Rate of convergence of a stochastic particle system for the Smoluchowski coagulation equation

Madalina DEACONU, Nicolas FOURNIER and Etienne TANRÉ
IECN - INRIA Lorraine, Campus Scientifique, BP 239
54506 Vandoeuvre lès Nancy Cedex, FRANCE
mdeaconu@loria.fr, fournier@iecn.u-nancy.fr, tanre@iecn.u-nancy.fr

Abstract

By continuing the probabilistic approach of [DFT01], we derive a stochastic particle approximation for the Smoluchowski coagulation equations. A convergence result for this model is obtained.
Under quite stringent hypothesis we obtain a central limit theorem associated with our convergence. In spite of these restrictive technical assumptions, the rate of convergence result is interesting because it is the first obtained in this direction and seems to hold numerically under weaker hypothesis. This result answers a question closely connected to the Open Problem 16 formulated by Aldous [Ald99].

Key words : Smoluchowski coagulation equation, interacting stochastic particle systems, Monte Carlo methods, central limit theorem.
MSC 2000 : 82C22, 60H30, 60J75, 35M10.

1 Introduction

In its famous review, Aldous [Ald99] presents the Marcus-Lushnikov particle system (Marcus [Mar68], Lushnikov [Lus78]) as an approximation for the solution of the Smoluchowski equation. Convergence results for this scheme have been obtained by Jeon [Jeo98] (for the discrete coagulation-fragmentation model approached by Markov chains), Norris [Nor99], [Nor00] (for the continuous coagulation equation).
In its Open Problem 16, Aldous wonders about the existence of a central limit theorem associated with this approximation.
More recently, Eibeck and Wagner [EW01] have introduced a new class of stochastic algorithms in which the number of particles is constant in time. This approach has been extended to the discrete coagulation-fragmentation case by Jourdain [Jou01].
In [DFT01], we introduced a nonlinear process $X = \{X_t\}_t$ such that $\{\mathcal{L}(X_t)\}_t$ is solution to the Smoluchowski equation. The nonlinear process X is a richer structure than the Smoluchowski equation, since it provides historical information on the particle behavior. This process is "physically correct", since it can be obtained as a weak limit of the size of the cluster containing a "marked" particle in the Marcus-Lushnikov process.
Linearizing the nonlinear process built in [DFT01] leads to a particle system, which is the

same as that introduced in [EW01].
In the present paper, we prove:
(i) A new convergence result for the particle system (Theorem 3.3 and Corollary 3.5), which excludes the case of gelation (treated in [EW01]). Our result however applies to the case of additive coagulation kernels, which is one of the most important examples. We furthermore allow the initial total concentration to be infinite. Finally, our convergence holds in a strong sense: we replace the "vague" topology of [EW01] by a weak topology, and the convergence holds in terms of laws of stochastic processes, instead of families of laws of random variables.
(ii) This convergence result leads to a new existence result (Corollary 3.4) for the Smoluchowski equation, allowing the initial total concentration to be infinite in the case of any continuous subadditive coagulation kernel.
(iii) We prove a propagation of chaos result (Proposition 3.7), in the strong variation norm: any k-uple of particles become independent as the total number of particles tends to infinity.
(iv) In the discrete case, for a bounded coagulation kernel, we prove a central limit theorem (Theorem 4.4) associated with our convergence result. The result is not completely satisfying, since the assumptions are strong. It is however the first one in that direction, concerning any Monte-Carlo scheme for the Smoluchowski equation, and seems important. To prove this fluctuation result, we follow the proof scheme of Méléard, [Mél98], who was concerned with a similar problem on the Boltzmann equation. We however can not apply directly the result of [Mél98] because she works in functional spaces adapted to the Boltzmann equation (weighted Sobolev spaces on $\mathbb{R}^d$), which can clearly not be used for the Smoluchowski equation.

Let us finally mention that the methods used in [DFT01] and also in the present paper are inspired by probabilistic works on Boltzmann equation. We refer to Tanaka [Tan79], Graham-Méléard [GM97] and Méléard [Mél98].

The present paper is structured as follows. In Section 2, we recall the Smoluchowski equation, define its weak solutions, and introduce a related nonlinear stochastic differential equation. Section 3 "linearizes" the nonlinear SDE. After linearization we obtain a particle system, easily simulable, which is the same as the one in Eibeck-Wagner [EW01]. We prove a convergence result and a propagation of chaos property for this particle system. In Section 4, we prove a central limit theorem associated to our Monte Carlo method. We consider only the discrete case, because the arguments are very technical. We obtain a very precise result, under the quite stringent assumption that the coagulation kernel K is bounded. Section 5 gives numerical results, which illustrate the results of the present paper, and show that the central limit theorem seems to apply also in cases of unbounded kernels. An Appendix lies at the end of the paper.

2 Notations and previous results

The Smoluchowski equation describes the time evolution of the average number of particles having certain mass, in a dynamic particle system where coagulation phenomena in pairs occur. The equation writes in the form of the following infinite dimensional system : for all $x \geq 1$

$$\frac{d}{dt}\mu_t(x) = \frac{1}{2}\sum_{y=1}^{x-1}\mu_t(y)\mu_t(x-y)K(y,x-y) - \mu_t(x)\sum_{y=1}^{\infty}\mu_t(y)K(x,y) \tag{2.1}$$

where $\mu_t(x)$ notes the densities of particles of mass $x \in \mathbb{N}^*$ at time t, and K marks the coagulation kernel, supposed to be symmetric and positive. A continuous version of this equation also exists, in which the particles' sizes may take their values in the whole $(0,\infty)$. We consider here a weak form of the equation (2.1), which gather together discrete and continuous versions of the Smoluchowski equation (we refer to Norris [Nor99] for more details on this approach). More precisely, let $K : \mathbb{R}_+^* \times \mathbb{R}_+^* \mapsto [0,\infty)$, be a symmetric coagulation kernel (*i.e.* $K(x,y) = K(y,x)$). Let also $C_b^1(\mathbb{R}_+)$ stand for the set of bounded C^1 functions on $\mathbb{R}_+$ with a bounded derivative.

Definition 2.1 *Let $T_0 \leq \infty$ be fixed. A family $\{\mu_t\}_{t\in[0,T_0)}$ of nonnegative measures on $\mathbb{R}_+^*$ is said to be a weak solution to the Smoluchowski equation if*
- *for all $t \in [0,T_0)$, $\int_{\mathbb{R}_+} x\mu_t(dx) = 1$,*
- *for all $t \in [0,T_0)$, $\sup_{s\in[0,t]}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} xyK(x,y)\mu_s(dx)\mu_s(dy) < \infty$,*
- *for all $t \in [0,T_0)$ and all test function φ on $\mathbb{R}_+$ such that $\varphi(x)/x \in C_b^1(\mathbb{R}_+)$,*

$$\int_{\mathbb{R}_+}\varphi(x)\mu_t(dx) = \int_{\mathbb{R}_+}\varphi(x)\mu_0(dx) + \frac{1}{2}\int_0^t\int_{\mathbb{R}_+^2}[\varphi(x+y)-\varphi(x)-\varphi(y)]\,K(x,y)\mu_s(dx)\mu_s(dy)ds. \tag{2.2}$$

With this definition, for an $\mathbb{N}^*$-supported initial condition, we recover the discrete case. For measures which are absolutely continuous with respect to the Lebesgue measure, we recover the continuous case. We refer to Norris [Nor99] and Deaconu-Fournier-Tanré [DFT01], for more details on this topic.

Let us recall the main notations and results for the probabilistic interpretation introduced in [DFT01]. The main property which allows us a probabilistic approach is that we have conservation of mass in the system. This means that for some $T_0 > 0$ and $\{\mu_t\}_{t\in[0,T_0)}$ a weak solution to the Smoluchowski equation, the quantity $Q_t(dx) = x\mu_t(dx)$, is a probability measure on $\mathbb{R}_+$ for all $t \in [0,T_0)$, and has to be understood as the distribution of particles' mass at some time t. Let $\mathcal{P}_1$ denote the set of probability measure on $\mathbb{R}_+^*$ having finite first order moment.
For $Q_0 \in \mathcal{P}_1$, we denote by $\mathcal{H}_{Q_0} = \overline{\{\sum_{i=1}^n x_i \ ; \ x_i \in \text{Supp } Q_0 \ , \ n \in \mathbb{N}^*\}}^{\mathbb{R}_+}$, the smallest closed subset of $\mathbb{R}_+$ in which the sizes of the particles will always take their values. Notice that in the discrete case $\mathcal{H}_{Q_0}$ is contained in $\mathbb{N}^*$, while in the continuous case, $\mathcal{H}_{Q_0}$ is contained in the whole $\mathbb{R}_+$. Notice also that in any case, any "physical" solution $\{\mu_t\}_t$ to the Smoluchowski equation satisfies, for each t, Supp $\mu_t \subset \mathcal{H}_{Q_0}$.
We introduce now the weak form corresponding to $Q_t(dx)$. As in [EW01], we call it the *mass flow equation.*

Definition 2.2 *Let Q_0 belong to $\mathcal{P}_1$ and $T_0 \leq \infty$. A family $\{Q_t\}_{t\in[0,T_0)}$, of probability measures on $\mathbb{R}_+^*$ is a weak solution to (MS) if:*
- *for all $t \in [0,T_0)$,* Supp $Q_t \subset \mathcal{H}_{Q_0}$,
- *for all $t \in [0,T_0)$, $\sup_{s\in[0,t]}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K(x,y)Q_s(dx)Q_s(dy) < \infty$,*

• *for all test function* $\varphi \in C_b^1(\mathbb{R}_+)$ *and all* $t \in [0, T_0)$,

$$(MS)\quad \int_{\mathbb{R}_+} \varphi(x) Q_t(dx) = \int_{\mathbb{R}_+} \varphi(x) Q_0(dx) + \int_0^t \int_{\mathbb{R}_+^2} [\varphi(x+y) - \varphi(x)] \frac{K(x,y)}{y} Q_s(dy) Q_s(dx) ds.$$

Clearly, if $\{Q_t\}_{t\in[0,T_0)}$ is a weak solution to (MS), then, by setting $\mu_t(dx) = x^{-1}Q_t(dx)$, we get a "physical" weak solution to the Smoluchowski equation, and reversely.
Equation (MS) can be interpreted as the evolution equation of the time marginals of a pure jump Markov process. In order to exploit this remark, we associate to (MS) a martingale problem. We introduce first some notations.

Notation 2.3 *Let* $T_0 \le \infty$ *and* $Q_0 \in \mathcal{P}_1$ *be fixed. We denote by* $\mathbb{D}^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$ *the set of positive non-decreasing càdlàg functions from* $[0,T_0)$ *into* $\mathcal{H}_{Q_0}$. *Let* $\mathcal{P}_1^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$ *be the set of probability measures* Q *on* $\mathbb{D}^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$ *such that* $Q(x(0) > 0) = 1$ *and*

$$(2.3)\qquad \forall t < T_0, \int_{x\in\mathbb{D}^{\uparrow}([0,T_0),\mathcal{H}_{Q_0})} x(t) Q(dx) < \infty.$$

Definition 2.4 *Let* $T_0 \le \infty$, *and* $Q_0 \in \mathcal{P}_1$ *be fixed. Consider* $Q \in \mathcal{P}_1^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$. *Let* Z *be the canonical process of* $\mathbb{D}^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$. *Denote by* Q_s *the law of* Z_s *under* Q. *We say that* Q *is a solution to the martingale problem* (MP) *on* $[0,T_0)$ *if*
• *for all* $t \in [0,T_0)$, $\sup_{s\in[0,t]} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K(x,y) Q_s(dx) Q_s(dy) < \infty$,
• *for all* $\varphi \in C_b^1(\mathbb{R}_+)$ *and all* $t \in [0,T_0)$,

$$(MP)\qquad \varphi(Z_t) - \varphi(Z_0) - \int_0^t \int_{\mathbb{R}_+} [\varphi(Z_s + y) - \varphi(Z_s)] \frac{K(Z_s,y)}{y} Q_s(dy) ds$$

is a Q-L^1-*martingale.*

Thus, by taking expectations in (MP), we obtain that, if Q is a solution to the martingale problem (MP) on $[0,T_0)$, then $\{Q_t\}_{t\in[0,T_0)}$ is a weak solution of (MS).
We are now seeking for a pathwise representation of the martingale problem (MP). To this aim, let us introduce some notations. The main ideas of the following notations and definitions are taken from Tanaka [Tan79].

Notation 2.5 *We consider two probability spaces:* $(\Omega, \mathcal{F}, \mathbb{P})$ *is an abstract space and* $([0,1], \mathcal{B}[0,1], d\alpha)$ *is an auxiliary space (here* $d\alpha$ *denotes the Lebesgue measure). In order to avoid confusions, the elements on* $[0,1]$ *will be called* α-*elements.*
Let $T_0 \le \infty$ *and* $Q_0 \in \mathcal{P}_1$ *be fixed. A non-decreasing positive càdlàg process* $\{X_t(\omega)\}_{t\in[0,T_0)}$ *is said to belong to* $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$ *if its law belongs to* $\mathcal{P}_1^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$.
In the same way, a non-decreasing positive càdlàg α-*process* $\{\tilde{X}_t(\alpha)\}_{t\in[0,T_0)}$ *is said to belong to* $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$-$\alpha$ *if its* α-*law belongs to* $\mathcal{P}_1^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$.

Definition 2.6 *Let* $T_0 \le \infty$ *and* $Q_0 \in \mathcal{P}_1$ *be fixed. We say that* $(X_0, X, \tilde{X}, N)$ *is a solution to the problem* (SDE) *on* $[0,T_0)$ *if:*

1. $X_0 : \Omega \to \mathbb{R}_+$ *is a random variable whose law is* Q_0.

2. $X_t(\omega) : [0,T_0) \times \Omega \to \mathbb{R}_+$ *is a* $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$-*process* .

3. $\tilde{X}_t(\alpha) : [0, T_0) \times [0, 1] \to \mathbb{R}_+$ *is a* $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$*-*$\alpha$*-process.*

4. $N(\omega, dt, d\alpha, dz)$ *is a Poisson measure on* $[0, T_0) \times [0, 1] \times \mathbb{R}_+$ *with intensity measure* $dt\, d\alpha\, dz$ *and independent of* X_0.

5. X *and* $\tilde{X}$ *have same law :* $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ *(this equality holds in* $\mathcal{P}_1^{\uparrow}([0, T_0), \mathcal{H}_{Q_0})$*).*

6. *For all* $t \in [0, T_0)$, $\sup_{s\in[0,t]} \mathbb{E}\mathbb{E}_\alpha[K(X_s, \tilde{X}_s)] < \infty$.

7. *Finally, the following S.D.E. is satisfied on* $[0, T_0)$*:*

$$(SDE) \qquad X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) 1\!\!1_{\left\{z \le \frac{K(X_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, dz).$$

We recall the following result (see [DFT01]) : if $(X_0, X, \tilde{X}, N)$ is a solution to (SDE) on $[0, T_0)$, then the law $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ satisfies the martingale problem (MP) on $[0, T_0)$ with initial condition $Q_0 = \mathcal{L}(X_0)$. Hence $\{\mathcal{L}(X_t)\}_{t\in[0,T_0)}$ is a weak solution to the mass flow equation (MS) with initial condition Q_0.
In the sequel, we will assume the following hypothesis.

Assumption (H_β): Q_0 belongs to $\mathcal{P}_1$, $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$, and the symmetric kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous on $(\mathcal{H}_{Q_0})^2$, and satisfies, for some constant $C_K < \infty$ and some $\beta \in [0, 1]$, $K(x, y) \le C_K(1 + x + y + x^\beta y^\beta)$.

Let us recall the main results of [DFT01].

Theorem 2.7 *Assume* (H_β) *and that* K *is locally Lipschitz continuous on* $(\mathcal{H}_{Q_0})^2$*. If* $\beta \le 1/2$*, set* $T_0 = \infty$*, else set* $T_0 = 1/C_K(1 + \int_{\mathbb{R}_+} xQ_0(dx))$*. Then there holds existence for* (SDE) *and* (MP) *on* $[0, T_0)$*.*
If furthermore, Supp $Q_0 \subset \mathbb{N}^*$ *and* $\beta \le 1/2$*, then uniqueness holds for* (MP)*, and uniqueness in law holds for* (SDE)*.*

It is also proved in [DFT01] that the solution X to (SDE) can be obtained as a weak limit of the size of the cluster containing a "marked" particle, in a Marcus-Lushnikov process. The solution of (2.1) has naturally different behaviors according to the value of β. In the case $\beta \in [0, 1/2]$, the solution is defined on the time interval $[0, \infty)$. For $\beta > 1/2$, we may have gelation in finite time. This means that $T_{gel} = \inf\{t \ge 0, \int_{\mathbb{R}_+} x^2 Q_t(dx) = \infty\}$ is finite. For physical reasons we consider here only solutions that preserve mass, so they are defined up to T_{gel}.

3 An associated particle system

The aim of this section is to solve numerically the Smoluchowski coagulation equation, by constructing an approximation scheme for $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$, where $(X_0, X, \tilde{X}, N)$ is a solution to (SDE). Due to $\tilde{X}$, the system is nonlinear so we cannot approach directly X. The natural way to get rid of this nonlinearity is to construct an interacting particle system. For technical (but rather serious) reasons, we restrict our study, for the moment, to the case (H_β) with $\beta = 1/2$. We explain at the end of this section how to treat $\beta = 1$. Let us define the "linearized" version of the nonlinear stochastic differential equation (SDE).

Definition 3.1 *Let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a symmetric kernel. Let $Q_0 \in \mathcal{P}_1$ and $n \in \mathbb{N}^*$ be fixed. Consider a family $\{X_0^{i,n}\}_{i\in\{1,\dots,n\}}$ of i.i.d. Q_0-distributed random variables. Consider also a family $\{N^i(ds,dj,dz)\}_{i\in\{1,\dots,n\}}$ of i.i.d. Poisson measures on $[0,\infty) \times \{1,\dots,n\} \times [0,\infty)$ with intensity measures $ds\left(\frac{1}{n}\sum_{k=1}^n \delta_k(dj)\right)dz$.*
A process $X^n = (X^{1,n},\dots,X^{n,n})$, with values in $\left[\mathbb{D}^\uparrow([0,\infty),\mathcal{H}_{Q_0})\right]^n$, is said to solve $(PS)_n$ if for all $i \in \{1,\dots,n\}$ and all $t \in [0,\infty)$

$$(PS)_n \qquad X_t^{i,n} = X_0^{i,n} + \int_0^t \int_j \int_0^\infty X_{s-}^{j,n} \mathbb{1}_{\left\{z \le \frac{K\left(X_{s-}^{i,n}, X_{s-}^{j,n}\right)}{X_{s-}^{j,n}}\right\}} N^i(ds,dj,dz).$$

For X^n a solution to $(PS)_n$, we will denote by $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}}$, the empirical distribution (it is a random probability measure on $\mathbb{D}^\uparrow([0,\infty),\mathcal{H}_{Q_0})$).

$(PS)_n$ is well-defined. Indeed we have the following proposition.

Proposition 3.2 *Let $Q_0 \in \mathcal{P}_1$. Assume (H_β) with $\beta = 1/2$ and let $n \in \mathbb{N}^*$ be fixed. Then there exists an unique solution $X^n = (X^{1,n},\dots,X^{n,n})$ to $(PS)_n$. This solution is exactly simulable.*

The proof is classical, since the $\mathbb{R}^n$-valued Markov process X^n is clearly piecewise constant, so that $(PS)_n$ is "self-solved". We will omit it.
We have also a tightness (weak relative compactness) and a convergence result of this system. More precisely :

Theorem 3.3 *Let $Q_0 \in \mathcal{P}_1$. Assume (H_β) with $\beta = 1/2$. Consider, for each n, the solution X^n to $(PS)_n$, and its associated empirical measure μ^n.*

1. *The sequence $(\mathcal{L}(\mu^n))_{n\ge1}$ is tight in $\mathcal{P}\left(\mathcal{P}\left(\mathbb{D}^\uparrow([0,\infty),\mathcal{H}_{Q_0})\right)\right)$ (the set $\mathcal{P}\left(\mathbb{D}^\uparrow([0,\infty),\mathcal{H}_{Q_0})\right)$ being endowed with the weak convergence topology associated with the Skorokhod topology on $\mathbb{D}^\uparrow([0,\infty),\mathcal{H}_{Q_0})$).*

2. *Any limiting point π of $(\mathcal{L}(\mu^n))_{n\ge1}$ satisfies* Supp $\pi \subset \{$*solutions to* $(MP)\}$.
This implies that there exists a subsequence $\{\mu^{n_k}\}_k$ which converges in law, for the weak topology of $\mathcal{P}\left(\mathbb{D}^\uparrow([0,\infty),\mathcal{H}_{Q_0})\right)$ (associated with the Skorokhod topology on $\mathbb{D}^\uparrow([0,\infty),\mathcal{H}_{Q_0})$), to some random probability measure μ, and that μ is a.s. a solution to (MP).

Let us first notice that this result implies a new existence theorem for the Smoluchowski equation, allowing the initial total concentration $\int_{\mathbb{R}_+} \mu_0(dx)$ to be infinite. This is only a remark, our main aim not being here to obtain new existence results.

Corollary 3.4 *Let μ_0 be a nonnegative measure on $\mathbb{R}_+^*$ such that $\int_{\mathbb{R}_+} x\mu_0(dx) = 1$. Let $Q_0(dx) = x\mu_0(dx)$, and assume (H_β) with $\beta = 1/2$. Then there exists a weak solution to the Smoluchowski equation in the sense of the Definition 2.1.*

The following corollary deals with the case where uniqueness holds for (MP). Notice that since the projections $x \mapsto x(t)$ are not continuous on $\mathbb{D}^{\uparrow}([0,\infty), \mathcal{H}_{Q_0})$, we cannot *a priori* conclude, only by Theorem 3.3, that for each t fixed, μ_t^n converges to μ_t.

Corollary 3.5 *Let $Q_0 \in \mathcal{P}_1$. Assume (H_β) with $\beta = 1/2$. Assume also that uniqueness holds for (MP) (see Theorem 2.7 and also Corollary 5.5 in [DFT01]). Then we know from Theorem 3.3 that the empirical measure μ^n goes to the unique solution Q of (MP) in probability. (Recall that the convergence in law to some deterministic object implies the convergence in probability.) Then the $\mathcal{P}(\mathbb{R}_+)$-valued process $\{\mu_t^n\}_{t\geq 0} = \{\frac{1}{n}\sum_{i=1}^n \delta_{X_t^{i,n}}\}_{t\geq 0}$, converges in probability to $\{Q_t\}_{t\geq 0}$ in $\mathbb{D}([0,\infty), \mathcal{P}(\mathbb{R}_+))$. Here $\mathbb{D}([0,\infty), \mathcal{P}(\mathbb{R}_+))$ is endowed with the topology of the uniform convergence on every compact associated with the weak topology of $\mathcal{P}(\mathbb{R}_+)$.*

Let us now point out the link between the particle system and the classical Marcus-Lushnikov process: it is proved in [DFT01] that under suitable assumptions, the law of the process defined as *the size of the cluster containing a marked particle*, converges weakly to the solution Q of the martingale problem (MP).
In our particle system, the process $X^{1,n}$ has to be understood as the size of a cluster containing a marked particle, in a sort of "asymmetric" Marcus Lushnikov process. This asymmetry allows to have always n particles, and furnish a good representation of the whole system, while it is well known that the Marcus-Lushnikov is reduced to *one* particle in finite (large) time.
Let us finally compare our convergence result with the one in Eibeck-Wagner [EW01].

Remark 3.6 *Since they are interested in gelation, Eibeck and Wagner [EW01] do not consider exactly the same particle system as we do: they introduce a cutoff procedure in the coagulation kernel. They assume that $K(x,y) \leq h(x)h(y)$ for some positive function h such that $\lim_{x+y\to\infty} K(x,y)/h(x)h(y) = 0$ and $h(x)/x$ is non-increasing, and that the initial condition Q_0 satisfies $\int_{\mathbb{R}_+} x^{-1}h(x)Q_0(dx) < \infty$. It seems that the standard additive kernel $K(x,y) = x+y$ does not satisfy these assumptions. Our assumptions are not better nor less good, but different. They prove the weak convergence (up to extraction) of the process $\{\mu_t^n\}_{t\geq 0}$ to some $\{\mu_t\}_{t\geq 0}$ in $\mathcal{D}([0,\infty), (\mathcal{M}_+, d))$. $\{\mu_t\}_t$ is a.s. solution to (MS) ($(\mathcal{M}_+, d)$ is the set of finite nonnegative measures on $[0,\infty)$ endowed with the vague topology). This is a less strong convergence than ours (see e.g. Corollary 3.5).*

We now give the proof of Theorem 3.3. Since it is quite standard, we just present the main steps.
Proof of Theorem 3.3. Notice that for obvious reasons of symmetry, $\mathcal{L}(X^{i,n})$ is independent of $i \in \{1,\dots,n\}$, $\mathcal{L}(X^{i,n}, X^{j,n})$ is independent of $\{(i,j) \in \{1,\dots,n\}^2 \; ; \; i \neq j\}$, etc.
We break the proof in several steps.
Step 1: Using (H_β) with $\beta = 1/2$, the fact that $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$, Gronwall's Lemma, the symmetry of the system and the fact that the processes $X^{i,n}$ is *a.s.* non-decreasing, it is easily checked that for all $T < \infty$,

$$\text{(3.1)} \qquad \sup_n \sup_{i\in\{1,\dots,n\}} \mathbb{E}\left[\sup_{s\in[0,T]} (X_s^{i,n})^2\right] = \sup_n \mathbb{E}\left[(X_T^{1,n})^2\right] < \infty.$$

Step 2: It is known (see Méléard [Mél96], Lemma 4.5), that the tightness of μ^n is equivalent to that of $X^{1,n}$. Using the Aldous criterion (see Jacod-Shiryaev, [JS87]) and Step 1, the tightness of $\{\mu^n\}_{n\geq 1}$ is easily obtained.
Step 3: Let us consider a convergent subsequence of μ^n, that we still denote by μ^n, whose weak limit is μ, a random probability measure on $\mathbb{D}^{\uparrow}([0,\infty),\mathcal{H}_{Q_0})$.
We want to prove that μ satisfies *a.s.* the martingale problem (MP). To this aim, we consider $\phi \in C_b^1(\mathbb{R}_+)$, $g_1,\dots,g_k \in C_b(\mathbb{R}_+)$ and $0 \leq s_1 \leq \cdots \leq s_k < s < t < T_0$. Let F be the map from $\mathbb{D}^{\uparrow}([0,\infty),\mathcal{H}_{Q_0}) \times \mathbb{D}^{\uparrow}([0,\infty),\mathcal{H}_{Q_0})$ into $\mathbb{R}$ defined by

$$
\begin{aligned}
(3.2) \qquad & F(x,y) = g_1(x(s_1)) \cdots g_k(x(s_k)) \\
& \times \left\{ \phi(x(t)) - \phi(x(s)) - \int_s^t \left[\phi(x(u)+y(u)) - \phi(x(u))\right] \frac{K(x(u),y(u))}{y(u)} du \right\}.
\end{aligned}
$$

We have to prove that *a.s.*, $\langle \mu \otimes \mu, F \rangle = 0$. The map $Q \mapsto < Q \otimes Q, F >$ is not continuous nor bounded from $\mathcal{P}_1^{\uparrow}([0,T_0),\mathcal{H}_{Q_0})$ into $\mathbb{R}$. However, classical arguments (using the fact that the limiting point μ is *a.s.* the law of a process with quasi-left limit, and the uniform integrability obtain in Step 1) show that $\mathbb{E}(< \mu \otimes \mu, F >^2) = \lim_n \mathbb{E}(< \mu^n \otimes \mu^n, F >^2)$. It thus only remains to prove that $\mathbb{E}(\alpha_n)$ goes to 0, where $\alpha_n =< Q^n \otimes Q^n, F^2 >$. Setting for i fixed,

$$
(3.3) \qquad M_t^{i,n}(\phi) = \int_0^t \int_j \int_0^\infty \left[\phi(X_{u_-}^{i,n} + X_{u_-}^{j,n}) - \phi(X_{u_-}^{i,n})\right] \mathbb{1}_{\left\{z \leq \frac{K\left(X_{u_-}^{i,n}, X_{u_-}^{j,n}\right)}{X_{u_-}^{j,n}}\right\}} \bar{N}^i(ds,dj,dz)
$$

where $\bar{N}^i(ds,dj,dz) = N^i(ds,dj,dz) - ds\left(\frac{1}{n}\sum_{k=1}^n \delta_k(dj)\right) dz$ is the compensated Poisson measure associated with N^i, an easy computation shows that

$$
\begin{aligned}
\mathbb{E}(\alpha_n) &= \mathbb{E}\left[\left\{\frac{1}{n}\sum_{i=1}^n g_1(X_{s_1}^{i,n}) \dots g_k(X_{s_k}^{i,n}) \left[M_t^{i,n}(\phi) - M_s^{i,n}(\phi)\right]\right\}^2\right] \\
&= \frac{1}{n^2}\mathbb{E}\left[\sum_{i=1}^n \left\{g_1(X_{s_1}^{i,n}) \dots g_k(X_{s_k}^{i,n}) \left[M_t^{i,n}(\phi) - M_s^{i,n}(\phi)\right]\right\}^2\right] \\
&\quad + \frac{1}{n^2}\mathbb{E}\Big[\sum_{i=1}^n \sum_{j\neq i} g_1(X_{s_1}^{i,n}) \dots g_k(X_{s_k}^{i,n}) \left[M_t^{i,n}(\phi) - M_s^{i,n}(\phi)\right] \\
&\quad \times g_1(X_{s_1}^{j,n}) \dots g_k(X_{s_k}^{j,n}) \left[M_t^{j,n}(\phi) - M_s^{j,n}(\phi)\right]\Big] \\
(3.4) \qquad &= \mathbb{E}(\alpha_n^1) + \mathbb{E}(\alpha_n^2)
\end{aligned}
$$

with obvious notations for α_n^1 and α_n^2. By using (3.3) and (H_β) with $\beta = 1/2$, we see that for some constant A, depending only on the test function ϕ and on t,

$$
(3.5) \qquad \mathbb{E}(\alpha_n^1) \leq \frac{1}{n^2} n ||g_1||_\infty \dots ||g_k||_\infty \mathbb{E}\left[\left\{M_t^{1,n}(\phi) - M_s^{1,n}(\phi)\right\}^2\right] \leq \frac{A}{n}.
$$

On the other hand, the bracket $< M^{i,n}(\phi), M^{j,n}(\phi) >$ vanishes identically for all $i \neq j$, since the Poisson measures N^i and N^j are independent. It is easily deduced that $\mathbb{E}(\alpha_n^2) = 0$ for all n. Hence $\mathbb{E}(\alpha_n)$ goes to 0, which concludes the proof. $\square$

We now give the proof of Corollary 3.5, which relies on the use of Lemma 6.1 of the Appendix (due to Méléard [Mél96]).

Proof of Corollary 3.5 Thanks to Lemma 6.1, we just have to prove that for Q the unique solution to (MP) and any $T < \infty$ we have

$$\sup_{t\in[0,T]} \int_{x\in\mathbb{D}^{\uparrow}([0,T],\mathcal{H}_{Q_0})} \sup_{s\in[t-r,t+r]} (|\Delta x(s)| \wedge 1)\, Q(dx) \underset{r\to 0}{\longrightarrow} 0. \tag{3.6}$$

To this end, we use the fact that Q is the law of X, $(X_0, X, \tilde{X}, N)$ denoting a solution to (SDE). We thus have to show that

$$\sup_{t\in[0,T]} \mathbb{E}\left(\sup_{s\in[t-r,t+r]} (|\Delta X_s| \wedge 1) \right) \underset{r\to 0}{\longrightarrow} 0. \tag{3.7}$$

Since X is *a.s.* non-decreasing, we obtain, by using (H_β) with $\beta = 1/2$,

$$\begin{aligned}\sup_{t\in[0,T]} \mathbb{E}\left(\sup_{s\in[t-r,t+r]} (|\Delta X_s| \wedge 1) \right) &\le \sup_{t\in[0,T]} \mathbb{E}\left(X_{t+r} - X_{t-r}\right) \\ &\le \sup_{t\in[0,T]} \int_{t-r}^{t+r} \mathbb{E}\mathbb{E}_\alpha \left[K(X_s, \tilde{X}_s)\right] ds \le 2r\, C_K \sup_{s\in[0,T+r]} \mathbb{E}\left(1 + 3X_s\right)\end{aligned}$$

which clearly goes to 0 as r goes to 0, thanks to the fact that for all $T < \infty$, $\sup_{s\in[0,T]} \mathbb{E}\left(X_s\right) < \infty$. The corollary is now proved. □

We would also like to mention that propagation of chaos, in the total variation norm, holds for the particle system, under suitable assumptions. The propagation of chaos yields a sort of asymptotical independence of the particles. For $T > 0$ fixed, $k \in \mathbb{N}^*$, we denote by $|\nu|_T$ the total variation norm, of a measure on $\mathbb{D}([0,\infty), \mathbb{R}^k)$, restricted to $[0,T]$.

Proposition 3.7 *Let Q_0 belong to $\mathcal{P}_1$, and assume (H_β) with $\beta = 1/2$. Assume furthermore that there exists a constant B_K such that for all $x, y \in (\mathcal{H}_{Q_0})^2$, $K(x,y)/y \le B_K(1+x)$ (this assumption is quite restrictive in the continuous case, but always holds in the discrete case, i.e. for $\mathcal{H}_{Q_0} \subset \mathbb{N}^*$). Consider, for each n, a solution $(X^{1,n}, \cdots, X^{n,n})$ to $(PS)_n$. Then there is propagation of chaos in the sense that for each $k \in \mathbb{N}^*$ fixed, all $T \in [0,\infty)$,*

$$\left|\mathcal{L}(X^{1,n}, \cdots, X^{k,n}) - \mathcal{L}(X^{1,n}) \otimes \cdots \otimes \mathcal{L}(X^{k,n})\right|_T \underset{n\to\infty}{\longrightarrow} 0. \tag{3.8}$$

Proof Let us sketch the proof of this result. Graham and Méléard proved in [GM97] a similar result for the case of a particle system associated with a non homogeneous Boltzmann equation. Although we can not apply their result directly, one can follow their proof line by line in the case where the total rate of jump per particle is finite.
We consider first the particle system $(X^{1,n}, \cdots, X^{n,n})$ associated with the initial conditions $(X_0^{1,n}, \cdots, X_0^{n,n})$ and with the Poisson measures $N^{1,n}, \cdots, N^{n,n}$.
For each $M < \infty$, we denote by K^M the truncated coagulation kernel $K^M(x,y) = K(x \wedge M, y)$. Then we denote by $(X^{1,n,M}, \cdots, X^{n,n,M})$ the particle system associated with K^M, the initial conditions $\{X_0^{i,n}\}$ and the Poisson measures $\{N^{i,n}\}$. Since the total rate of

coagulation of each particle $X^{i,n,M}$ is bounded by $\Lambda_M = B_K(1+M)$, one can follow line by line the proof of [GM97] and obtain that for each k and each T,
(3.9)

$$\left|\mathcal{L}(X^{1,n,M},\cdots,X^{k,n,M}) - \mathcal{L}(X^{1,n,M})\otimes\cdots\otimes\mathcal{L}(X^{k,n,M})\right|_T \le k(k-1)\frac{\Lambda_M T + (\Lambda_M T)^2}{n}.$$

On the other hand, one may check that the particle system with coagulation kernel K^M does not differ too much from the one with K, in the sense that for any l,

$$\begin{aligned}
&\left|\mathcal{L}(X^{1,n,M},\cdots,X^{l,n,M}) - \mathcal{L}(X^{1,n},\cdots,X^{l,n})\right|_T \\
&\quad\le 2\mathbb{P}\left[\exists t\in[0,T],\ (X_t^{1,n},\cdots,X_t^{l,n}) \ne (X_t^{1,n,M},\cdots,X_t^{l,n,M})\right] \\
&\quad\le 2\mathbb{P}\left[\exists i\in\{1,\cdots,l\},\, X_T^{i,n}\ge M\right] \\
&\quad\le 2l\mathbb{P}\left[X_T^{1,n}\ge M\right] \le 2l\mathbb{E}[X_T^{1,n}]/M \le l\frac{A_T}{M},
\end{aligned} \tag{3.10}$$

thanks to the symmetry of the system and to (3.1). Indeed, one may check that for each i, $\inf\{s>0, X_s^{i,n,M}\ne X_s^{i,n}\} = \inf\{s>0, X_s^{i,n}\ge M\}$. Combining (3.9) and (3.10) yields (3.8). □

In order to conclude this section, we present the approximation scheme for the solution of the Smoluchowski equation in the case of (H_β) with $\beta=1$ (by making use of previous results).

As a matter of fact, we are not able (and this might be false) to prove that the particle system is well-defined when $\beta > 1/2$, nor that it is simulable. That's why we are led to consider a double approximation.

Remark 3.8 *Assume (H_β) with $\beta=1$. Let $T_0 = 1/C_K(1+\int_{\mathbb{R}_+} xQ_0(dx))$. Consider a kernel with cutoff $K^M(x,y) = K(x\wedge M, y\wedge M)$. Then K^M satisfies (H_β) with $\beta = 1/2$, and we may define the associated particle system: we obtain an empirical measure $\mu^{M,n}$. We know from Theorem 3.3 that the sequence $\mathcal{L}(\mu^{M,n})$ is tight, and that any limiting point μ^M satisfies a.s. the martingale problem with cutoff $(MP)_M$, obtained for (MP) by replacing K with K^M.*
On the other hand, the following result holds (it can be proved by following line by line the proofs of Lemmas 3.6, 3.7 and 3.8 of [DFT01]): consider, for each $M>0$ fixed, a solution Q^M to (MP). Then the sequence Q^M is tight in $\mathcal{P}(\mathbb{D}^\uparrow([0,T_0),\mathcal{H}_{Q_0}))$, and any limiting point Q is a solution to (MP).
Hence, if M and n are large enough, $\mu^{M,n}$ approaches a solution Q of (MP).

4 A central limit theorem in the discrete case

Our aim in this section is to prove a central limit type theorem, *i.e.* to show that the rate of convergence associated with our approximation is of order $1/\sqrt{n}$, and that the limit of the fluctuations is Gaussian, in a certain sense. We are inspired by the works of Ferland-Fernique-Giroux [FFG92] and Méléard [Mél98] for similar results concerning the Boltzmann equation and its particle approximating system.

However, we cannot apply their results and we have to deal with other "functional" spaces. We consider only the discrete case for simplicity, but it seems reasonable that a similar result may hold in the continuous case. However, the technical arguments are clearly much more easy in the discrete case. We will assume the strong hypothesis:

Assumption (A)**:** Q_0 has its support in $\mathbb{N}^*$ and admits a second order moment. The nonnegative symmetric coagulation kernel K is bounded.

This assumption is clearly not satisfying: K is in general unbounded. However, we are not able to get rid of this stringent hypothesis.
Under (A), one knows (see Theorem 2.7) that uniqueness of a solution Q holds for (MP), and that its marginal flow $\{Q_t\}_{t\geq 0}$ is the unique solution to (MS).
We recall here the notations of Definition 3.1 and explain the topologies we use in each space. Consider, for each n, a solution $X^n = (X^{1,n}, \dots, X^{n,n})$ to $(PS)_n$, lying *a.s.* in $\left(\mathbb{D}^\uparrow([0,\infty),\mathbb{N}^*)\right)^n$, associated with the Poisson measures $\{N^i(ds,dj,dz)\}_{i\in\{1,\dots,n\}}$ and with the initial particles $\left(X_0^{i,n}\right)_{i\in\{1,\dots,n\}}$. Denote by $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}}$, which takes its values in $\mathcal{P}\left(\mathbb{D}^\uparrow([0,\infty),\mathbb{N}^*)\right)$. For each $t\in[0,\infty)$, denote by $\mu_t^n = \frac{1}{n}\sum_{i=1}^n \delta_{X_t^{i,n}}$, the time marginal of μ^n, which takes its values in $\mathcal{P}(\mathbb{N}^*)$.
We know from Theorem 3.3 and the uniqueness for (MP) that the empirical measures μ^n converge in law to Q, the set $\mathcal{P}(\mathbb{D}^\uparrow([0,\infty),\mathbb{N}^*))$ being endowed with the weak topology associated with the Skorokhod topology on $\mathbb{D}^\uparrow([0,\infty),\mathbb{N}^*)$. We also know from Corollary 3.5 that the $\mathcal{P}(\mathbb{N}^*)$-valued process $\{\mu_t^n\}_{t\geq 0}$ goes in probability to $\{Q_t\}_{t\geq 0}$, in $\mathbb{D}([0,\infty),\mathcal{P}(\mathbb{N}^*))$ endowed with the topology of the uniform convergence on every compact subset (of $[0,\infty)$) for the weak topology of $\mathcal{P}(\mathbb{N}^*)$.
Let us now consider the fluctuation process:

$$\eta^n = \sqrt{n}(\mu^n - Q) \tag{4.1}$$

which, for each n, can be seen as a stochastic process with values in the set $\mathcal{M}(\mathbb{N}^*)$ of signed measures on $\mathbb{N}^*$. The aim is to prove that η^n converges weakly to a Gaussian process η, as n goes to infinity. In order to obtain this convergence, we have to introduce a "better" space than $\mathcal{M}(\mathbb{N}^*)$.

Remark 4.1 *Consider the space* $l^2 = \{(u_k)_{k\geq 0},\, u_k\in\mathbb{R},\, \sum_k u_k^2 < \infty\}$*, endowed with its natural norm* $||u||_{l^2} = \sqrt{\sum_{k\geq 1} u_k^2}$*.* l^2 *is an Hilbert space.*
Notice that any bounded (signed) measure $\nu = \{\nu(k)\}_{k\geq 1}$ *can be seen as an element of* l^2*.*
Remark also that l^2 *is not Polish when endowed with the weak topology.*
Finally notice that for each n*, each* t*,* η_t^n *belongs a.s. to* l^2*, and that, since* μ_t^n *and* Q_t *are probability measures,* $||\eta_t^n||_{l^2} \leq \sqrt{2n}$*.*

We need to introduce some random objects in order to formulate properly the main result. We will use these objects to define the limit law of η^n.

Definition 4.2 *Assume* (A) *and let* $(Q_t,\, t\geq 0)$ *be the unique solution of* (MS)*. Let* $\eta_0 = (\eta_0(k))_{k\geq 0}$ *be an* l^2*-valued random variable, and consider an* l^2*-valued stochastic process,* $W = (W_s(k),\, k\in\mathbb{N}^*,\, s\geq 0)$*. We say that* (η_0, W) *is of law* $(\mathcal{GP})$ *if:*

1. *The random infinite vector* $(\eta_0(1),\dots,\eta_0(k),\dots)$ *is a centered Gaussian vector of covariance: for all* k*,* l *in* $\mathbb{N}^*$*,*

$$\mathbb{E}(\eta_0(k)\eta_0(l)) = \begin{cases} Q_0(k) - Q_0^2(k) & \text{if } k = l \\ -Q_0(k)Q_0(l) & \text{if } k \neq l. \end{cases} \tag{4.2}$$

2. *The process W is strongly continuous from $[0,\infty)$ into l^2. For all $k_0 \in \mathbb{N}^*$, the real-valued process $W_.(k_0)$ is an $(\mathcal{F}_t,\, t\geq 0)$-martingale starting from 0, where for each $t \geq 0$,*

$$\mathcal{F}_t = \sigma\left\{\eta_0(k)\,;\; k\in\mathbb{N}^*\right\} \vee \sigma\left\{W_s(k)\,;\; 0\leq s\leq t,\; k\in\mathbb{N}^*\right\}. \tag{4.3}$$

For all k_1, k_2 in $\mathbb{N}^*$, $W(k_1)$ and $W(k_2)$ have the following (deterministic) Doob-Meyer bracket

$$\langle W(k_1), W(k_2)\rangle_t = \int_0^t \sum_{i\geq 1}\sum_{j\geq 1} Q_s(i)Q_s(j)\left(1\!\!1_{\{i+j=k_1\}} - 1\!\!1_{\{i=k_1\}}\right)\left(1\!\!1_{\{i+j=k_2\}} - 1\!\!1_{\{i=k_2\}}\right)\frac{K(i,j)}{j}ds.$$

3. *The random objects η_0 and W are independent.*

Notice that (η_0, W) is a Gaussian object, and that the law $(\mathcal{GP})$ is uniquely and completely defined. Let us carry on with the definition of a limit S.D.E.

Proposition 4.3 *Assume (A). Let (η_0, W) be a process of law $(\mathcal{GP})$. Then existence and uniqueness of a strongly continuous l^2-valued process η, satisfying the following S.D.E.,*

$$\eta_t(.) = \eta_0(.) + W_t(.) + \int_0^t \sum_{i\geq 1}\sum_{j\geq 1} L(.)(i,j)\left[\eta_s(i)Q_s(j) + Q_s(i)\eta_s(j)\right] ds \tag{4.5}$$

holds. $L(k)$ is, for each $k\in\mathbb{N}^*$, a map on $\mathbb{N}^*\times\mathbb{N}^*$ defined by

$$L(k)(i,j) = \left(1\!\!1_{\{i+j=k\}} - 1\!\!1_{\{i=k\}}\right)\frac{K(i,j)}{j}. \tag{4.6}$$

Let us now state the main result.

Theorem 4.4 *Assume (A). Then for each n, the process η^n (see (4.1) is a.s. strongly càdlàg from $[0,\infty)$ into l^2. Furthermore, η^n converges weakly to the solution η of (4.5) as n goes to infinity.*
By "η^n converges weakly to η", we mean the convergence of the law of η^n to that of η in the weak topology of $\mathcal{P}(\mathbb{D}([0,\infty), l^2))$, the space $\mathbb{D}([0,\infty), l^2)$ being endowed here with the Skorokhod topology associated with the weak topology of l^2.

Let us now sketch briefly the main ideas of the proof. First, we will give an useful characterization of the law $(\mathcal{GP})$. After a technical lemma we will be able to prove the uniqueness result stated in Proposition 4.3.
As usual we split the process η^n into satisfying terms, study the tightness and weak convergence of these terms. This technique allows to conclude the proof of Theorem 4.4.
We begin with a characterization of the law $(\mathcal{GP})$. We now state a technical lemma. The hypothesis K bounded appears to be useful in this statement only.

Lemma 4.5 *Assume (A). There exists a constant A such that for every couple of probability measures q, μ in $\mathcal{P}(\mathbb{N}^*)$ and all $\alpha\in l^2$, we have the inequalities*

$$\left|\left|\sum_{i\geq 1}\sum_{j\geq 1} L(.)(i,j)\left[\alpha(i)q(j) + \mu(i)\alpha(j)\right]\right|\right|_{l^2} \leq A\left|\left|\alpha\right|\right|_{l^2} \text{ and} \tag{4.7}$$

$$\left|\left|\sum_{i\geq 1}\sum_{j\geq 1} L(.)(i,j)\alpha(i)\left[q(j) - \mu(j)\right]\right|\right|_{l^2} \leq A\left|\left|\alpha\right|\right|_{l^2}\left|\left|q-\mu\right|\right|_{l^2}. \tag{4.8}$$

Proof Let us check (4.7). First notice that

$$\left|\left|\sum_{i\geq 1}\sum_{j\geq 1} L(.)(i,j)\left[\alpha(i)q(j)+\mu(i)\alpha(j)\right]\right|\right|_{l^2}^2 \leq 2(I_1+I_2) \tag{4.9}$$

where

$$I_1 = \sum_{k\geq 1}\left\{\sum_{i\geq 1}\sum_{j\geq 1}\left(1\!\!1_{\{i+j=k\}}-1\!\!1_{\{i=k\}}\right)\frac{K(i,j)}{j}\alpha(i)q(j)\right\}^2 \tag{4.10}$$

$$I_2 = \sum_{k\geq 1}\left\{\sum_{i\geq 1}\sum_{j\geq 1}\left(1\!\!1_{\{i+j=k\}}-1\!\!1_{\{i=k\}}\right)\frac{K(i,j)}{j}\mu(i)\alpha(j)\right\}^2. \tag{4.11}$$

Thanks to the Cauchy-Schwartz inequality applied to the probability measure q,

$$\begin{aligned}
I_1 &\leq \sum_{k\geq 1}\sum_{j\geq 1} q(j)\left\{\sum_{i\geq 1}\left(1\!\!1_{\{i+j=k\}}-1\!\!1_{\{i=k\}}\right)\frac{K(i,j)}{j}\alpha(i)\right\}^2 \\
&= \sum_{k\geq 1}\sum_{j\geq 1}\frac{q(j)}{j^2}\left\{K(k-j,j)\alpha(k-j)1\!\!1_{\{k>j\}}-K(k,j)\alpha(k)\right\}^2 \\
&\leq A\sum_{j\geq 1} q(j)\sum_{k\geq 1}\left[\alpha^2(k-j)1\!\!1_{\{k>j\}}+\alpha^2(k)\right] \\
&\leq A\,||\alpha||_{l^2}^2 \qquad (4.12)
\end{aligned}$$

since K is bounded and since q is a probability measure. On the other hand, using twice the Cauchy-Schwartz inequality, one obtains

$$\begin{aligned}
I_2 &\leq \sum_{k\geq 1}\sum_{i\geq 1}\mu(i)\left\{\sum_{j\geq 1}\left(1\!\!1_{\{i+j=k\}}-1\!\!1_{\{i=k\}}\right)\frac{K(i,j)}{j}\alpha(j)\right\}^2 \\
&\leq A\sum_{k\geq 1}\sum_{i\geq 1}\mu(i)\left(\sum_{j\geq 1}\alpha^2(j)\right)\sum_{j\geq 1}\left(1\!\!1_{\{i+j=k\}}+1\!\!1_{\{i=k\}}\right)\frac{K(i,j)}{j^2} \\
&\leq A\,||\alpha||_{l^2}^2\sum_{i\geq 1}\mu(i)\sum_{j\geq 1}\frac{1}{j^2}\sum_{k\geq 1}\left(1\!\!1_{\{i+j=k\}}+1\!\!1_{\{i=k\}}\right) \\
&\leq A\,||\alpha||_{l^2}^2. \qquad (4.13)
\end{aligned}$$

Let us now check (4.8). Since K is bounded, we obtain, using the Cauchy-Schwartz inequality, that for some constant A, the square of the left hand side member of (4.8) is smaller than

$$\begin{aligned}
&A\sum_{k\geq 1}\left(\sum_{i\geq 1}\sum_{j\geq 1}\frac{1\!\!1_{\{i+j=k\}}+1\!\!1_{\{i=k\}}}{j}|\alpha(i)||q(j)-\mu(j)|\right)^2 \\
&\leq A\sum_{k\geq 1}\left(\sum_{j\geq 1}\frac{1}{j^2}\right)\sum_{j\geq 1}\left(\sum_{i\geq 1}\left(1\!\!1_{\{i+j=k\}}+1\!\!1_{\{i=k\}}\right)|\alpha(i)||q(j)-\mu(j)|\right)^2 \\
&\leq A\sum_{k\geq 1}\sum_{j\geq 1}\left[q(j)-\mu(j)\right]^2\left[\alpha^2(k-j)1\!\!1_{\{k>j\}}+\alpha^2(k)\right] \\
&\leq A\,||q-\mu||_{l^2}^2\,||\alpha||_{l^2}^2, \qquad (4.14)
\end{aligned}$$

the last inequality being obvious when exchanging the order of the sums. The lemma is proved. □

The uniqueness for (4.5) is now straightforward.

Proof of Proposition 4.3 Let us for example prove uniqueness. Consider two solutions η and η' of the equation (4.5), and let $T > 0$ be fixed. Then one gets immediately the existence of a constant A_T such that

(4.15)

$$\begin{aligned}\sup_{s\in[0,t]} ||\eta_s - \eta'_s||^2_{l^2} &\le A_T \int_0^t \left|\left| \sum_{i\ge 1}\sum_{j\ge 1} L(.)(i,j)\left[(\eta_s(i)-\eta'_s(i))Q_s(j)+Q_s(i)(\eta_s(j)-\eta'_s(j))\right]\right|\right|^2_{l^2} ds \\ &\le A_T \int_0^t ||\eta_s - \eta'_s||^2_{l^2}\, ds\end{aligned}$$

thanks to Lemma 4.5. Gronwall's Lemma allows to conclude. □

In order to prove Theorem 4.4, we have to split η^n into satisfying terms.

Notation 4.6 Thanks to Definition 3.1 and the fact that $(Q_t,\, t\ge 0)$ satisfies (MS), we can split η^n_t in 3 terms :

$$\eta^n_t(k) = \eta^n_0(k) + M^n_t(k) + F^n_t(k) \tag{4.16}$$

where

$$\eta^n_0(k) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(1\!\!1_{\{X_0^{i,n}=k\}} - Q_0(k)\right), \tag{4.17}$$

(4.18)

$$M^n_t(k) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \int_j \int_0^\infty \left(1\!\!1_{\{X_{s-}^{i,n}+X_{s-}^{j,n}=k\}} - 1\!\!1_{\{X_{s-}^{i,n}=k\}}\right) 1\!\!1_{\left\{z\le \frac{K(X_{s-}^{i,n},X_{s-}^{j,n})}{X_{s-}^{j,n}}\right\}} \bar{N}^i(ds,dj,dz),$$

and

$$F^n_t(k) = \int_0^t \sum_{i\ge 1}\sum_{j\ge 1} L(k)(i,j)\left[\eta^n_s(i)\mu^n_s(j) + Q_s(i)\eta^n_s(j)\right] ds, \tag{4.19}$$

$\bar{N}^i$ denotes the Poisson compensated measure associated with N^i.

Let us study first the asymptotic behavior of the initial condition.

Lemma 4.7 *Assume (A). Then, for all n, $\mathbb{E}\left[||\eta^n_0||^2_{l^2}\right] \le 1$. Furthermore, η^n_0 converges weakly (for the weak topology on l^2) to a random variable η_0 with the law defined in Definition 4.2 1.*

Proof The proof is completely standard, since it concerns the sequence $X^{1,n},\cdots,X^{n,n}$ of *i.i.d.* Q_0-distributed random variables. It requires only the use of the standard central limit theorem. □

We now state some easy moment calculus and trajectorial estimates for M^n and η^n.

Lemma 4.8 *Assume (A).*

1. *For all $T > 0$, there exists a constant A_T, such that for any n*

$$\mathbb{E}\left[\sup_{t\in[0,T]} ||M_t^n||_{l^2}^2\right] \leq \mathbb{E}\left[\sum_{k\geq 1} \sup_{t\in[0,T]} (M_t^n(k))^2\right] \leq A_T \tag{4.20}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} ||\eta_t^n||_{l^2}^2\right] \leq A_T. \tag{4.21}$$

2. *For all n, M^n and η^n are a.s. strongly càdlàg from $[0,\infty)$ into l^2.*

3. *For each $T > 0$, there exists a constant A_T, such that for any n and $k_0 \in \mathbb{N}^*$*

$$\mathbb{E}\left[\sup_{t\in[0,T]} (M_t^n(k_0))^4\right] \leq A_T. \tag{4.22}$$

Proof **1.** A simple computation using Doob's inequality and the expression of M^n shows that

$$\begin{aligned}
&\sum_{k\geq 1}\mathbb{E}\left[\sup_{t\in[0,T]} (M_t^n(k))^2\right] \\
&\leq A\sum_{k\geq 1}\frac{1}{n}\mathbb{E}\left[\sum_{i=1}^n \frac{1}{n}\sum_{j=1}^n \int_0^T \left(\mathbb{1}_{\{X_s^{i,n}+X_s^{j,n}=k\}} - \mathbb{1}_{\{X_s^{i,n}=k\}}\right)^2 \frac{K(X_s^{i,n},X_s^{j,n})}{X_s^{j,n}} ds\right] \\
&\leq A\frac{1}{n^2}\sum_{i,j=1}^n \int_0^T \sum_{k\geq 1}\left(\mathbb{1}_{\{X_s^{i,n}+X_s^{j,n}=k\}} + \mathbb{1}_{\{X_s^{i,n}=k\}}\right) ds \\
&\leq AT
\end{aligned} \tag{4.23}$$

since K is bounded, and since $X^{i,n} \geq 1$. (4.20) is now proved.
We deduce from (4.16), Lemmas 4.7 and 4.5 that for any n and $t \leq T$,

$$\mathbb{E}\left[\sup_{s\in[0,t]} ||\eta_s^n||_{l^2}^2\right] \leq A_T + A_T\int_0^t \mathbb{E}\left[||\eta_s^n||_{l^2}^2\right] ds. \tag{4.24}$$

Gronwall's Lemma allows to conclude.
2. Recall that $\eta_t^n = \eta_0^n + M_t^n + F_t^n$. Let us first notice that thanks to the Cauchy-Schwartz inequality and (4.7), we obtain, for all $s < t$

$$||F_t^n - F_s^n||_{l^2}^2 \leq (t-s)\int_s^t ||\eta_u^n||_{l^2}^2\, du \tag{4.25}$$

and it is clear from 1 that F^n is strongly continuous.
We thus just have to check that M^n is càdlàg. Let us for example show that it is right continuous. Let t_m be a sequence decreasing to t and $\varepsilon > 0$ be fixed. Then for all $q \in \mathbb{N}$,

$$\left|\left|M_{t_m}^n - M_t^n\right|\right|_{l^2}^2 \leq \sum_{k=1}^q \left(M_{t_m}^n(k) - M_t^n(k)\right)^2 + 2\sum_{k=q+1}^\infty \sup_{u\in[0,t+1]} (M_u^n(k))^2 \tag{4.26}$$

(at least if m is large enough, which ensures that $t_m \leq t+1$). Now, choosing q large enough will imply that the second term of the right hand side member is smaller than $\varepsilon/2$, thanks to 1. It is also clear that for each k fixed $M^n(k)$ is càdlàg, since it is a finite sum of integrals against Poisson measures. We can now conclude : it is sufficient to choose m large enough, in order to obtain that for all k in $\{1,\ldots,q\}$, $(M^n_{t_m}(k) - M^n_t(k))^2 \leq \varepsilon/(2q)$.
3. The proof is not difficult. Since k_0 is fixed, (4.22) relies on the use of the standard Burkholder-Davis-Gundy inequality. Notice that computing the quadratic variation of $M^n(k_0)$ is easy since the Poisson measures N^i are independent.
The Lemma 4.8 is proved. □

It remains to prove tightness results. Recall that l^2, endowed with the weak topology, is not Polish. We thus have to use a specific Lemma which is due to Fernique [Fer91] (see lemma 6.2 in the Appendix). This lemma allows us to state and prove the following result.

Lemma 4.9 *Assume* (A).

1. *The sequences of processes* η^n *and* M^n *are tight in* $\mathbb{D}([0,\infty), l^2)$ *(endowed with the Skorokhod topology associated with the weak topology of* l^2*).*
2. *Any limit point* η *(resp.* M*) of the sequence* η^n *(resp.* M^n*) is a.s. strongly continuous from* $[0,\infty)$ *into* l^2.

Proof **1.** We apply Lemma 6.2 to the sequences η^n and M^n. Condition (i) is satisfied thanks to Lemma 4.8, 1.
To prove the second condition, we fix k, and apply the Aldous criterion [JS87]. We just have to show, for example, that there exists a constant A_T such that for all n, all $\delta > 0$, all couple of stopping times $0 < S < S' < (S+\delta) \wedge T$,

$$\mathbb{E}\left[(M^n_{S'}(k) - M^n_S(k))^2\right] + \mathbb{E}\left[(\eta^n_{S'}(k) - \eta^n_S(k))^2\right] \leq A_T \delta. \tag{4.27}$$

We obtain it easily in the same spirit as Lemma 4.8, 1.
2. To prove the continuity results, it suffices to check that

$$\mathbb{E}\left[\sup_{t\in[0,\infty)} ||\Delta M^n_t||_{l^2}\right] \underset{n\to 0}{\longrightarrow} 0 \tag{4.28}$$

and that the same limit result holds for η^n. First notice that for any ω, n and t, $\Delta M^n_t = \Delta \eta^n_t$, so that it suffices to prove (4.28). Remark also that as the Poisson measures N^i are independent, it is obvious, using (4.18), that any jump of M^n is such that there exist i and j such that

$$\Delta M^n_t(.) = \frac{1}{\sqrt{n}}\left[\mathbb{1}_{\{X^{i,n}_{t-}+X^{j,n}_{t-}=.\}} - \mathbb{1}_{\{X^{i,n}_{t-}=.\}}\right] \tag{4.29}$$

from which we deduce that

$$||\Delta M^n_t||^2_{l^2} \leq \frac{1}{n}\sum_{k\geq 1}\left[\mathbb{1}_{\{X^{i,n}_{t-}+X^{j,n}_{t-}=k\}} - \mathbb{1}_{\{X^{i,n}_{t-}=k\}}\right]^2 \leq \frac{1}{n}, \tag{4.30}$$

which implies (4.28). The Lemma is now proved. □

Let us study now the law of the limiting points.

Lemma 4.10 *Assume (A). Consider a subsequence η^{n_k} of η^n, weakly convergent to a strongly continuous l^2-valued process η, in $\mathbb{D}([0,\infty),l^2)$ (endowed with the Skorokhod topology associated with the weak topology on l^2). Consider the processes*

$$\begin{aligned}(4.31)\quad M^n &= \eta^n - \eta^n_0 - F^n \\ (4.32)\quad W_t(.) &= \eta_t(.) - \eta_0(.) - \int_0^t \sum_{i\geq 1}\sum_{j\geq 1} L(.)(i,j)\left\{\eta_s(i)Q_s(j) + Q_s(i)\eta_s(j)\right\} ds.\end{aligned}$$

Finally set, for $t \geq 0$,

$$(4.33)\qquad \mathcal{F}_t = \sigma\left\{\eta_0(k)\ ;\ k\in\mathbb{N}^*\right\} \vee \sigma\left\{W_s(k)\ ;\ 0\leq s\leq t,\ k\in\mathbb{N}^*\right\}.$$

We have:

1. *$(\eta_0^{n_k}, M^{n_k})$ converges weakly to (η_0, W) in $l^2\times\mathbb{D}([0,\infty),l^2)$ (endowed with the product topology, l^2 being endowed with the weak topology and $\mathbb{D}([0,\infty),l^2)$ with the associated Skorokhod topology).*

2. *The process W is a.s. strongly continuous from $[0,\infty)$ into l^2. For each $k_0\in\mathbb{N}^*$, the real-valued process $W(k_0)$ is an $(\mathcal{F}_t,\ t\geq 0)$-martingale.*

3. *For each q_1, q_2 in $\mathbb{N}^*$, the Doob-Meyer bracket of $W(q_1)$ and $W(q_2)$ is given by*

(4.34)
$$\langle W(q_1),W(q_2)\rangle_t = \int_0^t \sum_{i\geq 1}\sum_{j\geq 1} Q_s(i)Q_s(j)\left(\mathbb{1}_{\{i+j=q_1\}} - \mathbb{1}_{\{i=q_1\}}\right)\left(\mathbb{1}_{\{i+j=q_2\}} - \mathbb{1}_{\{i=q_2\}}\right)\frac{K(i,j)}{j} ds.$$

Proof **1.** We split the proof of 1 in two steps. Consider the process

$$(4.35)\qquad G^n_t(.) = \eta^n_t(.) - \eta^n_0(.) - \int_0^t \sum_{i\geq 1}\sum_{j\geq 1} L(.)(i,j)\left\{\eta^n_s(i)Q_s(j) + Q_s(i)\eta^n_s(j)\right\} ds.$$

(i) We will first prove that for any $T>0$, the process $M^n - G^n$ goes to 0, as n tends to infinity, in L^1 (and thus in probability), for the uniform norm on $[0,T]$ and the strong norm of l^2.
(ii) Then we will check that $(\eta_0^{n_k}, G^{n_k})$ goes in law, as k tends to infinity, to (η_0, W) in $l^2\times\mathbb{D}([0,\infty),l^2)$, the space l^2 being endowed (twice) with the weak topology. Once the two points are proved, 1 is straightforward.
Let us check **(i)**. We deduce from the expression of F^n (see (4.19)) that

$$\mathbb{E}\left[\sup_{t\in[0,T]} ||M^n_t - G^n_t||_{l^2}\right] \leq \mathbb{E}\left[\int_0^T \left|\left|\sum_{i\geq 1}\sum_{j\geq 1} L(.)(i,j)\eta^n_s(i)\left\{\mu^n_s(j) - Q_s(j)\right\}\right|\right|_{l^2} ds\right].$$

By Lemma 4.5, we get the existence of a constant A_T such that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} ||M^n_t - G^n_t||_{l^2}\right] &\leq A_T\mathbb{E}\left[\int_0^T ||\eta^n_s||_{l^2}\,||\mu^n_s - Q_s||_{l^2}\, ds\right] \\ (4.37)\qquad &\leq \frac{1}{\sqrt{n}} A_T\mathbb{E}\left[\sup_{s\in[0,T]} ||\eta^n_s||^2_{l^2}\right]\end{aligned}$$

which goes to 0 as n tends to infinity thanks to Lemma 4.8.
To prove that $(\eta_0^{n_k}, M^{n_k})$ goes in law to (η_0, W), it suffices to prove that the map $\mathcal{G}$ from $\mathbb{D}([0,\infty[,l^2)$ into $l^2 \times \mathbb{D}([0,\infty),l^2)$, defined by

$$\begin{aligned} \mathcal{G}(\alpha) &= (\mathcal{G}^{(1)}(\alpha), \mathcal{G}^{(2)}(\alpha)) \\ &= \left(\alpha_0, \alpha - \alpha_0 - \int_0^{\cdot} \sum_{i\geq 1}\sum_{j\geq 1} L(\boldsymbol{.})(i,j)\left\{\alpha_s(i)Q_s(j) + Q_s(i)\alpha_s(j)\right\} ds\right) \end{aligned} \tag{4.38}$$

is continuous at any point α that is strongly continuous. Indeed, we know that η^{n_k} goes to η and that η is strongly continuous.
We thus consider a sequence α^n of $\mathbb{D}([0,\infty),l^2)$, converging (for the Skorokhod topology associated with the weak topology of l^2) to a strongly continuous l^2-valued function α. It is well-known that since the limit α is continuous, the convergence holds also for the topology of the uniform convergence on compacts, *i.e.* that for any $T>0$, any γ in l^2,

$$\sup_{t\in[0,T]}\left|\sum_{k\geq 1}\gamma(k)(\alpha_t^n(k) - \alpha_t(k))\right| \underset{n\to 0}{\longrightarrow} 0. \tag{4.39}$$

We first deduce that $\mathcal{G}^{(1)}(\alpha_n)$ tends to $\mathcal{G}^{(1)}(\alpha)$ for the weak topology of l^2. It remains to prove that for any β in l^2 and any $T>0$

$$\delta_n = \sup_{t\in[0,T]}\left|\sum_{k\geq 1}\beta(k)\left(\mathcal{G}_t^{(2)}(\alpha_n) - \mathcal{G}_t^{(2)}(\alpha)\right)\right| \underset{n\to\infty}{\longrightarrow} 0. \tag{4.40}$$

First, it is clear that

$$\begin{aligned} \delta_n &\leq 2\sup_{t\in[0,T]}\left|\sum_{k\geq 1}\beta(k)[\alpha_t^n(k) - \alpha_t(k)]\right| \\ &\quad + \int_0^T\left|\sum_{k\geq 1}\beta(k)\left(\sum_{i\geq 1}\sum_{j\geq 1}L(k)(i,j)[\alpha_s^n(i) - \alpha_s(i)]Q_s(j)\right)\right| ds \\ &\quad + \int_0^T\left|\sum_{k\geq 1}\beta(k)\left(\sum_{i\geq 1}\sum_{j\geq 1}L(k)(i,j)Q_s(i)[\alpha_s^n(j) - \alpha_s(j)]\right)\right| ds \\ &= 2\delta_n^{(1)} + \delta_n^{(2)} + \delta_n^{(3)} \end{aligned} \tag{4.41}$$

with obvious notations in the last equality. First, $\delta_n^{(1)}$ tends to 0 thanks to (4.39) applied with $\gamma = \beta$. On the other hand,

$$\begin{aligned} \delta_n^{(2)} &= \int_0^T\left|\sum_{k\geq 1}\beta(k)\sum_{i\geq 1}\sum_{j\geq 1}\left(\mathbb{1}_{\{i+j=k\}} - \mathbb{1}_{\{i=k\}}\right)\frac{K(i,j)}{j}\left[\alpha_s^n(i) - \alpha_s(i)\right]Q_s(j)\right| ds \\ &= \int_0^T\left|\sum_{i\geq 1}[\alpha_s^n(i) - \alpha_s(i)]\left\{\sum_{j\geq 1}\beta(i+j)\frac{K(i,j)}{j}Q_s(j) - \beta(i)\sum_{j\geq 1}\frac{K(i,j)}{j}Q_s(j)\right\}\right| ds. \end{aligned} \tag{4.42}$$

For each s, the integrand tends to 0, thanks to (4.39) applied with

$$\gamma(i) = \sum_{j\geq 1}\beta(i+j)\frac{K(i,j)}{j}Q_s(j) - \beta(i)\sum_{j\geq 1}\frac{K(i,j)}{j}Q_s(j), \tag{4.43}$$

which belongs to l^2. On the other hand, the Cauchy-Schwartz inequality and Lemma 4.5 allow to obtain that the integrand in (4.42) is smaller than $A\,||\beta||_{l^2}\,||\alpha_s^n - \alpha_s||_{l^2}$ which is clearly bounded (uniformly in n) on $[0,T]$. By Lebesgue theorem we get that $\delta_n^{(2)}$ tends to 0. One proves in the same way that $\delta_n^{(3)}$ tends to 0. This ends the proof of 1.

2. From Lemma 4.9 the process W is strongly continuous since it is a limit point of M^n. To check that $W(k_0)$ is a $(\mathcal{F}_t,\, t \geq 0)$-martingale, let us consider $0 \leq s_1 \leq \cdots \leq s_l \leq t$, $k_1, \ldots, k_l$, $m_1, \ldots, m_p$ in $\mathbb{N}^*$, and a family $\phi_1, \ldots, \phi_l$, $\psi_1, \ldots, \psi_p$ of continuous bounded functions from $\mathbb{R}$ into itself. We have to check that

(4.44)
$$\mathbb{E}\Big[\,\{W_t(k_0) - W_{s_l}(k_0)\}\,\psi_1(\eta_0(m_1))\ldots\psi_p(\eta_0(m_p))\phi_1(M_{s_1}(k_1))\ldots\phi_l(M_{s_l}(k_l))\Big] = 0.$$

By notation 4.6, we see that for each n,

(4.45)
$$\mathbb{E}\Big[\,\{M_t^n(k_0) - M_{s_l}^n(k_0)\}\,\psi_1(\eta_0^n(m_1))\ldots\psi_p(\eta_0^n(m_p))\phi_1(M_{s_1}^n(k_1))\ldots\phi_l(M_{s_l}^n(k_l))\Big] = 0,$$

so that we "just" have to take the limit. We know from 1 that $(\eta_0^{n_k}, M^{n_k})$ converges weakly to (η_0, W), and that W is strongly continuous. Furthermore, the map $F : l^2 \times \mathbb{D}([0,\infty), l^2) \to \mathbb{R}$, defined by

$$F(\zeta,\alpha) = \{\alpha_t(k_0) - \alpha_{s_l}(k_0)\}\,\psi_1(\zeta(m_1))\ldots\psi_p(\zeta(m_p))\phi_1(\alpha_{s_1}(k_1))\ldots\phi_l(\alpha_{s_l}(k_l)) \tag{4.46}$$

is continuous at any point (ζ,α), when α is strongly continuous. We deduce that for each $0 \leq B < \infty$,

$$\mathbb{E}\left[F(\eta_0^{n_k}, M^{n_k}) \wedge B \vee (-B)\right] \underset{k\to\infty}{\longrightarrow} \mathbb{E}\left[F(\eta_0, W) \wedge B \vee (-B)\right]. \tag{4.47}$$

Finally, using the fact that for some constant B, $|F(\zeta,\alpha)| \leq B \sup_{s\in[0,t]} |\alpha_s(k_0)|$ and the uniform integrability (concerning M^n) obtained in Lemma 4.8, we can make B go to infinity, and obtain that

$$\mathbb{E}\left[F(\eta_0^{n_k}, M^{n_k})\right] \underset{k\to\infty}{\longrightarrow} \mathbb{E}\left[F(\eta_0, W)\right]. \tag{4.48}$$

By using (4.45), we get $\mathbb{E}\left[F(\eta_0, W)\right] = 0$, which was our aim.

3. To prove (4.34) it suffices to check that, for Y the process denoting the right hand side member of (4.34), $W(q_1)W(q_2) - Y$ is a $(\mathcal{F}_t,\, t \geq 0)$-martingale. To this aim, we proceed exactly as in 2. We have to prove, with the same notations as in 2, that

$$\begin{aligned}&\mathbb{E}\Big[\,\{W_t(q_1)W_t(q_2) - Y_t - W_{s_l}(q_1)W_{s_l}(q_2) + Y_{s_l}\}\\ &\psi_1(\eta_0(m_1))\ldots\psi_p(\eta_0(m_p))\phi_1(M_{s_1}(k_1))\ldots\phi_l(M_{s_l}(k_l))\Big] = 0.\end{aligned} \tag{4.49}$$

This equality holds when replacing everywhere W by M^{n_k}, Y by $\langle M^{n_k}(q_1), M^{n_k}(q_2)\rangle$, and η_0 by $\eta_0^{n_k}$. We just have to make k tend to infinity. Using (4.18) we obtain for each n

(4.50)
$$\langle M^n(q_1), M^n(q_2)\rangle_t = \int_0^t \sum_{i\geq 1}\sum_{j\geq 1} \mu_s^n(i)\mu_s^n(j)\left(1\!\!1_{\{i+j=q_1\}} - 1\!\!1_{\{i=q_1\}}\right)\left(1\!\!1_{\{i+j=q_2\}} - 1\!\!1_{\{i=q_2\}}\right)\frac{K(i,j)}{j}ds.$$

A simple computation shows that, for any $T > 0$, there exists a constant A such that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |\langle M^n(q_1), M^n(q_2)\rangle_t - Y_t|\right] \\
&\le \mathbb{E}\Big[\int_0^T \sum_{i\ge1}\sum_{j\ge1} \left|1\!\!1_{\{i+j=q_1\}} - 1\!\!1_{\{i=q_1\}}\right| \left|1\!\!1_{\{i+j=q_2\}} - 1\!\!1_{\{i=q_2\}}\right| \frac{K(i,j)}{j} \\
&\qquad \times \{|Q_s(i)||Q_s(j) - \mu_s^n(j)| + |Q_s(j)||Q_s(i) - \mu_s^n(i)|\}\, ds\Big] \\
&\le A\mathbb{E}\Big[\int_0^T \sum_{j\ge1}\frac{1}{j}\sum_{i\ge1}\left\{1\!\!1_{\{i+j=q_1\}} + 1\!\!1_{\{i=q_1\}} + 1\!\!1_{\{i+j=q_2\}} + 1\!\!1_{\{i=q_2\}}\right\} \\
&\qquad \times \{|Q_s(i)||Q_s(j) - \mu_s^n(j)| + |Q_s(j)||Q_s(i) - \mu_s^n(i)|\}\, ds\Big] \\
&\le A\mathbb{E}\Big[\int_0^T \sum_{j\ge1}\frac{1}{j} 4\sup_{i\ge1}\Big\{|Q_s(i)||Q_s(j) - \mu_s^n(j)| + |Q_s(j)||Q_s(i) - \mu_s^n(i)|\Big\}ds\Big] \\
&\le A\mathbb{E}\Big[\int_0^T \sum_{j\ge1}\frac{1}{j}\{|Q_s(j) - \mu_s^n(j)| + Q_s(j)\, ||\mu_s^n - Q_s||_{l^2}\}\, ds\Big] \\
&\le A\mathbb{E}\left[\int_0^T ||\mu_s^n - Q_s||_{l^2}\, ds\right] \\
&\le A\,T\frac{1}{\sqrt{n}}\mathbb{E}\left[\sup_{s\in[0,T]} ||\eta_s^n||_{l^2}\right]
\end{aligned}
$$

which tends to 0 thanks to Lemma 4.8. Finally notice that for any $T > 0$,

(4.52)

$$
\begin{aligned}
&\sup_n \mathbb{E}\left[\sup_{t\in[0,T]} |M_t^n(q_1)M_t^n(q_2) - \langle M^n(q_1), M^n(q_2)\rangle_t|^2\right] \\
&\le 2\sup_n \mathbb{E}\left[\sup_{t\in[0,T]} (M_t^n(q_1))^4\right]\mathbb{E}\left[\sup_{t\in[0,T]} (M_t^n(q_2))^4\right] + 2\sup_n \mathbb{E}\left[|M_t^n(q_1)M_t^n(q_2)|^2\right] \\
&< \infty
\end{aligned}
$$

thanks to Lemma 4.8, 3, and since one obviously deduces from (4.50) (recall that μ^n is a probability measure) that for all $t \in [0, T]$,

$$
| < M^n(q_1), M^n(q_2) >_t | \le AT,
$$

since K is bounded. Using the convergence in L^1 of $< M^n(q_1), M^n(q_2) >$ to Y, the convergence in law of $(\eta_0^{n_k}, M^{n_k})$ to (η_0, W), the uniform integrability obtained in (4.52) and the equality

(4.53)

$$
\begin{aligned}
\mathbb{E}\Big[\Big\{&M_t^{n_k}(q_1)M_t^{n_k}(q_2) - \langle M^{n_k}(q_1), M^{n_k}(q_2)\rangle_t - M_{s_l}^{n_k}(q_1)M_{s_l}^{n_k}(q_2) + \langle M^{n_k}(q_1), M^{n_k}(q_2)\rangle_{s_l}\Big\} \\
&\times\psi_1(\eta_0^{n_k}(m_1))\dots\psi_p(\eta_0^{n_k}(m_p))\phi_1(M_{s_1}^{n_k}(k_1))\dots\phi_l(M_{s_l}^{n_k}(k_l))\Big] = 0,
\end{aligned}
$$

we obtain (4.49) by letting k go to infinity. This concludes the proof of the lemma. □

We finally are able to provide the:

Proof of Theorem 4.4 Recall the Notation 4.6. We know from Lemma 4.9 that the sequence η^n is tight, and that any limit point is strongly continuous. Let us consider a converging subsequence η^{n_k}, going to some process η. From Lemma 4.10 we know that $\eta_t^{n_k} - \eta_0^{n_k} - F_t^{n_k}$ goes to the process

$$W_{.}(\bullet) = \eta_{.}(\bullet) - \eta_0(\bullet) - \int_0^{.} \sum_{i\geq 1}\sum_{j\geq 1} L(\bullet)(i,j)\,\{\eta_s(i)Q_s(j) + Q_s(i)\eta_s(j)\}\,ds. \tag{4.54}$$

It is clear from Lemmas 4.10, 4.7, and Definition 4.2 that (η_0, W) has the law $(\mathcal{GP})$. Hence, η can be written as a solution of the equation (4.5). Since the uniqueness in law for this S.D.E. holds (thanks to Proposition 4.3), we deduce that the whole sequence η^n goes, in law, to the solution η of equation (4.5), which concludes the proof of Theorem 4.4. □

5 Numerical results

In this section we test numerically the present approximation scheme algorithm. We refer to [EW01] for further numerical results. We would like here to treat the following points.
(i) Is our algorithm better than the classical Marcus-Lushnikov procedure?
(ii) We illustrate our theoretical results.
(iii) Is the particle system still simulable when the coagulation kernel K does not satisfy $(H_{1/2})$ but only (H_1) ? Does the convergence result of Theorem 3.3 remain valid under (H_1) ? Does the central limit type result, obtained in Theorem 4.4, hold when the coagulation kernel does not any more satisfy assumption (A), but only $(H_{1/2})$ or even (H_1) ?

We begin with some notations. Let us recall that $\mu_t^n = \frac{1}{n}\sum_{i=1}^n \delta_{X_t^{i,n}}$ is defined in Definition 3.1, and approaches the solution Q_t to the equation (MS).
We consider the case where $Q_0 = \delta_1$. We will essentially compare $m(t) = \int_{\mathbb{R}_+} xQ_t(dx)$ to $m_n(\omega, t) = \int_{\mathbb{R}_+} x\mu_t^n(\omega, dx) = \frac{1}{n}\sum_{i=1}^n X_t^{i,n}$. This will give a "global idea" of the rate of convergence. The function $m(t)$ can be explicitly computed in any case where $K(x,y)$ is of the form $a + b(x+y) + cxy$, for a, b and c given constants.

Let us also mention that in the sequel, we simulate exactly the particle system when $K(x,y) = xy$: it seems that even in this (explosive) case, the particle system is directly simulable (up to gelation).

We first compare the result obtained with the particle system to that obtained with a Marcus-Lushnikov procedure. It is well known that the Marcus-Lushnikov algorithm is not good for large times, since there are less and less particles. However, this problem can be solved by using a standard trick, consisting in making a copy of the system as soon as the number of particles is too small. We do not use this trick here.
On Figure 1, we plot $m(t)$ for t in $[0,3]$, for the additive kernel $K(x,y) = x + y$. This quantity is compared with $m_{600}(t)$ (obtained with one simulation of the particle system), and with the approximation obtained with the Marcus-Lushnikov procedure (for a similar time of computation, of order $0.800u$). Note that the corresponding Marcus-Lushnikov procedure starts with 10000 particles and ends with 504 particles. We see that as soon as time is slightly large, the present numerical scheme looks better.

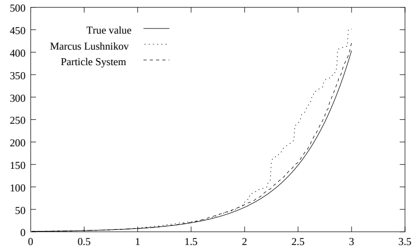

Figure 1: $K(x, y) = x + y$.

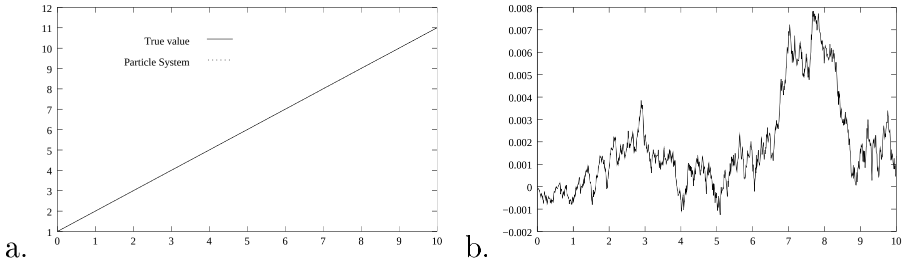

Figure 2: $K(x, y) = 1$.

We now treat points (*ii*) and (*iii*). We first consider the case $K(x, y) = 1$. Figure 2a. represents $m(t)$ and $m_n(t)$ (obtained with one simulation) for $n = 10^6$ particles, as functions of $t \in [0, 10]$. On 2 b., we draw the error $m_n(t, \omega) - m(t)$: it really looks like one path of a continuous "Brownian kind" process, which illustrates Theorem 4.4.
In Figure 3, we study the multiplicative kernel $K(x, y) = xy$. In this case, one has an explicit expression of the solution $(n(k, t),\ 0 \le t < 1,\ k \in \mathbb{N}^*)$ to (2.1). The first part 3 a. represents $\frac{1}{2}Q_t(\{2\})$ and its approximation $\frac{1}{2n}\sum_{i=1}^{n} \mathbb{1}_{\{X_t^{i,n}=2\}}$ as functions of $t \in [0, 0.98]$, for $n = 10^5$ particles. The second part 3 b. represents the corresponding error $\frac{1}{2}Q_t(\{2\}) - \frac{1}{2n}\sum_{i=1}^{n} \mathbb{1}_{\{X_t^{i,n}=2\}}$, which clearly also looks "Brownian".

We study now the rate of convergence of our scheme, as the number of particles increases. On Figure 4 a. each cross is obtained for one simulation, and represents the normalized error $\sqrt{n}[m_n(\omega, t) - m(t)]$, for $t = 1$ and $K(x, y) = 1$, as a function of the number $n \in \{10, \dots, 10^6\}$ of particles. We remark that the obtained "cloud" is almost surely contained in $[-C, C]$, for some constant C. We remark also that the distribution of the normalized error does not seem to depend on the number of particles. This illustrates Theorem 4.4.

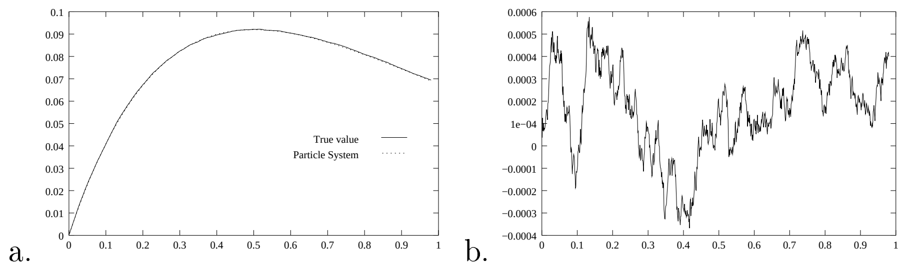

Figure 3: $K(x, y) = xy$.

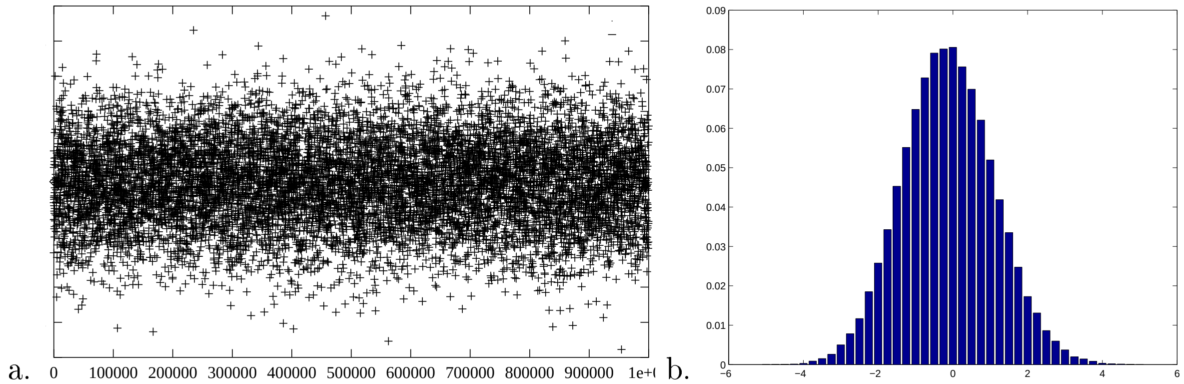

Figure 4: $K(x,y) = 1$

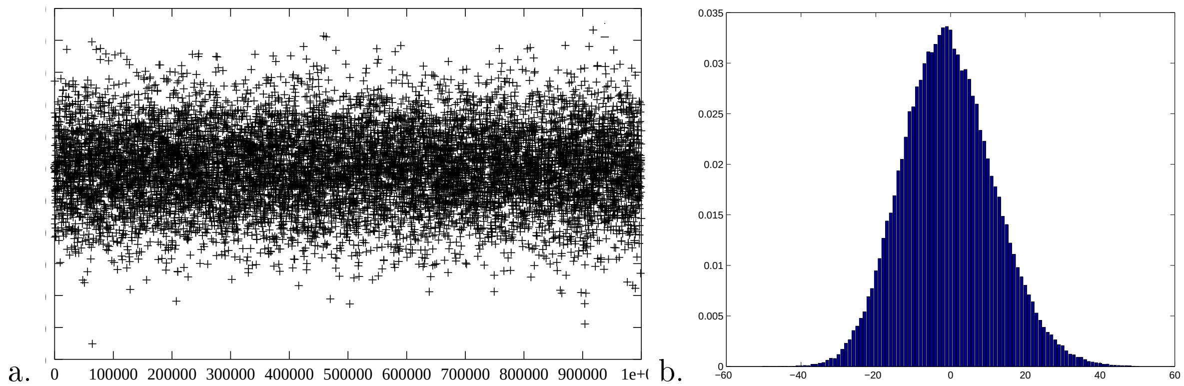

Figure 5: $K(x,y) = x + y$

On Figure 4 b., we represent the (empirical) distribution of the preceding normalized error we $n = 10^3$ and we observe a Gaussian distribution. This figure is obtained with 10^5 simulations.

Figure 5 treats the same problem as Figure 4 in the case where $K(x,y) = x+y$, $t = 1.0$ and $n \in \{10, \ldots, 10^6\}$ (5 a.), $n = 10^3$ (5 b.).
Figure 6 treats the same problem as Figure 4 in the case where $K(x,y) = xy$, $t = 0.5$ and $n \in \{10, \ldots, 10^6\}$ (6 a.), $n = 10^3$ (6 b.).

The main conclusion of this numerical study is that our assumptions in Theorems 3.3 and 4.4 are too stringent, and that these results seem to hold true in a more general context. Indeed, the particle system really seems to be simulable under (H_1) up to gelation (we have not used any cutoff procedure to obtain Figures 3, 5, 6. The convergence result of Theorem 3.3 seems to hold also under (H_1).
Finally, Figures 2 b., 3 b. ,5 and 6 show that the result of Theorem 4.4 may hold, at least in the discrete case, under (H_1).

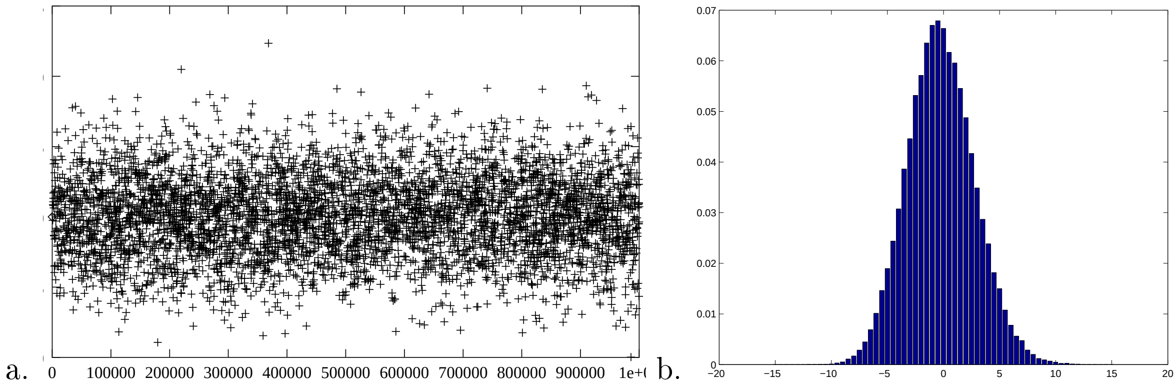

Figure 6: $K(x,y) = xy$

6 Appendix

We begin with a Lemma which can be found in Méléard [Mél96].

Lemma 6.1 *Let $T > 0$ be fixed and let ν^n be a sequence of random probability measures on $\mathbb{D}([0,T],\mathbb{R})$, which converges in law to a deterministic probability measure $R \in \mathcal{P}(\mathbb{D}([0,T],\mathbb{R}))$. Assume moreover that*

$$\sup_{t\in[0,T]} \int_{x\in\mathbb{D}([0,T],\mathbb{R})} \sup_{s\in[t-r,t+r]} \left(|\Delta x(s)| \wedge 1\right) R(dx) \underset{r\to 0}{\longrightarrow} 0. \tag{6.1}$$

Then $\{\nu^n_t\}_{t\in[0,T]}$ converges in probability to $\{R_t\}_{t\in[0,T]}$ in $\mathbb{D}([0,T],\mathcal{P}(\mathbb{R}))$ endowed with the topology of the uniform convergence.

We state here a Lemma due to Fernique [Fer91], which we restrict to the particular situation of l^2-valued processes.

Lemma 6.2 *Let α^n be a sequence of strongly càdlàg processes with values in l^2. Then the sequence α^n is tight in $\mathbb{D}([0,\infty), l^2)$ (endowed with the Skorokhod topology associated with the weak topology of l^2) if the following conditions are satisfied:*
(i) For any $T < \infty$, there exists a sequence of weakly compact subsets K_m of l^2 such that for any n, any m,

$$\mathbb{P}\left(\forall\, t \in [0,T],\ \alpha^n_t \in K_m\right) \geq 1 - 2^{-m}. \tag{6.2}$$

In particular this condition is always satisfied if for all T,

$$\sup_n \mathbb{E}\left[\sup_{t\in[0,T]} ||\alpha^n_t||^2_{l^2}\right] < \infty. \tag{6.3}$$

(ii) For each $k \geq 1$, the sequence of real-valued processes $\alpha^n(k)$ is tight (for the usual Skorokhod topology on $\mathbb{D}([0,\infty),\mathbb{R})$.

Acknowledgments The work of the third author was supported by INRIA Lorraine and Région Lorraine.

References

[Ald99] D. J. Aldous. Deterministic and stochastic models for coalescence (aggregation and coagulation): a review of the mean-field theory for probabilists. *Bernoulli*, 5(1):3–48, 1999.

[DFT01] M. Deaconu, N. Fournier, and E. Tanré. A pure jump Markov process associated with Smoluchowski's coagulation equation (to appear in Ann. Probab.). *Prépublication de l'Institut Élie Cartan de Nancy*, 6, 2001.

[EW01] A. Eibeck and W. Wagner. Stochastic particle approximations for Smoluchoski's coagulation equation. *Ann. Appl. Probab.*, 11(4):1137–1165, 2001.

[Fer91] X. Fernique. Convergence en loi de fonctions aléatoires continues ou càdlàg, propriétés de compacité des lois. In *Séminaire de Probabilités, XXV*, volume 1485 of *Lecture Notes in Math.*, pages 178–195. Springer, Berlin, 1991.

[FFG92] R. Ferland, X. Fernique, and G. Giroux. Compactness of the fluctuations associated with some generalized nonlinear Boltzmann equations. *Canad. J. Math.*, 44(6):1192–1205, 1992.

[GM97] C. Graham and S. Méléard. Stochastic particle approximations for generalized Boltzmann models and convergence estimates. *Ann. Probab.*, 25(1):115–132, 1997.

[Jeo98] I. Jeon. Existence of gelling solutions for coagulation-fragmentation equations. *Comm. Math. Phys.*, 194(3):541–567, 1998.

[Jou01] B. Jourdain. Nonlinear process associated with the discrete Smoluchowski coagulation fragmentation equation. *Prépublication ENPC-CERMICS*, 207, 2001.

[JS87] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1987.

[Lus78] A. A. Lushnikov. Some new aspects of coagulation theory. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosfer. I Okeana*, 14(10):738–743, 1978.

[Mar68] A. H. Marcus. Stochastic coalescence. *Technometrics*, 10:133–143, 1968.

[Mél96] S. Méléard. Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, volume 1627 of *Lecture Notes in Math.*, pages 42–95. Springer, Berlin, 1996.

[Mél98] S. Méléard. Convergence of the fluctuations for interacting diffusions with jumps associated with Boltzmann equations. *Stochastics Stochastics Rep.*, 63(3-4):195–225, 1998.

[Nor99] J. R. Norris. Smoluchowski's coagulation equation: uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent. *Ann. Appl. Probab.*, 9(1):78–109, 1999.

[Nor00] J. R. Norris. Cluster coagulation. *Comm. Math. Phys.*, 209(2):407–435, 2000.

[Tan79] H. Tanaka. Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Z. Wahrsch. Verw. Gebiete*, 46(1):67–105, 1978/79.