) - n(s)dy\Big)\mathbf{1}_{\{x\neq y\}}U(x-y)\Big) + \alpha n^2(s), \end{aligned} \tag{4.5}$$

we obtain from (4.4)

$$\begin{aligned} n(t) \quad &= \quad n(0) + (\gamma - \mu - \alpha U(0))\int_0^t dsn(s) - \alpha \int_0^t dsn^2(s) \\ &\quad -\alpha \int_0^t dsE\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_s(dx)\Big(\nu_s(dy) - n(s)dy\Big)\mathbf{1}_{\{x\neq y\}}U(x-y)\Big). \end{aligned} \tag{4.6}$$

That allows us to define, following the terminology of Bolker-Pacala, a covariance measure C_t on $\mathbb{R}^d$ for each time t. Let τ_{-y} denote the translation by the vector $-y$. We set

$$C_t(dr) = E\Big(\int_{y\in\mathbb{R}^d} \mathbf{1}_{\{r\neq 0\}}\nu_t \circ \tau_{-y}^{-1}(dr) \otimes \nu_t(dy)\Big) - n^2(t)dr, \tag{4.7}$$

In other words, for each measurable bounded function ϕ with compact support in $\mathbb{R}^d$ by

$$\begin{aligned} \int_{\mathbb{R}^d} C_t(dr)\phi(r) \quad &= \quad E\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_t(dx)\nu_t(dy)\mathbf{1}_{\{x\neq y\}}\phi(x-y)\Big) - n^2(t)\int_{\mathbb{R}^d} dr\phi(r) \\ &= \quad E\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_t(dx)\Big(\nu_t(dy) - n(t)dy\Big)\mathbf{1}_{\{x\neq y\}}\phi(x-y)\Big). \end{aligned} \tag{4.8}$$

By using these notations, we obtain the mean equation obtained by Bolker-Pacala [2] (formula (6) p 183), with a rigorous sense for the quadratic term:

$$\frac{dn(t)}{dt} = n(t)(\gamma - \mu - \alpha n(t)) - \alpha U(0)n(t) - \alpha \int_{\mathbb{R}^d} C_t(dr)U(r). \tag{4.9}$$

Let us finally remark that we are also able to derive an evolution equation for the covariance measure. In other words, one may write differential equations solved by $\int_{\mathbb{R}^d} C_t(dr)\phi(r)$ for all measurable bounded functions ϕ on $\mathbb{R}^d$ (we however do not obtain the same equation as in [2]). Of course moments of higher order are involved in such equations. So an issue is to find reasonable *moment closures*, as developed in [4]. These closures are at the moment not rigorously justified.

5 Infinite particle approximations

Our aim in this section is to describe the effect of two different re-normalizations on the BPDL process. In both cases, we will make the initial number of plants grow to infinity. We will first consider the case where the birth and death rates are unchanged. Will show that the random measure $(\nu_t)_{t\geq 0}$ tends to a deterministic measure $(\xi_t)_{t\geq 0}$. Then $(\xi_t)_{t\geq 0}$ solves a nonlinear integro-differential equation. We propose this limiting object as a rigorous interpretation of the *number density.*

The second re-normalization consists in addition in accelerating the rates in a convenient way. Then $(\nu_t)_{t\geq 0}$ converges to a superprocess $(X_t)_{t\geq 0}$. This measure-valued process has been introduced by Etheridge in [6]. She called it the *superprocess version of the Bolker-Pacala model.*

Let us first consider the most general situation.

Notation 5.1 *For each $n \in \mathbb{N}^*$, we consider a set of parameters $(\mu_n, \gamma_n, \alpha_n, U_n, D_n)$ as in Notation 2.1, satisfying for each n Assumption (A), and an initial condition $\nu_0^n \in \mathcal{M}$. Then, we denote by $(\nu_t^n)_{t\geq 0}$ the Bolker-Pacala process (see Definition 2.5) with the corresponding coefficients. We consider the subset $\mathcal{M}^n$ of $M_F(\bar{\mathcal{X}})$ defined by*

$$\mathcal{M}^n = \left\{ \frac{1}{n}\nu,\ \nu \in \mathcal{M} \right\}. \tag{5.10}$$

We finally consider the càdlàg $\mathcal{M}^n$-valued Markov process $(X_t^n)_{t\geq 0}$ defined by $X_t^n = \frac{1}{n}\nu_t^n$.

The generator of $(X_t^n)_{t\geq 0}$ is then given, for any measurable map ϕ from $\mathcal{M}^n$ into $\mathbb{R}$ by

$$\begin{aligned} L^n\phi(\nu) \quad &= \quad n\int_{\bar{\mathcal{X}}} \nu(dx) \int_{\mathbb{R}^d} dz \gamma_n(x) D_n(x,z) \left[\phi(\nu + \frac{1}{n}\delta_{x+z}) - \phi(\nu) \right] \\ &\quad + n\int_{\bar{\mathcal{X}}} \nu(dx) \left\{ \mu_n(x) + n\alpha_n(x) \int_{\bar{\mathcal{X}}} \nu(dy) U_n(x,y) \right\} \left[\phi(\nu - \frac{1}{n}\delta_x) - \phi(\nu) \right]. \end{aligned} \tag{5.11}$$

Indeed, the generator $\tilde{L}^n$ of $(\nu_t^n)_{t\geq 0}$ is given by (2.3), replacing $(\mu, \gamma, \alpha, U, D)$ by $(\mu_n, \gamma_n, \alpha_n, U_n, D_n)$. Hence,

$$L^n\phi(\nu) = \partial_t E_\nu \left[\phi(X_t^n) \right]_{t=0} = \partial_t E_{n\nu} \left[\phi(\nu_t^n / n) \right]_{t=0} = \tilde{L}^n \phi^n(n\nu), \tag{5.12}$$

where $\phi^n(\mu) = \phi(\mu/n)$. The conclusion follows from a straightforward computation. We now restate Lemma 3.4 for the renormalized model.

Lemma 5.2 *Let $n \geq 1$ be fixed, consider the process $(X_t^n)_{t\geq 0}$ defined in Notation 5.1. Assume that for some $p \geq 2$, $E\left[\langle X_0^n, 1 \rangle^p \right] < \infty$.*

(i) For all measurable functions ϕ from $\mathcal{M}^n$ into $\mathbb{R}$ such that for some constant C, for all $\nu \in \mathcal{M}^n$, $|\phi(\nu)| + |L^n\phi(\nu)| \leq C(1 + \langle \nu, 1\rangle^p)$, the process

$$\phi(X_t^n) - \phi(X_0^n) - \int_0^t ds L^n \phi(X_s^n) \tag{5.13}$$

is a càdlàg L^1-martingale starting from 0.
(ii) Point (i) applies to any measurable ϕ satisfying $|\phi(\nu)| \leq C(1 + \langle \nu, 1\rangle^{p-2})$.
(iii) Point (i) applies to any function $\phi(\nu) = \langle \nu, f\rangle^q$, with $0 \leq q \leq p-1$ and with f bounded and measurable on $\mathcal{M}$.
(iv) For any f bounded and measurable on $\bar{\mathcal{X}}$, the process

$$\begin{aligned} M_t^{n,f} &= \langle X_t^n, f\rangle - \langle X_0^n, f\rangle - \int_0^t ds \int_{\bar{\mathcal{X}}} X_s^n(dx) \int_{\mathbb{R}^d} dz \gamma_n(x) D_n(x,z) f(x+z) \\ &\quad + \int_0^t ds \int_{\bar{\mathcal{X}}} X_s^n(dx) \left\{ \mu_n(x) + n\alpha_n(x) \int_{\bar{\mathcal{X}}} X_s^n(dy) U_n(x,y) \right\} f(x) \end{aligned} \tag{5.14}$$

is a càdlàg L^2-martingale with (predictable) quadratic variation

$$\begin{aligned} \langle M^{n,f} \rangle_t &= \frac{1}{n} \int_0^t ds \int_{\bar{\mathcal{X}}} X_s^n(dx) \int_{\mathbb{R}^d} dz \gamma_n(x) D_n(x,z) f^2(x+z) \\ &\quad + \frac{1}{n} \int_0^t ds \int_{\bar{\mathcal{X}}} X_s^n(dx) \left\{ \mu_n(x) + n\alpha_n(x) \int_{\bar{\mathcal{X}}} X_s^n(dy) U_n(x,y) \right\} f^2(x). \end{aligned} \tag{5.15}$$

We endow $M_F(\bar{\mathcal{X}})$ with the weak topology.

5.1 Convergence to a nonlinear integro-differential equation

Let us now consider the mean-field approximating case in which the initial number of particles n tends to infinity, the parameters of seed production and intrinsic death stay unchanged, whereas the mortality competition parameter tends to zero as $\frac{1}{n}$. We will show that the BPDL process can be approximated by a deterministic nonlinear integro-differential equation. This might be a better deterministic way than the moment equations of [2] to describe the model. In particular, it allows us to deal with space-dependent parameters.

> <u>Assumption $(C1)$:</u>
> 1) The initial conditions X_0^n converge in law and for the weak topology on $M_F(\bar{\mathcal{X}})$ to some deterministic finite measure $\xi_0 \in M_F(\bar{\mathcal{X}})$, and $\sup_n E(\langle X_0^n, 1\rangle^3) < +\infty$.
> 2) There exist some continuous non-negative functions α, γ, μ on $\bar{\mathcal{X}}$, bounded by $\bar{\alpha}, \bar{\gamma}, \bar{\mu}$, such that $\gamma_n(x) = \gamma(x)$, $\mu_n(x) = \mu(x)$, $\alpha_n(x) = \alpha(x)/n$.
> 3) There exists a bounded non-negative symmetric continuous function U on $\bar{\mathcal{X}} \times \bar{\mathcal{X}}$ bounded by $\bar{U}$ such that $U_n(x,y) = U(x,y)$.
> 4) There exists a continuous non-negative function D on $\bar{\mathcal{X}} \times \mathbb{R}^d$, satisfying for each $x \in \bar{\mathcal{X}}$, $\int_{z \in \mathbb{R}^d,\, x+z \in \bar{\mathcal{X}}} dz D(x,z) = 1$, $D(x,z) = 0$ as soon as $x + z \notin \bar{\mathcal{X}}$, and such that $D(x,z) \leq C\tilde{D}(z)$ for a constant $C > 0$ and a probability density $\tilde{D}$ on $\mathbb{R}^d$. We set $D_n(x,z) = D(x,z)$.

The first assertion of Assumption (C1) is satisfied for example if $X_0^n = \frac{1}{n}\sum_{i=1}^n \delta_{Z^i}$ where the random variables Z^i are independent, with law ξ_0. In this case, the number n can be seen as the *volume* of particles at initial time, and the limit of $X_t^n = \frac{1}{n}\nu_t^n$ may give a rigorous sense to the *number density*.

Theorem 5.3 *Assume (C1), and consider the sequence of processes X^n defined in Notation 5.1. Then for all $T>0$, the sequence (X^n) converges in law, in $\mathbb{D}([0,T],M_F(\bar{\mathcal{X}}))$, to a deterministic continuous function $(\xi_t)_{t\geq 0}\in C([0,T],M_F(\bar{\mathcal{X}}))$. This measure-valued function ξ is the unique solution, satisfying $\sup_{t\in[0,T]}\langle \xi_t,1\rangle<\infty$, of the integro-differential equation written in its weak form: for all bounded and measurable functions f from $\bar{\mathcal{X}}$ into $\mathbb{R}$,*

$$\begin{aligned}\langle \xi_t,f\rangle &= \langle \xi_0,f\rangle+\int_0^t ds\int_{\bar{\mathcal{X}}}\xi_s(dx)\gamma(x)\int_{\mathbb{R}^d}dzD(x,z)f(x+z)\\ &\quad -\int_0^t ds\int_{\bar{\mathcal{X}}}\xi_s(dx)f(x)\left\{\mu(x)+\alpha(x)\int_{\bar{\mathcal{X}}}\xi_s(dy)U(x,y)\right\}. \end{aligned}\tag{5.16}$$

Note that the link between (2.8) and (5.16) is the same as the one between the continuous-time binary Galton-Watson process with birth rate γ and death rate μ and the deterministic differential equation $f'(t)=(\gamma-\mu)f(t)$.

Proof We divide the proof in several steps. Let us fix $T>0$.

Step 1 Let us first show the uniqueness for the equation (5.16). We consider two solutions $(\xi_t)_{t\geq 0}$ and $(\bar{\xi}_t)_{t\geq 0}$ of (5.16) satisfying $\sup_{t\in[0,T]}\langle \xi_t+\bar{\xi}_t,1\rangle=A_T<+\infty$. We consider the variation norm defined for μ_1 and μ_2 in $M_F(\bar{\mathcal{X}})$ by

$$||\mu_1-\mu_2||=\sup_{f\in L^\infty(\bar{\mathcal{X}}),\,||f||_\infty\leq 1}|\langle \mu_1-\mu_2,f\rangle|. \tag{5.17}$$

Then, we consider some $f\in L^\infty(\bar{\mathcal{X}})$ such that $||f||_\infty\leq 1$ and obtain

$$\begin{aligned}|\langle \xi_t-\bar{\xi}_t,f\rangle| &\leq \int_0^t ds\left|\int_{\bar{\mathcal{X}}}[\xi_s(dx)-\bar{\xi}_s(dx)]\left(\gamma(x)\int_{\mathbb{R}^d}dzD(x,z)f(x+z)-\mu(x)f(x)\right)\right|\\ &\quad +\int_0^t ds\left|\int_{\bar{\mathcal{X}}}[\xi_s(dx)-\bar{\xi}_s(dx)]\alpha(x)f(x)\int_{\bar{\mathcal{X}}}\xi_s(dy)U(x,y)\right|\\ &\quad +\int_0^t ds\left|\int_{\bar{\mathcal{X}}}[\xi_s(dy)-\bar{\xi}_s(dy)]\int_{\bar{\mathcal{X}}}\bar{\xi}_s(dx)\alpha(x)f(x)U(x,y)\right|. \end{aligned}\tag{5.18}$$

But since $||f||_\infty\leq 1$, for all $x\in\bar{\mathcal{X}}$, $\left|\gamma(x)\int_{\mathbb{R}^d}dzD(x,z)f(x+z)-\mu(x)f(x)\right|\leq\bar{\gamma}+\bar{\mu}$ while $\left|\alpha(x)f(x)\int_{\bar{\mathcal{X}}}\xi_s(dy)U(x,y)\right|\leq\bar{\alpha}\bar{U}A_T$, and $\left|\int_{\bar{\mathcal{X}}}\bar{\xi}_s(dx)\alpha(x)f(x)U(x,y)\right|\leq\bar{\alpha}\bar{U}A_T$ almost everywhere. We deduce that

$$|\langle \xi_t-\bar{\xi}_t,f\rangle|\leq\left[\bar{\gamma}+\bar{\mu}+2\bar{\alpha}\bar{U}A_T\right]\int_0^t ds||\xi_s-\bar{\xi}_s||. \tag{5.19}$$

Taking the supremum over all functions f such that $||f||_\infty\leq 1$, and using then the Gronwall Lemma, we finally deduce that for all $t\leq T$, $||\xi_t-\bar{\xi}_t||=0$. Uniqueness holds.

Step 2 Let us prove some moment estimates. By $(C1)$, it is easy to check that for all $T>0$,

$$\sup_n E\left(\sup_{t\in[0,T]}\langle X_t^n,1\rangle^3\right)<+\infty. \tag{5.20}$$

Indeed, recalling that $X_t^n=\frac{1}{n}\nu_t^n$, one can prove, following line by line the proof of Theorem 3.1 (ii) with $p=3$, that $E[\sup_{t\in[0,T]}\langle\nu_t^n,1\rangle^3]\leq C_TE[\langle\nu_0^n,1\rangle^3]$, noting that the constant C_T does not depend on n. One easily concludes, using assumption $(C1)$-1.

Step 3 We first endow $M_F(\bar{\mathcal{X}})$ with the vague topology, the extension to the weak topology being handled in Step 6 below. To show the tightness of the sequence of the laws $Q^n=\mathcal{L}(X^n)$ in

$\mathcal{P}(\mathbb{D}([0,T],M_F(\bar{\mathcal{X}})))$, it suffices, following Roelly [15], to show that for any continuous bounded function f on $\bar{\mathcal{X}}$, the sequence of laws of the processes $\langle X^n, f\rangle$ is tight in $\mathbb{D}([0,T],\mathbb{R})$. To this end, we use the Aldous criterion [1] and the Rebolledo criterion (see [7]). We have to show that

$$\sup_n E\left(\sup_{t\in[0,T]} |\langle X^n_s, f\rangle|\right) < \infty, \tag{5.21}$$

and the tightness respectively of the laws of the martingale part and of the drift part of the semimartingales $\langle X^n, f\rangle$.
Since f is bounded, (5.21) is a consequence of (5.20). Let us thus consider a couple (S,S') of stopping times satisfying a.s. $0\le S\le S'\le S+\delta\le T$. Using Lemma 5.2, we get

$$\begin{aligned} E\left[|M^{n,f}_{S'} - M^{n,f}_S|\right] &\le E\left[|M^{n,f}_{S'} - M^{n,f}_S|^2\right]^{1/2} \le E\left[\left\langle M^{n,f}\right\rangle_{S+\delta} - \left\langle M^{n,f}\right\rangle_S\right]^{1/2} \\ &\le E\left[(\bar\gamma+\bar\mu+\bar\alpha\bar U)\int_S^{S+\delta} ds\left(\langle X^n_s,1\rangle + \langle X^n_s,1\rangle^2\right)\right]^{1/2} \le C\sqrt{\delta}, \end{aligned} \tag{5.22}$$

the last inequality coming from (5.20). The finite variation part of $\langle X^n_{S'}, f\rangle - \langle X^n_S, f\rangle$ is bounded by

$$\int_S^{S+\delta} ds\left[(\bar\gamma+\bar\mu)\langle X^n_s,1\rangle + \bar\alpha\bar U\langle X^n_s,1\rangle^2\right] \le \delta C\left[1+\sup_{t\in[0,T]}\langle X^n_s,1\rangle^2\right]. \tag{5.23}$$

Hence, formula (5.20) allows us to conclude that the sequence $Q^n = \mathcal{L}(X^n)$ in tight.
Step 4 Let us now denote by Q the limiting law of a subsequence of Q^n. We still denote this subsequence by Q^n. Let $X = (X_t)_{t\ge 0}$ a process with law Q. We remark that by construction, almost surely,

$$\sup_{t\in[0,T]}\ \sup_{f\in L^\infty(\bar{\mathcal{X}}),||f||_\infty\le 1} |\langle X^n_t, f\rangle - \langle X^n_{t^-}, f\rangle| \le 1/n. \tag{5.24}$$

Then implies that the process X is a.s. strongly continuous.
Step 5 Let us now check that a.s. X is the unique solution of (5.16) satisfying, for each T, $\sup_{t\in[0,T]}\langle X_t,1\rangle$ (which is clear from (5.20)). Standard density arguments show that it in fact suffices to check that X solves (5.16) for all $f\in C_b(\bar{\mathcal{X}})$ and all $t\ge 0$.
Let thus $f\in C_b(\bar{\mathcal{X}})$ and $t\ge 0$ be fixed. For $\nu\in C([0,\infty),M_F(\bar{\mathcal{X}}))$, denote by

$$\begin{aligned} \Psi_t(\nu) = \langle \nu_t, f\rangle - \langle \nu_0, f\rangle &- \int_0^t ds\int_{\bar{\mathcal{X}}}\nu_s(dx)\gamma(x)\int_{\mathbb{R}^d} D(x,z)f(x+z)dz \\ &+ \int_0^t ds\int_{\bar{\mathcal{X}}}\nu_s(dx)f(x)\left\{\mu(x)+\alpha(x)\int_{\bar{\mathcal{X}}}\nu_s(dy)U(x,y)\right\}. \end{aligned} \tag{5.25}$$

We have to show that

$$E_Q\left[|\Psi_t(X)|\right] = 0. \tag{5.26}$$

But Lemma 5.2 and $(C1)$ imply that for each n,

$$M^{n,f}_t = \Psi_t(X^n). \tag{5.27}$$

A straightforward computation using Lemma 5.2, $(C1)$, and (5.20) shows that

$$E\left[|M^{n,f}_t|^2\right] = E\left[\left\langle M^{n,f}\right\rangle_t\right] \le \frac{C_f}{n} E\left[\int_0^t ds\left\{1+\langle X^n_s,1\rangle^2\right\}\right] \le \frac{C_{f,t}}{n}, \tag{5.28}$$

which goes to 0 as n tends to infinity. On the other hand, since X is a.s. strongly continuous, since f is continuous and thanks to $(C1)$, the function Ψ_t is a.s. continuous at X. Furthermore, for any $\nu \in \mathbb{D}([0,T], M_F(\bar{\mathcal{X}}))$,

$$|\Psi_t(\nu)| \le C_{f,t} \sup_{s\in[0,t]} \left(1 + \langle \nu_s, 1\rangle^2\right). \tag{5.29}$$

Hence using (5.20), we see that the sequence $(\Psi_t(X^n))_n$ is uniformly integrable, and thus

$$\lim_n E\left(|\Psi_t(X^n)|\right) = E\left(|\Psi_t(X)|\right). \tag{5.30}$$

Associating (5.27), (5.28) and (5.30), we conclude that (5.26) holds.
Step 6 The previous steps imply that the sequence (X^n) converges to ξ in $\mathbb{D}([0,T], M_F(\bar{\mathcal{X}}))$, where $M_F(\bar{\mathcal{X}})$ is endowed with the vague topology. To extend the result to the case where $M_F(\bar{\mathcal{X}})$ is endowed with the weak topology, we use a criterion proved in Méléard-Roelly [12]: since the limiting process is continuous, it suffices to prove that the sequence $(\langle X^n, 1\rangle)$ converges to $\langle \xi, 1\rangle$ in law, in $\mathbb{D}([0,T], \bar{\mathcal{X}}))$. But one may of course apply Step 5 with $f \equiv 1$, which concludes the proof. $\square$

Proposition 5.4 *Assume that ξ_0 in $M_F(\bar{\mathcal{X}})$ has a density with respect to the Lebesgue measure. Consider the associated solution $(\xi_t)_{t\ge0}$ to (5.16). Then for every $t \ge 0$, the finite measure ξ_t has a density with respect to the Lebesgue measure.*

Proof The proof is similar as the one of Proposition 3.3. We consider a Borel subset A of $\bar{\mathcal{X}}$ with measure zero. We apply (5.16) with $f = \mathbf{1}_A$. The RHS expression is equal to 0 since the first term is zero by hypothesis, the second one is zero since for all x, $\int_{\mathbb{R}^d} \mathbf{1}_{x+z\in A} D(x,z)dz = 0$, and the last term is non-positive. $\square$

Remark 5.5 *(i) Equation (5.16) is the weak form of: for all $x \in \bar{\mathcal{X}}$, $t \ge 0$,*

$$\partial_t \xi_t(x) = \int_{\bar{\mathcal{X}}} dy \xi_t(y)\gamma(y)D(y, x-y) - \mu(x)\xi_t(x) - \alpha(x)\xi_t(x) \int_{\bar{\mathcal{X}}} dy \xi_t(y) U(x,y). \tag{5.31}$$

(ii) Assume now that $\bar{\mathcal{X}} = \mathbb{R}^d$, that the competition kernel is of the form $U(x,y) = U(x-y)$, and that $D(x,z) = D(z)$ does not depend on x. Then (5.16) is the weak form of: for all $x \in \mathbb{R}^d$, $t \ge 0$,

$$\partial_t \xi_t(x) = \left[\gamma \xi_t \star D\right](x) - \mu(x)\xi_t(x) - \alpha(x)\xi_t(x)\left[\xi_t \star U\right](x), \tag{5.32}$$

where for example, $\left[\gamma \xi_t \star D\right](x) = \int_{\mathbb{R}^d} \xi_t(dy)\gamma(y)D(x-y)$.

5.2 Convergence to a superprocess

We would like in this section to show the relation between the original BPDL model (rigorously written in Definition 2.5) and the *superprocess version of the Bolker-Pacala model* introduced by Etheridge [6]. More precisely, we will show in this section that accelerating the rates of production and natural death by a factor n makes the BPDL processes converge to a continuous random measure-valued process which generalizes the one studied in [6].

Assumption $(C2)$:
1) The space $\bar{\mathcal{X}} = \mathbb{R}^d$. The initial conditions X_0^n converge in law, for the weak topology on $M_F(\mathbb{R}^d)$, to a measure $X_0 \in M_F(\mathbb{R}^d)$. Furthermore, $\sup_n E(\langle X_0^n, 1\rangle^3) < +\infty$.

2) There exist some continuous positive functions $\sigma(x), \alpha(x), \gamma(x), \beta(x)$ on $\mathbb{R}^d$ respectively bounded by $\bar{\sigma}, \bar{\alpha}, \bar{\gamma}, \bar{\beta}$, a non-negative symmetric continuous function $U(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ bounded by $\bar{U}$, such that

$$\gamma_n(x) = n\gamma(x) + \beta(x),\ \mu_n(x) = n\gamma(x),\ \alpha_n(x) = \alpha(x)/n,\ U_n(x, y) = U(x, y),$$
$$D_n(x, z) = \left(\tfrac{n}{2\pi\sigma(x)}\right)^{d/2} \exp\left(-n|z|^2/2\sigma(x)\right). \tag{5.33}$$

Note that $D_n(x, z)$ is the density of a Gaussian vector with mean 0 and variance $\frac{\sigma(x)}{n} I_d$.
With these coefficients and when n tends to infinity, one has more and more seed production and natural death, less and less competition. Each seed falls more and more close to its *mother*.

Theorem 5.6 *Assume $(C2)$, and consider the sequence of processes X^n defined in Notation 5.1. Then for all $T > 0$, the sequence (X^n) converges in law, in $\mathbb{D}([0, T], M_F(\mathbb{R}^d))$, to the unique (in law) superprocess $X \in C([0, T], M_F(\mathbb{R}^d))$, defined by the following conditions:*

$$\sup_{t\in[0,T]} E\left[\langle X_t, 1\rangle^3\right] < \infty, \tag{5.34}$$

and for any $f \in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned}\bar{M}_t^f \quad = \quad & \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t ds \left\langle X_s, \frac{1}{2}\sigma\gamma\Delta f\right\rangle \\ & - \int_0^t ds \int_{\mathbb{R}^d} X_s(dx) f(x) \left[\beta(x) - \alpha(x)\int_{\mathbb{R}^d} X_s(dy) U(x, y)\right]\end{aligned} \tag{5.35}$$

is a continuous martingale with quadratic variation

$$\left\langle \bar{M}^f \right\rangle_t = \int_0^t ds 2 \left\langle X_s, \gamma f^2\right\rangle. \tag{5.36}$$

Proof We break the proof in several steps.
Step 1 Let us first prove the uniqueness of the solution of the martingale problem defined by (5.34), (5.35) and (5.36), that is the uniqueness of a probability measure P on $C([0, T], M_F(\mathbb{R}^d))$ under which the canonical process X satisfies (5.34), (5.35) and (5.36). This result is well-known for the super-Brownian process (defined by a similar martingale problem, but with $\alpha = \beta = 0$, and $\sigma\gamma = 1$). As noted in [6], we may use the version of Dawson's Girsanov transform obtained in Evans-Perkins [5] Theorem 2.3, to deduce the uniqueness in our situation, provided the condition below is satisfied:

$$E_P\left(\int_0^t ds \int_{\mathbb{R}^d} X_s(dx) \left[\beta(x) - \alpha(x)\int X_s(dy) U(x, y)\right]^2\right) < +\infty.$$

But this is easily obtained from (5.34) since the coefficients are bounded.
Step 2 Next, we would like to obtain some moment estimates. First, we check that for all $T < \infty$,

$$\sup_n \sup_{t\in[0,T]} E\left[\langle X_t^n, 1\rangle^3\right] < \infty. \tag{5.37}$$

To this end, we use Lemma 5.2-(i) with $\phi(\nu) = \langle \nu, 1\rangle^3$. (To be completely rigorous, one should first use $\phi(\nu) = \langle \nu, 1\rangle^3 \wedge A$, and then make A tend to infinity). We obtain, using $(C2)$, that for all

$t \geq 0$, all n,

$$E\left[\langle X_t^n,1\rangle^3\right] = E\left[\langle X_0^n,1\rangle^3\right]$$
$$+\int_0^t dsE\left[\int_{\mathbb{R}^d} X_s^n(dx)[n^2\gamma(x)+\beta(x)]\left\{[\langle X_s^n,1\rangle+\frac{1}{n}]^3-\langle X_s^n,1\rangle^3\right\}\right] \quad (5.38)$$
$$+\int_0^t dsE\left[\int_{\mathbb{R}^d} X_s^n(dx)\left\{n^2\gamma(x)+n\alpha(x)\int_{\mathbb{R}^d} X_s^n(dy)U(x,y)\right\}\left\{[\langle X_s^n,1\rangle-\frac{1}{n}]^3-\langle X_s^n,1\rangle^3\right\}\right].$$

Neglecting the non-positive competition term, we get

$$E\left[\langle X_t^n,1\rangle^3\right] \leq E\left[\langle X_0^n,1\rangle^3\right]$$
$$+\int_0^t dsE\left[\int_{\mathbb{R}^d} X_s^n(dx)n^2\gamma(x)\left\{[\langle X_s^n,1\rangle+\frac{1}{n}]^3+[\langle X_s^n,1\rangle-\frac{1}{n}]^3-2\langle X_s^n,1\rangle^3\right\}\right]$$
$$+\int_0^t dsE\left[\int_{\mathbb{R}^d} X_s^n(dx)n\beta(x)\left\{[\langle X_s^n,1\rangle+\frac{1}{n}]^3-\langle X_s^n,1\rangle^3\right\}\right]. \quad (5.39)$$

But for all $x \geq 0$, all $\epsilon \in (0,1]$, $(x+\epsilon)^3 - x^3 \leq 6\epsilon(1+x^2)$ and $|(x+\epsilon)^3+(x-\epsilon)^3-2x^3| = 6\epsilon^2 x$. We finally obtain

$$E\left[\langle X_t^n,1\rangle^3\right] \leq E\left[\langle X_0^n,1\rangle^3\right]$$
$$+6\bar{\gamma}\int_0^t dsE\left[\langle X_s^n,1\rangle^2\right]+6\bar{\beta}\int_0^t dsE\left[\langle X_s^n,1\rangle+\langle X_s^n,1\rangle^3\right]. \quad (5.40)$$

Assumption $(C2)$-1 and the Gronwall Lemma allow us to conclude that (5.37) holds.
Next, we have to check that

$$\sup_n E\left(\sup_{t\in[0,T]}\langle X_t^n,1\rangle\right) < \infty. \quad (5.41)$$

Applying Lemma 5.2-(iv) with $f \equiv 1$ and $(C2)$, we obtain

$$\langle X_t^n,1\rangle = \langle X_0^n,1\rangle + \int_0^t ds\int_{\mathbb{R}^d} X_s^n(dx)\left[\beta(x)-\alpha(x)\int_{\mathbb{R}^d} X_s^n(dy)U(x,y)\right]+M_t^{n,1}. \quad (5.42)$$

Hence

$$\sup_{s\in[0,t]}\langle X_s^n,1\rangle \leq \langle X_0^n,1\rangle + \bar{\beta}\int_0^t ds\,\langle X_s^n,1\rangle + \sup_{s\in[0,t]}|M_s^{n,1}|. \quad (5.43)$$

Thanks to the Doob inequality, $(C2)$-1 and the Gronwall Lemma, there exists a constant C_t not depending on n such that

$$E\left(\sup_{s\in[0,t]}\langle X_t^n,1\rangle\right) \leq C_t(1+E\left[\langle M^{n,1}\rangle_t\right]^{1/2}). \quad (5.44)$$

Using now (5.15) and $(C2)$, we obtain, for some other constant C_t not depending on n,

$$E\left[\langle M^{n,1}\rangle_t\right] \leq (2\bar{\gamma}+\bar{\beta})\int_0^t dsE\left[\langle X_s^n,1\rangle\right]+\bar{\alpha}\bar{U}\int_0^t dsE\left[\langle X_s^n,1\rangle^2\right] \leq C_t \quad (5.45)$$

thanks to (5.37). This concludes the proof of (5.41).
Step 3 We first endow $M_F(\mathbb{R}^d)$ with the vague topology The extension to the weak topology

will be handled in Step 5 below. We prove the tightness of the sequence of the laws $(\mathcal{L}(X^n))_n$ in $\mathcal{P}(\mathbb{D}([0,\infty),M_F(\mathbb{R}^d)))$, following the same approach as in Theorem 5.3. First, we deduce from Step 2 that $\sup_n E\left[\sup_{s\in[0,T]}|\langle X_s^n,f\rangle|\right]<\infty$, for any bounded f. We thus have to prove that for any $f\in C_b^2(\mathbb{R}^d)$, the sequence $\langle X_t^n,f\rangle$ satisfies the Aldous-Rebolledo criterion. Let us consider a couple (S,S') of stopping times satisfying a.s. $0\le S\le S'\le S+\delta\le T$. Using Lemma 5.2, $(C2)$ and the fact that $\left|\int_{\mathbb{R}^d} dzD_n(x,z)f(x+z)-f(x)\right|\le\bar{\sigma}||\Delta f||_\infty/2n$, we deduce the existence of a constant C independent of n such that the finite variation part of $\langle X_{S'}^n,f\rangle-\langle X_S^n,f\rangle$ is bounded by

$$\int_S^{S+\delta}ds\int_{\mathbb{R}^d}X_s^n(dx)\bar{\beta}||f||_\infty+\int_S^{S+\delta}ds\int_{\mathbb{R}^d}X_s^n(dx)n\gamma(x)\left|\int_{\mathbb{R}^d}dzD_n(x,z)f(x+z)-f(x)\right|$$
$$+\int_S^{S+\delta}ds\int_{\mathbb{R}^d}X_s^n(dx)\bar{\alpha}\bar{U}||f||_\infty\int_{\mathbb{R}^d}X_s^n(dy)\le C\int_S^{S+\delta}ds\left(\langle X_s^n,1\rangle+\langle X_s^n,1\rangle^2\right). \tag{5.46}$$

Using now (5.15) and $(C2)$, we deduce that for some constant C,

$$E\left[\left\langle M^{n,f}\right\rangle_{S+\delta}-\left\langle M^{n,f}\right\rangle_S\right]\le CE\left[\int_S^{S+\delta}ds\left(\langle X_s^n,1\rangle+\langle X_s^n,1\rangle^2\right)\right]. \tag{5.47}$$

Using the moment estimate (5.37), we finally obtain by that the laws of $(M^{n,f})$ and the laws of the drift parts of $\langle X^n,f\rangle$ are tight, and then, by Rebolledo's criterion, the laws of $\langle X^n,f\rangle$ are tight.

Step 4 Let us identify the limit. Let us call $Q^n=\mathcal{L}(X^n)$ and denote by Q a limiting value of the tight sequence Q^n, and by $X=(X_t)_{t\ge0}$ a process with law Q. Exactly as in the proof of Theorem 5.3, one can show that X belongs a.s. to $C([0,T],M_F(\mathbb{R}^d))$. We have to show that X satisfies the conditions (5.34), (5.35) and (5.36). First note that (5.34) is straightforward from (5.37). Then, we show that for any function f in $C_b^3(\mathbb{R}^d)$, the process $\bar{M}_t^f$ defined by (5.35) is a martingale (the extension to every function in C_b^2 is not hard). We consider $0\le s_1\le...\le s_k<s<t$, some continuous bounded maps $\phi_1,...\phi_k$ on $M_F(\mathbb{R}^d)$, and our aim is to prove that, if the function Ψ from $\mathbb{D}([0,T],M_F(\mathbb{R}^d))$ into $\mathbb{R}$ is defined by

$$\Psi(\nu)=\phi_1(\nu_{s_1})...\phi_k(\nu_{s_k})\Big\{\langle\nu_t,f\rangle-\langle\nu_s,f\rangle-\int_s^t du\,\langle\nu_u,\gamma\sigma\Delta f/2\rangle$$
$$-\int_s^t du\int_{\mathbb{R}^d}\nu_u(dx)f(x)\left[\beta(x)-\int_{\mathbb{R}^d}\nu_u(dy)\alpha(x)U(x,y)\right]\Big\}, \tag{5.48}$$

then

$$E(\Psi(X))=0. \tag{5.49}$$

We know from Lemma 5.2 that using $(C2)$,

$$0=E\left[\phi_1(X_{s_1}^n)...\phi_k(X_{s_k}^n)\left\{M_t^{n,f}-M_s^{n,f}\right\}\right]=E\left[\Psi(X^n)\right]-A_n, \tag{5.50}$$

where A_n is defined by

$$A_n=E\Big[\int_s^t du\int_{\mathbb{R}^d}X_u^n(dx)\Big\{\gamma(x)n\left[\int_{\mathbb{R}^d}dzD_n(x,z)f(x+z)-f(x)-\frac{\sigma(x)}{2n}\Delta f(x)\right]$$
$$+\beta(x)\left[\int_{\mathbb{R}^d}dzD_n(x,z)f(x+z)-f(x)\right]\Big\}\phi_1(X_{s_1}^n)...\phi_k(X_{s_k}^n)\Big]. \tag{5.51}$$

First, an easy computation using $(C2)$, the fact that f is C_b^3, and (5.37) shows that

$$|A_n|\le\frac{C_f}{n}\int_s^t duE\left[\langle X_u^n,1\rangle\right]\longrightarrow 0 \tag{5.52}$$

as n grows to infinity. Next, it is clear from assumption $(C2)$, the fact that f is C_b^3, and that Q only charges the space of continuous processes, that the map Ψ is Q-a.s. continuous. Furthermore,

$$|\Psi(\nu)| \le C \left(1 + \langle \nu_s, 1 \rangle + \langle \nu_t, 1 \rangle + \int_s^t du \, \langle \nu_u, 1 \rangle^2 \right), \tag{5.53}$$

and one easily deduces from (5.37) that the sequence $(|\Psi(X^n)|)_n$ is uniformly integrable. Hence,

$$\lim_n E\left(|\Psi(X^n)|\right) = E_Q\left(|\Psi(X)|\right). \tag{5.54}$$

Associating (5.50), (5.52), (5.54) allows us to conclude that (5.49) holds, and thus $\bar{M}^f$ is a martingale.

We finally have to show that the bracket of $\bar{M}^f$ is given by (5.36). To this end, we first check that

$$\begin{aligned} \bar{N}_t^f \;=\; & \langle X_t, f \rangle^2 - \langle X_0, f \rangle^2 - \int_0^t ds \int_{\mathbb{R}^d} X_s(dx) 2\gamma(x) f^2(x) \\ & - \int_0^t ds 2 \langle X_s, f \rangle \int_{\mathbb{R}^d} X_s(dx) f(x) \left[\beta(x) - \alpha(x) \int_{\mathbb{R}^d} X_s(dy) U(x,y) \right] \\ & - \int_0^t ds 2 \langle X_s, f \rangle \int_{\mathbb{R}^d} X_s(dx) \frac{1}{2} \sigma(x) \gamma(x) \Delta f(x) \end{aligned} \tag{5.55}$$

is a martingale. This can be done exactly as for $\bar{M}_t^f$, using that thanks to Lemma 5.2-(iii) (with $q = 2$),

$$\begin{aligned} N_t^{n,f} \;=\; & \langle X_t^n, f \rangle^2 - \langle X_0^n, f \rangle^2 - \int_0^t ds \int_{\mathbb{R}^d} X_s^n(dx) \gamma(x) \left[\int_{\mathbb{R}^d} dz f^2(x+z) D_n(x,z) + f^2(x) \right] \\ & - \int_0^t ds 2 \langle X_s^n, f \rangle \int_{\mathbb{R}^d} X_s^n(dx) \Big[\beta(x) \int_{\mathbb{R}^d} dz f(x+z) D_n(x,z) - \alpha(x) f(x) \int_{\mathbb{R}^d} X_s^n(dy) U(x,y) \Big] \\ & - \int_0^t ds 2 \langle X_s^n, f \rangle \int_{\mathbb{R}^d} X_s^n(dx) \gamma(x) n \left[\int_{\mathbb{R}^d} dz f(x+z) D_n(x,z) - f(x) \right] \\ & - \frac{1}{n} \int_0^t ds \int_{\mathbb{R}^d} X_s^n(dx) \beta(x) \int_{\mathbb{R}^d} dz f^2(x+z) D_n(x,z) \\ & - \frac{1}{n} \int_0^t ds \int_{\mathbb{R}^d} X_s^n(dx) \alpha(x) \int_{\mathbb{R}^d} X_s^n(dy) U(x,y) f^2(x) \end{aligned} \tag{5.56}$$

is a martingale for each n. Next, using the Itô formula in the definition (5.35) of $\bar{M}_t^f$, we deduce that

$$\begin{aligned} & \langle X_t, f \rangle^2 - \langle X_0, f \rangle^2 - \langle \bar{M}^f \rangle_t - \int_0^t ds 2 \langle X_s, f \rangle \int_{\mathbb{R}^d} X_s(dx) f(x) \left[\beta(x) - \alpha(x) \int_{\mathbb{R}^d} X_s(dy) U(x,y) \right] \\ & - \int_0^t ds 2 \langle X_s, f \rangle \int_{\mathbb{R}^d} X_s(dx) \frac{1}{2} \sigma(x) \gamma(x) \Delta f(x) \end{aligned} \tag{5.57}$$

is a martingale. Comparing this formula with (5.55) allows us to conclude that (5.36) holds.

Step 5 The extension to the case where $M_F(\mathbb{R}^d)$ is endowed with the weak topology uses similar arguments as in Step 6 of the proof of Theorem 5.3. □

6 About extinction and survival

First of all, we would like to recall a result of Ethridge, [6]. Consider the superprocess X obtained in Theorem 5.6, assume that σ, γ, β and α are constant on $\mathbb{R}^d$. Suppose also that $U(x,y) = h(|x-y|)$,

for some non-negative decreasing function h on $\mathbb{R}_+$ satisfying $\int_0^\infty h(r)r^{d-1}dr < \infty$. Then, if β is sufficiently small, and α is sufficiently large, X does not survive: almost surely, there exists a $t \geq 0$ such that for all $s \geq 0$, $X_{t+s} = 0$.
One can also find a complementary result in [6], which shows non-extinction with positive probability for another model, the *stepping-stone* version of the Bolker Pacala process. Let us now come back to the BPDL process, defined as the solution of (2.8). The techniques used in [6] are specific to continuous processes and can not be generalized to the BPDL discontinuous process.
Before giving our results, let us point out the following obvious remark.

Remark 6.1 *Assume (A), and that $E[\langle \nu_0, 1\rangle] < \infty$. Consider the BPDL process $(\nu_t)_{t\geq 0}$. Assume also that there exist some constants $\gamma_0 \leq \mu_0$ such that for all $x \in \bar{\mathcal{X}}$, $\mu(x) \geq \mu_0$ and $\gamma(x) \leq \gamma_0$. Then $(\nu_t)_{t\geq 0}$ does a.s. not survive, that is $P[\exists s > 0\ \langle \nu_s, 1\rangle = 0] = 1$.*

The proof of this remark is not hard: in such a case, the process $Z_t = \langle \nu_t, 1\rangle$ can be bounded from above by a standard continuous time binary Galton-Watson process Y_t with death rate μ_0 and birth rate γ_0. Since $\mu_0 \geq \gamma_0$, extinction a.s. occurs.

In this section, we will first prove almost sure extinction in a case where the state space $\bar{\mathcal{X}}$ is compact. Then, we will show non-extinction in the case of a discrete version of the BPDL process, with a specific (and not quite realistic) competition kernel U.

6.1 Extinction in the compact case

We will check a result which essentially says that if the state space $\bar{\mathcal{X}}$ is compact, then the population does almost surely not survive. Let us assume

Assumption (E):
(i) The maps $\alpha(x)$ and $\mu(x) + \alpha(x)U(x, x)$ are bounded below.
(ii) There exists a non-decreasing function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, satisfying $\varphi(0) = 0$, such that $\lim_{x\to\infty} \varphi(x) = \infty$, such that the map $x\varphi(x)$ is convex on $[0, \infty)$ and such that for all $\nu \in \mathcal{M}$,

$$\langle \nu \otimes \nu, U\rangle \geq \langle \nu, 1\rangle\, \varphi(\langle \nu, 1\rangle). \tag{6.1}$$

Remark 6.2 *Assumption (E)-(ii) holds if $\bar{\mathcal{X}}$ is compact in $\mathbb{R}^d$, and if there exist $\epsilon > 0$ and $\delta > 0$ such that $U(x, y) \geq \epsilon \mathbf{1}_{\{|x-y|\leq\delta\}}$*

Theorem 6.3 *Assume (A), (E), $\nu_0 \in \mathcal{M}$ and $E(\langle \nu_0, 1\rangle) < \infty$. Consider the corresponding unique BPDL process $(\nu_t)_{t\geq 0}$ obtained in Theorem 3.1. Then there is almost surely extinction, that is $P(\ \exists t \geq 0,\ \langle \nu_t, 1\rangle = 0\ ) = 1$.*

Proof of Remark 6.2 First of all, we cover $\bar{\mathcal{X}}$ with a family $\{C_l\}_{l\in\{1,...,L\}}$ of disjoint cubes of $\mathbb{R}^d$, with side $\delta/\sqrt{d}$. Note that L is clearly finite and that for each l, each $x, y \in C_l$, $|x - y| \leq \delta$. Recall the following consequence of the Cauchy-Schwarz inequality, which says that for all $L \geq 1$, all $\{\alpha_1, ..., \alpha_L\}$ in $\mathbb{R}$, $\sum_{l=1}^L \alpha_l^2 \geq \frac{1}{L}\left[\sum_{l=1}^L \alpha_l\right]^2$. Hence for all $n \geq 1$, all $x_1, ..., x_n \in \bar{\mathcal{X}}$,

$$\begin{aligned}\sum_{i,j=1}^n U(x_i, x_j) &\geq \sum_{i,j=1}^n \epsilon \mathbf{1}_{\{|x_i - x_j|\leq\delta\}} \geq \epsilon \sum_{i,j=1}^n \sum_{l=1}^L \mathbf{1}_{C_l}(x_i)\mathbf{1}_{C_l}(x_j) \\ &= \epsilon \sum_{l=1}^L \left[\sum_{i=1}^n \mathbf{1}_{C_l}(x_i)\right]^2 \geq \epsilon \frac{1}{L}\left[\sum_{l=1}^L \sum_{i=1}^n \mathbf{1}_{C_l}(x_i)\right]^2 = \epsilon \frac{1}{L} n^2.\end{aligned} \tag{6.2}$$

One immediately deduces that for any $\nu \in \mathcal{M}$, since ν is atomic, $\langle \nu \otimes \nu, U \rangle \geq \epsilon \frac{1}{L} \langle \nu, 1 \rangle^2$. Hence (E)-(ii) holds with $\varphi(n) = \epsilon \frac{1}{L} n$. □

Proof of Theorem 6.3 We break the proof into several steps.
Step 1 We first of all prove that

$$A = \sup_{t \geq 0} E\left(\langle \nu_t, 1 \rangle\right) < +\infty. \tag{6.3}$$

To this end, we set $f(t) = E\left(\langle \nu_t, 1 \rangle\right)$, and we use Proposition 3.4 with $\phi(\nu) = \langle \nu, 1 \rangle$ to obtain

$$f(t) = f(0) + \int_0^t ds E\left[\langle \nu_s, \gamma - \mu \rangle - \int_{\bar{\mathcal{X}}} \int_{\bar{\mathcal{X}}} \nu_s(dx)\nu_s(dy)\alpha(x)U(x,y)\right]. \tag{6.4}$$

Hence f is differentiable, and if we set $\delta = ||\gamma - \mu||_\infty$ and $\alpha_0 = \inf_{x \in \bar{\mathcal{X}}} \alpha(x)$, we deduce that for any $t \geq 0$,

$$f'(t) \leq \delta f(t) - \alpha_0 E\left(\langle \nu_t \otimes \nu_t, U \rangle\right). \tag{6.5}$$

Using then assumption (E) and then the Jensen inequality, we obtain that

$$f'(t) \leq \delta f(t) - \alpha_0 f(t) \varphi(f(t)). \tag{6.6}$$

Let now x_0 be the greatest solution of $\delta x_0 = \alpha_0 x_0 \varphi(x_0)$ (recall that $\varphi(x)$ is non-decreasing, goes to infinity with x, and that $\varphi(0) = 0$). Then one deduces from (6.6) that for any $t \geq 0$, $f(t) \leq f(0) \vee x_0$. This concludes the first step.
Step 2 We now check that almost surely,

$$\liminf_{t \to \infty} \langle \nu_t, 1 \rangle \in \{0, \infty\} \tag{6.7}$$

Since $\langle \nu_t, 1 \rangle$ is $\mathbb{N}$-valued, it suffices to check that for any $M \in \mathbb{N}^*$, $P\left[\liminf_{t \to \infty} \langle \nu_t, 1 \rangle = M\right] = 0$. But this is clear: if $\liminf_{t \to \infty} \langle \nu_t, 1 \rangle = M$, then $\langle \nu_t, 1 \rangle$ reaches the state M infinitely often, but reaches the state $M - 1$ only a finite number of times. This is (a.s.) impossible, because each time $\langle \nu_t, 1 \rangle$ reaches the state M, the probability that its next state is $M - 1$ is bounded below by

$$\frac{M \epsilon_0}{M \bar{\gamma} + M \bar{\mu} + \bar{\alpha} \bar{U} M^2} > 0, \tag{6.8}$$

where $\epsilon_0 = \inf_{x \in \bar{\mathcal{X}}} [\mu(x) + \alpha(x) U(x, x)] > 0$.
Step 3 We immediately deduce from (6.7), the fact that $\langle \nu_t, 1 \rangle$ is $\mathbb{N}$-valued, and that 0 is an absorbing state, that almost surely, $\lim_{t \to \infty} \langle \nu_t, 1 \rangle$ exists and

$$\lim_{t \to \infty} \langle \nu_t, 1 \rangle \in \{0, \infty\}. \tag{6.9}$$

Step 4 By Fatou's lemma and Step 1,

$$E\left[\lim_{t \to \infty} \langle \nu_t, 1 \rangle\right] = E\left[\liminf_{t \to \infty} \langle \nu_t, 1 \rangle\right] \leq \liminf_{t \to \infty} E\left[\langle \nu_t, 1 \rangle\right] \leq A. \tag{6.10}$$

Hence $\lim_{t \to \infty} \langle \nu_t, 1 \rangle < \infty$ a.s., and we deduce from (6.9) that $\lim_{t \to \infty} \langle \nu_t, 1 \rangle = 0$ a.s. This concludes the proof. □

6.2 Survival in a simplified case

Next, we would like to show that in some cases, the BPDL process survives with positive probability. We are not able to handle a proof in a general case, because the problem seems very difficult. It actually looks much more difficult than the problem of survival for the contact process, which has been studied by many mathematicians (see Liggett [11]). The only result we are able to prove is deduced from a comparison with the contact process.

Assumption (S):

(i) The state space $\bar{\mathcal{X}} = \mathbb{Z}^d$.
(ii) The competition kernel U is pointwise, i.e. $U(x, y) = \mathbf{1}_{\{x=y\}}$.
(iii) The dispersion measure $D(x, dz) = D(dz) = \frac{1}{2^d} \sum_{u \in \mathbb{Z}^d, |u|=1} \delta_u(dz)$.
(iv) γ, μ, and α are positive constants satisfying:

$$\frac{\gamma 2^{-d}}{\mu + \alpha} > 2. \tag{6.11}$$

Note that $\bar{\mathcal{X}} = \mathbb{Z}^d$ was not covered by our construction. The adaptation is however immediate.

Proposition 6.4 *Assume (S), assume that $\nu_0 \in \mathcal{M}$, $\langle \nu_0, 1 \rangle \geq 1$ almost surely, and that $E[\langle \nu_0, 1 \rangle] < \infty$. Consider the corresponding BPDL process $(\nu_t)_{t \geq 0}$. This process survives with positive probability. That means that $P(\ \inf_{t \geq 0} \langle \nu_t, 1 \rangle \geq 1\ ) > 0$.*

We do not handle a completely rigorous proof. One would have to build a rigorous coupling between the contact process and the Bolker-Pacala process.

Proof We split the proof in two steps.
Step 1 Let us first recall definitions and results about the contact process (see [11] Chapter VI). First, denote by M_F^s the set of non-negative finite measures η on $\mathbb{Z}^d$ such that for all $x \in \mathbb{Z}^d$, $\eta(\{x\}) \in \{0, 1\}$. The contact process, with parameters $\lambda_d > 0$ and $\lambda_m > 0$ is a Markov process $(\eta_t)_{t \geq 0}$, taking its values in M_F^s and with generator K, defined for all bounded and measurable maps ϕ from $M_F(\mathbb{Z}^d)$ into $\mathbb{R}$, all $\eta \in M_F(\mathbb{Z}^d)$ by

$$\begin{aligned} K\phi(\eta) &= \lambda_d \int_{\mathbb{Z}^d} \eta(dx) \sum_{u \in \mathbb{Z}^d, |u|=1} \mathbf{1}_{\{\eta(\{x+u\})=0\}} \left[\phi(\eta + \delta_{x+u}) - \phi(\eta)\right] \\ &\quad + \lambda_m \int_{\mathbb{Z}^d} \eta(dx) \mathbf{1}_{\{\eta(\{x\})=1\}} \left[\phi(\eta - \delta_x) - \phi(\eta)\right]. \end{aligned} \tag{6.12}$$

Consider a (possibly random) initial state η_0 in M_F^s satisfying $\langle \eta_0, 1 \rangle \geq 1$ a.s. Then it is known, (see [11] Chapter VI), that the contact process $(\eta_t)_{t \geq 0}$ with parameters $\lambda_d > 0$, $\lambda_m > 0$ and initial state η_0 exists, is unique (in law), and that under the condition $\lambda_d > 2\lambda_m$, it survives with positive probability.
Step 2 Consider now the BPDL process $(\nu_t)_{t \geq 0}$, which takes its values in the integer-valued measures on $\mathbb{Z}^d$. Denote by $\tilde{\eta}_t = \sum_{x \in \mathbb{Z}^d} \mathbf{1}_{\{\nu_t(\{x\}) \geq 1\}} \delta_x$. Note that $\tilde{\eta}_t$ is always dominated by ν_t. Then $(\tilde{\eta}_t)_{t \geq 0}$ is a process with values in M_F^s and one can observe that $(\tilde{\eta}_t)_{t \geq 0}$ is a sort of contact process with time and space dependent, random parameters $\lambda_d(t, x, \omega) = \gamma 2^{-d}[1 \vee \nu_t(\{x\})]$ and $\lambda_m(t, x, \omega) = \mathbf{1}_{\nu_t(\{x\}) \leq 1}(\mu + \alpha)$. Under (S), $\lambda_d(t, x, \omega)$ is uniformly bounded from below by $\underline{\lambda}_d = \gamma 2^{-d}$ while $\lambda_m(t, x, \omega)$ is uniformly bounded from above by $\overline{\lambda}_m = \mu + \alpha$. Hence, the process $(\tilde{\eta}_t)_{t \geq 0}$ is bounded below by a contact process with parameters $\underline{\lambda}_d$ and $\overline{\lambda}_m$. Since (6.11) ensures that $2\overline{\lambda}_m < \underline{\lambda}_d$, the conclusion follows from Step 1. □

Note that the previously described method may not apply to the continuous-state BPDL process, since we really need the interaction to be strictly local. In fact, the only case we could treat by such a method is the case where the competition kernel is *completely local*, and can not propagate: for example, $\bar{\mathcal{X}} = \mathbb{R}^d$, and $U(x, y) \leq \sum_{p \in \mathbb{Z}^d} \mathbf{1}_{C_p}(x) \mathbf{1}_{C_p}(y)$, where, for $p \in \mathbb{Z}^d$, $C_p = [p_1, p_1 + 1] \times \ldots \times [p_d, p_d + 1]$.

7 On equilibria

An interesting question is that of existence of non trivial equilibria for the BPDL process. Since this question seems very complicated, we will first try to give some results about the deterministic equation (5.16). Then, we will show that there exists a nontrivial equilibrium for the BPDL process related to the carrying capacity, under a detailed balance condition which is unfortunately very restrictive. We will finally present some simulations. We will assume (B) (see Section 4) in the whole section.

7.1 Equilibrium of the deterministic equation

We first of all point out a trivial remark.

Remark 7.1 *Assume (B) and that $\gamma < \mu$, and consider a non-negative finite measure ξ_0 on $\mathbb{R}^d$. Consider the corresponding unique solution $(\xi_t)_{t\geq 0} \in C([0,\infty), M_F(\mathbb{R}^d))$ of (5.16). Then ξ_t tends to 0 as t grows to infinity, in the sense that $\langle \xi_t, 1\rangle \leq \langle \xi_0, 1\rangle\, e^{-(\mu-\gamma)t}$.*

This remark follows from a straightforward application of (5.16) with $f = 1$ and of the Gronwall Lemma. We next generalize existence of solutions to equation (5.16) to the case of possibly non integrable initial conditions.

Proposition 7.2 *Assume (B). Consider a non-negative bounded measurable function ξ_0 on $\mathbb{R}^d$.*
1) There exists a unique function $(\xi_t(x))_{t\geq 0, x\in\mathbb{R}^d}$ such that:
(i) for all $t \geq 0$, all $x \in \mathbb{R}^d$, $\xi_t(x) \geq 0$,
(ii) for all $T < \infty$, $\sup_{t\in[0,T], x\in\mathbb{R}^d} \xi_t(x) < \infty$,
(iii) for all $t \geq 0$, all $x \in \mathbb{R}^d$,

$$\xi_t(x) = \xi_0(x) + \int_0^t ds\, [\gamma(\xi_s \star D)(x) - \mu\xi_s(x) - \alpha\xi_s(x)(\xi_s \star U)(x)]\,, \tag{7.1}$$

where, for example, $[\xi_t \star D](x) = \int_{\mathbb{R}^d} dy D(x-y)\xi_t(y)$.
2) For all $x \in \mathbb{R}^d$, the map $t \mapsto \xi_t(x)$ is of class C^1 on $[0,\infty)$, and for all $T < \infty$, $|\partial_t \xi_t(x)|$ is bounded on $[0,T]\times\mathbb{R}^d$.
3) If furthermore $\int_{\mathbb{R}^d} \xi_0(x)dx < \infty$, then for all $T < \infty$, $\sup_{t\in[0,T]} \int_{\mathbb{R}^d} dx\xi_t(x) < \infty$, and the finite measure-valued function $(\xi_t(x)dx)_{t\geq 0}$ is the unique solution to (5.16).

Since this proposition is quite unsurprising, we only sketch the proof.

Proof First note that point 2 is an immediate consequence of (7.1) and of the fact that ξ is bounded, obtained in (i) and (ii). Point 3 is also easily deduced from point 1. To check the uniqueness part of point 1, it suffices to consider two solutions $(\xi_t(x))_{t\geq 0, x\in\mathbb{R}^d}$ and $(\tilde{\xi}_t(x))_{t\geq 0, x\in\mathbb{R}^d}$ to (i), (ii), (iii), both bounded by some constant A_T on $[0,T]\times\mathbb{R}^d$. A straightforward computation shows that, for $\phi(t) = \sup_{s\leq t, x\in\mathbb{R}^d} |\xi_s(x) - \tilde{\xi}_s(x)|$, for $t \leq T$,

$$\phi(t) \leq (\gamma + \mu + 2\alpha A_T)\int_0^t ds\phi(s). \tag{7.2}$$

(Recall that since $\int_{\mathbb{R}^d} U(x)dx = 1$, $\sup_{x\in\mathbb{R}^d}(\xi_s \star U)(x) \leq \sup_{x\in\mathbb{R}^d} \xi_s(x)$). The Gronwall Lemma allows to conclude that $\xi \equiv \tilde{\xi}$.
The existence part follows from an *implicit* Picard iteration. Define $\xi_t^0(x) = \xi_0(x)$ and construct by induction a sequence of functions $(\xi_t^n)_{t\geq 0}$ such that for each $x \in \mathbb{R}^d$, $t \mapsto \xi_t^n(x)$ is of class C^1 on $\mathbb{R}^+$ and satisfies for $n \geq 1$,

$$\xi_t^{n+1}(x) = \xi_0(x) + \int_0^t ds\, \left[\gamma(\xi_s^n \star D)(x) - \mu\xi_s^{n+1}(x) - \alpha\xi_s^{n+1}(x)(\xi_s^n \star U)(x)\right]. \tag{7.3}$$

One can moreover check at each step that ξ^n is well-defined, non-negative and bounded on $[0,T]\times\mathbb{R}^d$ for each n, each T. A straightforward computation shows that for all $t\geq 0$, $\sup_n \sup_{x\in\mathbb{R}^d} \xi_t^n(x) \leq \sup_{x\in\mathbb{R}^d} \xi_0(x)e^{\gamma t}$, and next that for any T, there exists a constant B_T such that for all $t\leq T$,

$$\sup_{x\in\mathbb{R}^d} |\xi_t^{n+1}(x)-\xi_t^n(x)| \leq B_T \int_0^t ds \left[\sup_{x\in\mathbb{R}^d} |\xi_s^{n+1}(x)-\xi_s^n(x)| + \sup_{x\in\mathbb{R}^d} |\xi_s^n(x)-\xi_s^{n-1}(x)|\right]. \tag{7.4}$$

Thanks to the Gronwall Lemma, we deduce that for all T, all $t\leq T$ and all n,

$$\sup_{x\in\mathbb{R}^d} |\xi_t^{n+1}(x)-\xi_t^n(x)| \leq B_T e^{TB_T} \int_0^t ds \sup_{x\in\mathbb{R}^d} |\xi_s^n(x)-\xi_s^{n-1}(x)|. \tag{7.5}$$

The Picard Lemma allows us to conclude that for all T,

$$\sum_{n\geq 1} \sup_{t\in[0,T],x\in\mathbb{R}^d} |\xi_t^{n+1}(x)-\xi_t^n(x)| < \infty. \tag{7.6}$$

Hence, there exists a function $(\xi_t(x))_{t\geq 0,x\in\mathbb{R}^d}$ such that for any T, $\sup_{t\in[0,T],x\in\mathbb{R}^d} |\xi_t(x)-\xi_t^n(x)|$ tends to 0. One easily checks that this function satisfies points (i), (ii), (iii). ☐

We may now define the equilibria.

Definition 7.3 *Assume* (B)*. For* f *a non-negative bounded continuous function on* $\mathbb{R}^d$*, define the function* Ff *on* $\mathbb{R}^d$ *by*

$$Ff(x) = \frac{\gamma\,[f\star D]\,(x)}{\mu+\alpha\,[f\star U]\,(x)}. \tag{7.7}$$

Then equation (7.1) can be rewritten as

$$\xi_t(x) = \xi_0(x) + \int_0^t ds(\mu+\alpha\,[\xi_s\star U]\,(x))(F\xi_s(x)-\xi_s(x)). \tag{7.8}$$

This leads us to define the equilibria in the following sense. A continuous bounded non-negative function c *on* $\mathbb{R}^d$ *is said to be a reasonable equilibrium of equation (7.1) if for all* $x\in\mathbb{R}^d$*,*

$$c(x) = Fc(x). \tag{7.9}$$

This definition is slightly restrictive, but we may note that if D and U are continuous, then any solution to (7.9) such that $\limsup_{|x|\to\infty}[c\star D](x)/[c\star U](x) < \infty$ will be continuous and bounded.

Remark 7.4 *Assume* (B)*, that* $\gamma > \mu$*, and that* $\alpha > 0$*. Then the constant function* $c_0(x) \equiv (\gamma-\mu)/\alpha$ *is a reasonable equilibrium of (7.1). The constant function* $c(x)\equiv 0$ *is also, of course, a reasonable equilibrium of (7.1).*

Note that the quantity $(\gamma-\mu)/\alpha$ appears in [2], and is called the *carrying capacity*, which can be understood as a sort of *maximum number of plants per unit of volume.* We will use the following estimate.

Lemma 7.5 *Assume* (B)*, that* $\gamma > \mu$ *and that* $\alpha > 0$*. Define the signed function* R *on* $\mathbb{R}^d$ *by* $R(x) = D(x) + \frac{\gamma-\mu}{\mu}(D(x)-U(x))$*. Then, for all bounded functions* f*, all* $x\in\mathbb{R}^d$*,*

$$Ff(x) - Fc_0(x) = \frac{\mu}{\mu+\alpha[f\star U](x)}[(f-c_0)\star R](x). \tag{7.10}$$

This result is immediately proved, using simply the expression of F. We now state an assumption which ensures that $R(x)dx$ is a probability measure, and hence that F is a contraction around c_0 in the space of bounded functions.

> Assumption (C): $\gamma > \mu$ and for all $x \in \mathbb{R}^d$, $\gamma D(x) \geq (\gamma - \mu)U(x)$. This implies that $R(x)dx$ is a probability measure on $\mathbb{R}^d$.

Let us now describe a situation for which the constant function c_0 is the unique nontrivial reasonable equilibrium.

Proposition 7.6 *Assume (B), (C), that $\gamma > 2^d\mu$, and that $\alpha > 0$. Suppose also that $D(x) = D(|x|)$, where the map D is non-increasing on $[0, \infty)$. (This hypothesis is physically reasonable, see [2]). Then any nontrivial reasonable equilibrium c of equation (7.1) identically equals c_0.*

Proof Let thus c be a nontrivial reasonable equilibrium for (7.1).
Step 1 Since c is nontrivial, there exists x_0 such that $c(x_0) > 0$. Since c is continuous, we deduce that c is bounded below on a neighborhood of x_0. Then (7.9) and the fact that D charges any neighborhood of 0 (since it is non-increasing) ensure that c does never vanish.
Step 2 We now show that there exists a constant $\epsilon_0 > 0$ such that for all $x \in \mathbb{R}^d$, $c(x) \geq \epsilon_0$. To this end, we first consider $\epsilon > 0$ such that $\gamma(1/2^d - \epsilon) > \mu$, and then $a > 0$ such that $\int_{[0,a]^d} D(x)dx \geq 1/2^d - \epsilon$, which is possible since D is radial.
Consider now any point $x = (x_1, ..., x_d) \in \mathbb{R}^d$ and the box $B = [x_1, x_1 + a] \times ... \times [x_d, x_d + a]$. Denote by $m = \inf_{x\in B} c(x)$, which is positive since c is continuous and does never vanish. Our aim is to show that $m \geq g(m)$, the C^1 function g being defined on $[0, \infty)$ by

$$g(u) = f[u(1/2^d - \epsilon)] \quad ; \quad f(u) = \frac{\gamma u}{\mu + \frac{\alpha\gamma}{\gamma-\mu} u}. \tag{7.11}$$

This will conclude the proof of Step 2, since one may check that $g'(0) = (1/2^d - \epsilon)\gamma/\mu > 1$, so that $m \geq \epsilon_0 > 0$, ϵ_0 being the smallest positive solution to $u = g(u)$.
We thus check that $m \geq g(m)$. Let $y \in B$. Using (7.9) and (C), we deduce that $c(y) \geq f([c\star D](y))$. But f is nondecreasing, so that $c(y) \geq f\left(m \int_B dz D(y - z)\right)$. Using the symmetry and the non-increasing properties of D, one easily deduces that since $y \in B$, $\int_B dz D(y - z) \geq \int_{[0,a]^d} dz D(z) \geq 1/2^d - \epsilon$. Thus for all $y \in B$, $c(y) \geq f\left(m(1/2^d - \epsilon)\right) = g(m)$, which ends Step 2.
Step 3 The conclusion is immediate: using (7.10), Step 2, and (C), we obtain

$$\sup_{x\in\mathbb{R}^d} |c(x) - c_0| = \sup_{x\in\mathbb{R}^d} |Fc(x) - Fc_0(x)| \leq \frac{\mu}{\mu + \alpha\epsilon_0} \sup_{x\in\mathbb{R}^d} |c(x) - c_0|. \tag{7.12}$$

This implies that $\sup_{x\in\mathbb{R}^d} |c(x) - c_0| = 0$. □

Although the above uniqueness result seems quite promising, we are at the moment not able to prove that under the conditions of the previous proposition, any solution $(\xi_t)_{t\geq 0}$ to (7.1) starting from a non trivial initial condition converges to c_0 in some sense. One may however obtain two partial results.

> Assumption (DBC): $\alpha > 0$, $\gamma > 0$, $\mu = 0$ and $D = U$.

This assumption is a *detailed balance condition*. Indeed, under this condition, the equilibrium $c_0(x) \equiv \gamma/\alpha$ ensures that for any couple of points x and y, the rate of appearance of plants at x due to seed productions at y equals the rate of disappearance of plants at x because of competition of plants at y. In other words, $\gamma D(x-y)c_0(y) = \alpha c_0(x)c_0(y)U(x-y)$. Unfortunately, this condition is very restrictive.

Proposition 7.7 *Assume (B) and (DBC). Let ξ_0 be a positive bounded and measurable function on $\mathbb{R}^d$. Consider the associated unique solution $(\xi_t)_{t\geq 0}$ of (7.1) starting from ξ_0 obtained in Proposition 7.2. Then ξ_t tends to $c_0 = \gamma/\alpha$ as t grows to infinity in the sense that for all x, all t,*

$$[\xi_t(x) - c_0]^2 \leq [\xi_0(x) - c_0]^2 e^{-2\alpha[(\xi_0 \wedge c_0)\star D](x)t}. \tag{7.13}$$

We will furthermore see in the proof below that the behavior of ξ_t is quite simple: if $\xi_0(x) < c_0$, then $\xi_t(x)$ increases to c_0, while if $\xi_0(x) > c_0$, then $\xi_t(x)$ decreases to c_0.

Proof Since in this case, $\partial_t \xi_t(x) = -\alpha \xi_t \star D(x)(\xi_t(x) - c_0)$, we easily show that for all $t \geq 0$, all $x \in \mathbb{R}^d$,

$$\partial_t [\xi_t(x) - c_0]^2 = -2\alpha[\xi_t(x) - c_0]^2[\xi_t \star D](x). \tag{7.14}$$

Since ξ is non-negative, we deduce that $[\xi_t(x) - c_0]^2$ is non-increasing in t for each x. Since furthermore $\xi_t(x)$ is continuous in t for each x, we deduce that for any t, x, $\xi_t(x) \geq \xi_0(x) \wedge c_0$. Hence

$$\partial_t [\xi_t(x) - c_0]^2 \leq -2\alpha[\xi_t(x) - c_0]^2[(\xi_0 \wedge c_0) \star D](x), \tag{7.15}$$

from which the conclusion follows. □

We now treat quite a general case of coefficients α, γ, μ, U, D, but we consider an initial condition which is only a *small perturbation* of c_0.

Proposition 7.8 *Assume (B), (C), that $\alpha > 0$, and that U is bounded below by a positive continuous function h on $\mathbb{R}^d$. Consider a non-negative bounded measurable function ξ_0 on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} [\xi_0(x) - c_0]^2 dx < \infty$. Consider the associated unique solution $(\xi_t)_{t\geq 0}$ of (7.1) starting from ξ_0 obtained in Proposition 7.2. Then ξ_t tends to c_0 as t grows to infinity in the sense that there exists $a > 0$ such that for all t,*

$$\int_{\mathbb{R}^d} [\xi_t(x) - c_0]^2 dx \leq e^{-at} \int_{\mathbb{R}^d} [\xi_0(x) - c_0]^2 dx. \tag{7.16}$$

Proof We break the proof into three steps.

Step 1 A straightforward computation using Proposition 7.2-2, (7.8) and (7.10) shows that for all $t \geq 0$, all $x \in \mathbb{R}^d$,

$$\begin{aligned}
\partial_t [\xi_t(x) - c_0]^2 &= 2[\xi_t(x) - c_0]\partial_t \xi_t(x) \\
&= 2[\xi_t(x) - c_0][\mu + \alpha(\xi_t \star U)(x)][F\xi_t(x) - \xi_t(x)] \\
&= 2[\xi_t(x) - c_0][\mu + \alpha(\xi_t \star U)(x)][F\xi_t(x) - Fc_0(x)] \\
&\quad +2[\xi_t(x) - c_0][\mu + \alpha(\xi_t \star U)(x)][c_0 - Fc_0(x)] \\
&= 2\mu[\xi_t(x) - c_0][(\xi_t - c_0) \star R](x) \\
&\quad -2[\xi_t(x) - c_0]^2[\mu + \alpha(\xi_t \star U)(x)] \\
&= -2\alpha[\xi_t(x) - c_0]^2(\xi_t \star U)(x) \\
&\quad -2\mu[\xi_t(x) - c_0][(\xi_t(x) - c_0) - \{(\xi_t - c_0) \star R\}(x)]
\end{aligned} \tag{7.17}$$

Integrating this differential inequality against time, we obtain

$$\begin{aligned}
[\xi_t(x) - c_0]^2 = [\xi_0(x) - c_0]^2 - 2\int_0^t ds \alpha[\xi_s(x) - c_0]^2[\xi_s \star U](x) \\
-2\int_0^t ds \mu[\xi_s(x) - c_0]\left\{[\xi_s(x) - c_0] - [(\xi_s - c_0) \star R](x)\right\} ds.
\end{aligned} \tag{7.18}$$

Thanks to (C), R is a probability measure. We furthermore know that ξ_t, and thus $\xi_t \star U$ is bounded on $[0, T] \times \mathbb{R}^d$ for each T. Thus an application of the Cauchy-Schwarz and Young inequalities yields

$$\int_{\mathbb{R}^d} dx[\xi_t(x) - c_0][(\xi_t(x) - c_0) \star R(x)] \leq \int_{\mathbb{R}^d} dx[\xi_t(x) - c_0]^2. \tag{7.19}$$

One easily deduces that for all $T \geq 0$, $\sup_{[0,T]} \int_{\mathbb{R}^d} dx[\xi_t(x) - c_0]^2 < \infty$. Hence equation (7.18) may be integrated on $x \in \mathbb{R}^d$, and we get that for all $t \geq 0$,

$$\partial_t \int_{\mathbb{R}^d} dx[\xi_t(x) - c_0]^2 \leq -2\alpha \int_{\mathbb{R}^d} dx[\xi_t(x) - c_0]^2[\xi_t \star U](x). \tag{7.20}$$

Step 2 We now wish to bound $[\xi_t \star U](x)$ from below. First, we deduce from (7.20) that $\int_{\mathbb{R}^d} dx[\xi_t(x) - c_0]^2$ is non-increasing. Hence there exists a constant $b < \infty$ such that for all $t \geq 0$,

$$\int_{\mathbb{R}^d} dx \mathbf{1}_{\{\xi_t(x) \leq c_0/2\}} \leq b. \tag{7.21}$$

But since $U(x) \geq h(x)$, for some positive continuous function h, there exists a constant $d > 0$ such that

$$\inf_{A \in \mathcal{B}(\mathbb{R}^d),\, \int_A dx \leq b} \int_{\mathbb{R}^d / A} dz U(z) \geq bd. \tag{7.22}$$

Indeed, choose any compact subset K of $\mathbb{R}^d$ whose Lebesgue measure equals $2b$, and set $d = \inf_{x \in K} h(x)$. Note that for all $A \in \mathcal{B}(\mathbb{R}^d)$ such that $\int_A dx \leq b$, one also has $\int_{K/A} dx \geq b$, so that

$$\int_{\mathbb{R}^d / A} dz U(z) \geq \int_{K/A} dz h(z) \geq bd. \tag{7.23}$$

Using finally (7.22) with $A = A_{t,x} = \{y \in \mathbb{R}^d,\ \xi_t(x-y) \geq c_0/2\}$, of which the Lebesgue measure is smaller that b thanks to (7.21), we obtain for all $x \in \mathbb{R}^d$, all $t \geq 0$,

$$[\xi_t \star U](x) = \int_{\mathbb{R}^d} dy \xi_t(x-y) U(y) \geq \frac{c_0}{2} \int_{A_{t,x}} dy U(y) \geq \frac{bdc_0}{2}. \tag{7.24}$$

Step 3 Gathering (7.20) and (7.24), we finally obtain

$$\partial_t \int_{\mathbb{R}^d} dx[\xi_t(x) - c_0]^2 \leq -bdc_0\alpha \int_{\mathbb{R}^d} dx[\xi_t(x) - c_0]^2 \tag{7.25}$$

from which the conclusion follows. □

7.2 Equilibrium of the BPDL process

We now would like to show that it might be possible to find an equilibrium for the BPDL processes. This is a first step to study the long time behavior of the Bolker-Pacala process $(\nu_t)_{t \geq 0}$ defined in Definition 2.5 conditioned on non-extinction. We will unfortunately be able to treat only the case where the detailed balance condition holds. Of course, such an equilibrium will be infinite (that is the number of plants is infinite). One may however state the following rigorous result.

We first of all denote by $\bar{\mathcal{M}}$ the set of non-negative (possibly infinite) integer-valued measures on $\mathbb{R}^d$. We also denote by $\mathcal{A}$ the set of functions ϕ from $\bar{\mathcal{M}}$ into $\mathbb{R}$ of the form $\phi(\nu) = F(\langle \nu, f \rangle)$, for some bounded measurable function F on $\mathbb{R}$ and some function f with compact support on $\mathbb{R}^d$.

Proposition 7.9 *Assume* (B) *and* (DBC) *(see Subsection 7.1), and that* $U(0) = 0$. *Consider a Poisson measure* π *on* $\mathbb{R}^d$ *with intensity measure* $c_0 dx$, *where* $c_0 = \gamma/\alpha$. *Then* π *is a stationary BPDL process, in the sense that for all* $\phi \in \mathcal{A}$, $L\phi(\pi)$ *a.s. exists, belongs to* L^1, *and* $E[L\phi(\pi)] = 0$, *where* L *is defined in (2.3).*

Note that assuming (DBC) and that $U(0) = 0$ implies that there is no *natural death*. Remark also that this result is somewhat surprising, since it suggests that at equilibrium, the plants locations are independent. Let us finally mention that a similar result without assumption (DBC) would be much more interesting. However, the stationary process π does not seem to be Poisson in such a case.
The proof relies on the following lemma, known as Slivnyak's formula in Moller [13] and also obtained from Palm measure considerations (see Kallenberg [8], chap. 10).

Lemma 7.10 *Let ν be a Poisson measure on $\mathbb{R}^d$ with intensity $m(dx)$. Denote by $\{x_i\}_{i\geq 1}$ the points of ν, that is $\nu = \sum_{i\geq 1}\delta_{x_i}$. Then for all measurable functions h from $\mathbb{R}^d \times \bar{\mathcal{M}}$ into $\mathbb{R}$ such that $\int_{\mathbb{R}^d} m(dx)E[|h(x,\nu+\delta_x)|] < \infty$,*

$$E\left[\sum_{i\geq 1} h(x_i,\nu)\right] = \int_{\mathbb{R}^d} m(dx)E[h(x,\nu+\delta_x)]. \tag{7.26}$$

Proof of Proposition 7.9 Let ϕ belong to $\mathcal{A}$. The fact that $L\phi(\pi)$ a.s. exists and belongs to L^1 for $\phi \in \mathcal{A}$ can be easily checked, using the explicit expression of L, and standard results about Poisson measures. We thus only prove that $E[L\phi(\pi)] = 0$. Denote by $\{x_i\}_{i\geq 1}$ the points of π, that is $\pi = \sum_{i\geq 1}\delta_{x_i}$. Hence, we obtain, using (DBC),

$$\begin{aligned} E[L\phi(\pi)] &= \gamma E\left[\sum_{i\geq 1}\int_{\mathbb{R}^d} dzD(z)\left\{\phi(\pi+\delta_{x_i+z})-\phi(\pi)\right\}\right] \\ &\quad +\alpha E\left[\sum_{i\geq 1}\left\{\phi(\pi-\delta_{x_i})-\phi(\pi)\right\}\sum_{j\geq 1} D(x_i-x_j)\right] \\ &= \gamma A_1 + \alpha A_2, \end{aligned} \tag{7.27}$$

where the last equality stands for a definition. We first use Lemma 7.10 with the function $h_1(x,\nu) = \int_{\mathbb{R}^d} dzD(z)\left\{\phi(\nu+\delta_{x+z})-\phi(\nu)\right\}$.

$$A_1 = E\left[\sum_{i\geq 1} h_1(x_i,\pi)\right] = \int_{\mathbb{R}^d} c_0 dxE\left[\int_{\mathbb{R}^d} dzD(z)\left\{\phi(\pi+\delta_x+\delta_{x+z})-\phi(\pi+\delta_x)\right\}\right]. \tag{7.28}$$

Next, with $h_2(x,\nu) = \{\phi(\nu-\delta_x)-\phi(\nu)\}\int_{\mathbb{R}^d}\nu(dy)D(x-y)$, we obtain

$$A_2 = E\left(\sum_{i\geq 1} h_2(x_i,\pi)\right) = \int_{\mathbb{R}^d} dxc_0E\left[\{\phi(\pi)-\phi(\pi+\delta_x)\}\int_{\mathbb{R}^d}(\pi+\delta_x)(dy)D(x-y)\right]. \tag{7.29}$$

Since $D(0) = U(0) = 0$, we obtain, setting $h_3^x(y,\nu) = D(x-y)\left\{\phi(\nu)-\phi(\nu+\delta_x)\right\}$,

$$A_2 = \int_{\mathbb{R}^d} dxc_0E\left(\sum_{j\geq 1} h_3^x(x_j,\pi)\right). \tag{7.30}$$

Using Lemma 7.10 again, we obtain

$$\begin{aligned} A_2 &= \int_{\mathbb{R}^d} dxc_0\int_{\mathbb{R}^d} dyc_0E\left[D(x-y)\left\{\phi(\pi+\delta_y)-\phi(\pi+\delta_x+\delta_y)\right\}\right] \\ &= c_0^2\int_{\mathbb{R}^d} dx\int_{\mathbb{R}^d} dzE\left[D(z)\left\{\phi(\pi+\delta_x)-\phi(\pi+\delta_{x+z}+\delta_x)\right\}\right], \end{aligned} \tag{7.31}$$

where we have used in the last equality the substitution $(y,x) \mapsto (x, x+z)$. Since $\alpha c_0^2 = \gamma c_0$, we deduce that $\gamma A_1 = -\alpha A_2$, which ends the proof. □

7.3 Simulations

The previous results suggest that the BPDL process, conditioned on non extinction, should converge as time tends to infinity, to a random measure ν_∞, quite well-distributed (not far from the Lebesgue measure), with $(\gamma - \mu)/\alpha$ plants per unit of volume on average. We would like to present simulations about this fact.

We assume that $\bar{\mathcal{X}} = \mathbb{R}$, that $\gamma = 5$, $\mu = 1$, and $\alpha = 1$. We consider the case where $U(x,y) = \mathbf{1}_{\{|x-y|\leq 1/2\}}$ and $D(z) = \frac{1}{6}\mathbf{1}_{\{|z|\leq 3\}}$. Then we we compare the BPDL process $(\nu_t)_{t\geq 0}$ with the stationary solution $c_0(dx) = [(\gamma-\mu)/\alpha]dx$ of (7.1).
On Figure 1, we assume that $\nu_0 = \delta_0$. The boxes represent the empirical density of the BPDL process at times $t = 3$ (1a) and then $t = 25$ (1b), obtained with one simulation, while the dotted line is the density of c_0, i.e. $(\gamma - \mu)/\alpha$. One checks that after some time, the BPDL process is quite well-approximated by c_0.

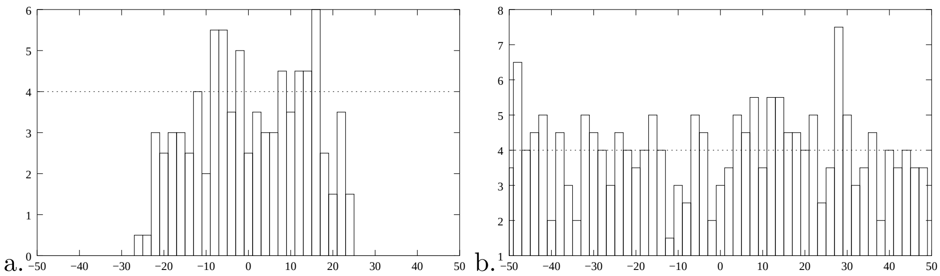

Figure 1: a. $t = 3$, b. $t = 25$

On Figure 2, we show the evolution in time of $\nu_t([-5,5])$ (full line), either starting from $\nu_0 = \delta_0$ (2a) or from $\nu_0 = 60\delta_0$ (2b), and compare it with $c_0([-5,5]) = 10(\gamma-\mu)/\alpha$ (dotted line).

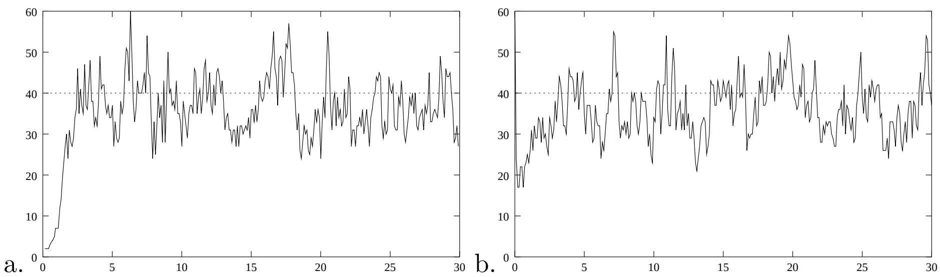

Figure 2: a. $\nu_0 = \delta_0$, b. $\nu_0 = 60\delta_0$

Finally, we would like to measure the power of competition. To this end, we compare the evolution in time of the rate of interaction of all particles on particles located in a ball, in the case of the BPDL process and in the case of c_0. We assume that $\nu_0 = \delta_0$. Figure 3a represents, in full line, the evolution in time of $\int_{\mathbb{R}} \nu_t(dx) \int_{\mathbb{R}} \nu_t(dy) \mathbf{1}_{|x|\leq 5} U(x,y)$, obtained by one simulation. The constant value (dotted line) is $\int_{\mathbb{R}} c_0(dx) \int_{\mathbb{R}} c_0(dy) \mathbf{1}_{|x|\leq 5} U(x,y) = 2 * 5 * [(\gamma-\mu)/\alpha]^2$. Figure 3b shows the same quantities replacing 5 by 50.

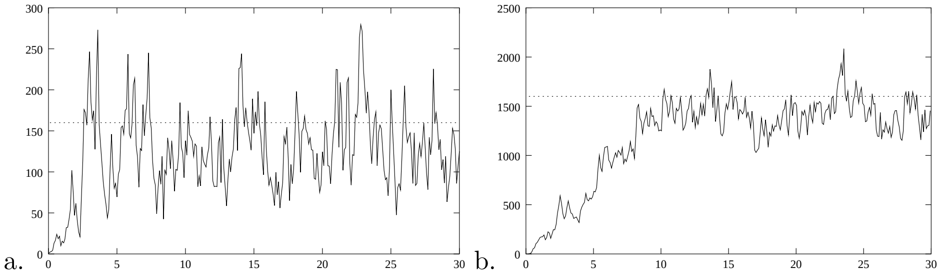

Figure 3: Rate of interaction endured by all particles in $[-5, 5]$ (a) or $[-50, 50]$ (b).

As a conclusion, one can say that on one hand, c_0 seems a good deterministic approximation of the BPDL process after a long time, but on the other hand, there are clearly stochastic fluctuations around the deterministic approximation, that it could be interesting to study.

Acknowledgments The authors wish to thank Régis Ferrière and Bernard Roynette for numerous fruitful discussions. We are very grateful to the anonymous referee for his many suggestions.

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