Probabilistic approach of some discrete and continuous coagulation equations with diffusion

Madalina Deaconu, Nicolas Fournier
Institut Elie Cartan, INRIA Lorraine
Campus Scientifique, BP 239
54506 Vandoeuvre-lès-Nancy Cedex, France
mdeaconu@loria.fr, fournier@iecn.u-nancy.fr

Abstract

The diffusive coagulation equation models the evolution of the local concentration $n(t, x, z)$ of particles having position $x \in \mathbb{R}^p$ and size z at time t, for a system in which a coagulation phenomenon occurs. The aim of this paper is to introduce a probabilistic approach and a numerical scheme for this equation. We first delocalise the interaction, by considering a "mollified" model. This mollified model is naturally related to a $\mathbb{R}^p \times \mathbb{R}_+$-valued nonlinear stochastic differential equation, in a certain sense. We get rid of the non linearity of this S.D.E. by approximating it with an interacting stochastic particle system, which is (exactly) simulable. By using propagation of chaos techniques, we show that the empirical measure of the system converges to the mollified diffusive equation.
Then we use the smoothing properties of the heat kernel to obtain the convergence of the mollified solution to the true one. Numerical results are presented at the end of the paper.

Key words : Non spatially homogeneous coagulation equations, nonlinear stochastic differential equations, interacting stochastic particle systems.
MSC 2000 : 60H30, 60K35.

1 Introduction

Coagulation models govern the dynamic of clusters growth and illustrate the mechanism by which clusters can coalesce to form bigger ones. This model applies in many physical context, in medicine or population dynamics.
Mostly, in the literature on this subject, the clusters are assumed to be entirely determined by their size which is a real positive number. We consider here the *coagulation equation with diffusion*, which is a more natural model that takes into account the position of the clusters. This equation deals with infinite systems of particles which move according to Brownian motions, the diffusion coefficient depends on their size, and in which a coagulation phenomenon occurs. We accept in this model only coalescence in pairs.
Let us first consider the discrete case, that is the particle's size is an positive integer number. We denote by $n(t, x, k)$ the local concentration of particles of size $k \in \mathbb{N}^*$

situated at $x \in \mathbb{R}^p$ ($p \in \mathbb{N}^*$ denotes the space dimension) at time $t \geq 0$. Then n is clearly a $\mathbb{R}_+$-valued function, and satisfies the following discrete diffusive coagulation equation:

$$\begin{cases} \partial_t n(t,x,k) = & d(k)\Delta_x n(t,x,k) + \dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1} K(j,k-j)n(t,x,j)n(t,x,k-j) \\ & -n(t,x,k)\displaystyle\sum_{j=1}^{\infty} K(j,k)n(t,x,j), \qquad (t,x,k) \in (0,\infty)\times\mathbb{R}^p\times\mathbb{N}^* \\ n(0,x,k) = & n_0(x,k), \qquad (t,x) \in \mathbb{R}^p \times \mathbb{N}^*. \end{cases} \tag{DS}$$

The coagulation kernel $K : \mathbb{N}^* \times \mathbb{N}^* \mapsto \mathbb{R}_+$ is positive and symmetric. The diffusion part d depends only on the size of the cluster and is a map from $\mathbb{N}^*$ into $\mathbb{R}_+$. The initial condition n_0 is a positive map from $\mathbb{R}^p \times \mathbb{N}^*$ into $\mathbb{R}_+$.
The interpretation of this equation is: every particle moves according to a Brownian motion, with a diffusion coefficient which depends on its size. This explains the first term. If two particles of sizes j and $k-j$ are at the same place $x \in \mathbb{R}^p$, at the same time t, they may coagulate and produce a new particle of size k, at position x. The frequency of this phenomenon is proportional to $n(t,x,j)n(t,x,k-j)$, and also to the proportionality constant $K(j,k-j)$. This is expressed in the second term. Finally, the last term shows that sometimes, at x, a particle of size k disappears because it has coagulated with another particle (whose position was also x).
Uniqueness for equation (DS) is an open problem, and the better existence result seems to be the one of Laurençot-Mischler [LM01a] who considered a more general diffusive coagulation-fragmentation equation in a bounded domain. Reduced to the coagulation situation their existence result holds under the assumptions: $\sum_{k\geq 1} k \int n_0(x,k)dx < \infty$, and for all $i \geq 1$, $\lim_{j\to\infty} K(i,j)/j = 0$ and $d(i) > 0$.
Similarly when the particles have their sizes in the whole space $\mathbb{R}_+$, we obtain the continuous model which describes the local average number per unit of volume, $n(t,x,z)$ of particles having position x and size z at time t. It writes

$$\begin{cases} \partial_t n(t,x,z) = & d(z)\Delta_x n(t,x,z) + \dfrac{1}{2}\displaystyle\int_0^z K(z',z-z')n(t,x,z')n(t,x,z-z')dz' \\ & -n(t,x,z)\displaystyle\int_0^\infty K(z,z')n(t,x,z')dz', \quad (t,x,z) \in [0,\infty)\times\mathbb{R}^p\times\mathbb{N}^* \\ n(0,x,z) = & n_0(x,z), \qquad (x,z) \in \mathbb{R}^p \times \mathbb{N}^*. \end{cases} \tag{CS}$$

In this case, the coagulation kernel K is a map from $\mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$, the diffusion part d is a map from $\mathbb{R}_+$ into $\mathbb{R}_+$ and the initial condition goes from $\mathbb{R}^p \times \mathbb{R}_+$ into $\mathbb{R}_+$. Once again, no uniqueness is known, and the better existence result was obtained by Laurençot-Mischler [LM01b] for a more general model of coagulation and fragmentation in an open bounded and smooth domain Ω. Their result applies in the coagulation situation under the hypothesis : $d + 1/d \in L^\infty_{loc}((0,\infty))$, $K \in L^\infty_{loc}([0,\infty)^2)$, $\lim_{z'\to\infty} \sup_{z\in(0,R)} K(z,z')/(1+z') = 0$, for all $R < \infty$, $n_0(x,z) \in L^1(\Omega \times \mathbb{R}_+, (1+z)dxdz)$ and $K(z', z-z') \leq K(z',z)$, for all $z' \geq z \geq 0$.
Spatially homogeneous versions of (DS) and (CS) have been much investigated, since the discrete model has been introduced by Smoluchowski in 1916. Let us mention the review of Aldous [Ald99], the probabilistic interpretations of Jéon [Jeo98] and Norris [Nor99, Nor00], see also Deaconu-Fournier-Tanré [DFTab] for another point of view con-

taining both discrete and continuous cases.
In particular, the use of stochastic particle systems was proved to be very efficient to solve numerically homogeneous coagulation equations: the famous Marcus-Lushnikov model (Marcus [Mar68], Lushnikov [Lus78]) is now in competition with a new particle system in which the number of particles is constant in time, first introduced by Eibeck-Wagner [EW00], and also studied in Deaconu-Fournier-Tanré [DFT01], or Jourdain [Jou01]. The literature on deterministic numerical schemes is quite poor.
Due to its difficulty, the non homogeneous case is of course much less treated. Our probabilistic interpretation and numerical scheme is essentially inspired from works of Tanaka [Tan78] and Graham-Méléard [GM97], concerning Boltzmann equations, and is an extension of the non homogeneous case of the previous papers of Eibeck-Wagner [EW00] and Deaconu-Fournier-Tanré [DFTab, DFT01].
Let us now describe the present work. We use the *a priori* conservation of mass to rewrite equations (DS) and (CS) in terms of probability measures. After, we introduce a mollified equation, which delocalises the coagulation phenomena: instead of allowing only coagulation of particles which have the same position, we allow coagulation of particles between which the distance is smaller than some ε. Then, we relate the mollified equation to a nonlinear jumping stochastic differential equation, whose solution $(X_t, Z_t)_{t\geq 0}$ is related to (DS) or (CS). More precisely its law is solution, in a certain sense, to the mollified diffusive coagulation equation. The Markov process $(X_t, Z_t)_{t\geq 0}$, with values in $\mathbb{R}^p \times \mathbb{R}_+$, can be seen as the evolution of the couple of characteristics (position, size) of a sort of typical particle.
In order to prove an existence result for the nonlinear S.D.E., and to obtain a numerical scheme for our equations, we "linearise" the S.D.E. by using an (exactly simulable) stochastic particle system. We prove for this system, a propagation of chaos result. Finally, we obtain the convergence of the mollified solution, up to extraction, to the true solution. This is proved under quite stringent assumptions, by using some ideas of a forthcoming paper by Norris [Nor01].
The paper is organised as follows: in Section 2, we introduce successively the notations and equations: weak form, mollified equation, nonlinear S.D.E. and particle system. Section 3 is devoted to the statements of our main assumptions and results. The proofs are given in Section 4. Finally, we present numerical results in Section 5.

2 Framework

In this section, we write in an unified equation the discrete and the continuous models. After this we introduce a "mollified" form of the equation, which allows us to present a probabilistic approach and to construct the corresponding particle system.

2.1 A mollified approximation

An *a priori* assumption for equations (DS) and (CS) is the conservation of mass, which writes in the discrete case $\sum_{k\geq 1}\int_{\mathbb{R}^p} kn(t,x,k)dx = \sum_{k\geq 1}\int_{\mathbb{R}^p} kn_0(x,k)dx = 1$ for all $t \geq 0$, while in the continuous case, $\int_{\mathbb{R}^p\times\mathbb{R}_+} zn(t,x,z)dxdz = \int_{\mathbb{R}^p\times\mathbb{R}_+} zn_0(x,z)dxdz = 1$. It is well-known that even in the spatially homogeneous context, the conservation of mass is far from holding in any case, for example it is classical that for $K(x,y) = xy$, there exists

a first instant $T_{gel} < \infty$ after which the mass decreases strictly. In this paper we are not dealing with such a phenomenon (called gelation). We assume that the total mass is preserved over time.
This conservation of mass allows us to adopt the following point of view. In the discrete case, for each t, $Q_t(dx, dz) = \sum_{k\geq 1} kn(t, x, k)\delta_k(dz)dx$ is a probability measure on $\mathbb{R}^p \times \mathbb{N}^*$, while in the continuous case, $Q_t(dx, dz) = zn(t, x, z)dxdz$ is a probability measure on $\mathbb{R}^p \times \mathbb{R}_+$ for each t. In both cases, the space part of Q_t has a density for each t, so that we can disintegrate Q_t as $Q_t(dx, dz) = \gamma_t(x, dz)dx$. Writing (DS) or (CS) in terms of Q_t leads to the same equation, and we unify the study of these equations. We have thus to consider initial distributions of the form $Q_0(dx, dz)$ which belong to $\mathcal{P}(\mathbb{R}^p \times \mathbb{R}_+)$, and which have to be thought in the discrete (respectively continuous) case, of the form $Q_0(dx, dz) = \sum_{k\geq 1} kn_0(x, k)\delta_k(dz)dx$ (respectively $Q_0(dx, dz) = zn_0(x, z)dxdz$). We first need the following definition.

Definition 2.1 *Let $Q_0(dx, dz)$ be a probability measure on $\mathbb{R}^p \times \mathbb{R}_+$. We denote by Q_0^z the "size" marginal of Q_0: for $A \subset \mathbb{R}_+$, $Q_0^z(A) = Q_0(\mathbb{R}^p \times A)$. Then we define*

$$\mathcal{H}_{Q_0} = \overline{\left\{ \sum_{i=1}^{n} z_i \ ; \ n \in \mathbb{N}^*, \ z_i \in \text{Supp } Q_0^z \right\}}^{\mathbb{R}_+}. \tag{2.1}$$

The space $\mathcal{H}_{Q_0}$ represents the space in which the particles have their sizes. Essentially, $\mathcal{H}_{Q_0}$ will be equal to $\mathbb{N}^*$ in the discrete case and to $\mathbb{R}_+$ in the continuous situation. Rewriting (DS) and (CS) in terms of probability measures leads to the following equation:

Definition 2.2 *Let Q_0 be in $\mathcal{P}(\mathbb{R}^p \times \mathbb{R}_+)$. $\{Q_t(dx, dz)\}_{t\geq 0}$ is a solution to (WS) if:*

1. *For each $t > 0$, $Q_t(dx, dz)$ is a probability measure on $\mathbb{R}^p \times \mathcal{H}_{Q_0}$ which can be disintegrated as $Q_t(dx, dz) = \gamma_t(x, dz)dx$.*

2. *For each $T > 0$,*

$$\sup_{t\in[0,T]} \left[\int_{\mathbb{R}^p\times\mathbb{R}_+} d(z)Q_t(dx, dz) + \int_{\mathbb{R}^p\times\mathbb{R}_+} Q_t(dx, dz) \int_{\mathbb{R}_+} K(z, z')\gamma_t(x, dz') \right] < \infty.$$

3. *For all $\phi \in C_c^2(\mathbb{R}^p \times (0, \infty))$ and all $t \geq 0$*

$$\int_{\mathbb{R}^p\times\mathbb{R}_+} \phi(x, z)Q_t(dx, dz) = \int_{\mathbb{R}^p\times\mathbb{R}_+} \phi(x, z)Q_0(dx, dz) + \int_0^t \int_{\mathbb{R}^p\times\mathbb{R}_+} \Delta_x\phi(x, z)d(z)Q_s(dx, dz)ds$$
$$+ \int_0^t \int_{\mathbb{R}^p\times\mathbb{R}_+} \int_{\mathbb{R}_+} [\phi(x, z + z') - \phi(x, z)] \frac{K(z, z')}{z'} \gamma_s(x, dz')Q_s(dx, dz)ds. \tag{WS}$$

Let us now write precisely the connections between (DS), (CS) and (WS).

Remark 2.3 *Consider a solution $\{Q_t\}_{t\geq 0}$ to (WS) with initial condition Q_0, admitting, for all $t > 0$, the disintegration $Q_t(dx, dz) = \gamma_t(x, dz)dx$.*

1. *Assume that Q_0 has its support contained in $\mathbb{R}^p \times \mathbb{N}^*$. Then clearly $\mathcal{H}_{Q_0}$ is included in $\mathbb{N}^*$, and hence γ_t can be written, for any $t > 0$, as $\gamma_t(x, dz) = \sum_{k\geq 1} \gamma_t(x, \{k\})\delta_k(dz)$. Then $n(t, x, k) = \gamma_t(x, \{k\})/k$ is a solution (in a weak sense) to (DS) with initial condition $n_0(x, k)dx = Q_0(dx, \{k\})/k$.*

2. *If now $Q_0(dx,dz)$ has a density, i.e. if it can be written as $Q_0(dx,dz) = \mu_0(x,z)dzdx$, and if for all $t \geq 0$, $Q_t(dx,dz) = \mu(t,x,z)dxdz$, then $n(t,x,z) = \mu(t,x,z)/z$ is a solution to (CS) (in a weak sense) with initial condition $n_0(x,z) = \mu_0(x,z)/z$.*

These remarks can be proved exactly as in the homogeneous case, and rely on very simple computations. We refer to Deaconu-Fournier-Tanré [DFTab].
We now introduce a mollified version of the equation (WS). It simply means that we want to delocalise the interactions. Similar approximations are often used for the non homogeneous Boltzmann equation, see *e.g.* Graham-Méléard [GM97]. This leads us to define a last equation.

Definition 2.4 *Let Q_0 be a probability measure on $\mathbb{R}^p \times \mathbb{R}_+$, and let $\varepsilon > 0$ be fixed. We say that $\{Q_t^\varepsilon(dx,dz)\}_{t\geq 0}$ is a solution to $(WS(\varepsilon))$ if for each $t > 0$, $Q_t^\varepsilon(dx,dz)$ is a probability measure on $\mathbb{R}^p \times \mathcal{H}_{Q_0}$, and if*
1. For all $T > 0$,

$$\sup_{t\in[0,T]} \left[\int_{\mathbb{R}^p\times\mathbb{R}_+} d(z) Q_t^\varepsilon(dx,dz) + \int_{\mathbb{R}^p\times\mathbb{R}_+} Q_t^\varepsilon(dx,dz) \int_{\mathbb{R}^p\times\mathbb{R}_+} Q_t^\varepsilon(dx',dz') K(z,z') \right] < \infty,$$

2. For all $\phi \in C_c^2(\mathbb{R}^p \times (0,\infty))$, and all $t \geq 0$,

$$\begin{aligned}\int_{\mathbb{R}^p\times\mathbb{R}_+} \phi(x,z) Q_t^\varepsilon(dx,dz) = & \int_{\mathbb{R}^p\times\mathbb{R}_+} \phi(x,z) Q_0(dx,dz) + \int_0^t \int_{\mathbb{R}^p\times\mathbb{R}_+} \Delta_x \phi(x,z) d(z) Q_s^\varepsilon(dx,dz) ds \\ & + \int_0^t \int_{\mathbb{R}^p\times\mathbb{R}_+} \int_{\mathbb{R}^p\times\mathbb{R}_+} [\phi(x,z+z') - \phi(x,z)] \frac{K(z,z')}{z'} \frac{1\!\!1_{\{|x-x'|\leq\varepsilon\}}}{v_p \varepsilon^p} Q_s^\varepsilon(dx',dz') Q_s^\varepsilon(dx,dz) ds\end{aligned} \tag{$WS(\varepsilon)$}$$

where $v_p := \frac{\pi^{p/2}}{\Gamma(1+p/2)}$ denotes the volume of the unit ball in $\mathbb{R}^p$.

2.2 Probabilistic approach

Our aim is now to present a stochastic differential equation which is "equivalent", in a certain sense, to $(WS(\varepsilon))$. In other words, we want to construct a stochastic process $(X_t^\varepsilon, Z_t^\varepsilon)_{t\geq 0}$, with values in $\mathbb{R}^p \times \mathbb{R}_+$, whose time marginals $\{Q_t^\varepsilon(dx,dz)\}_{t\geq 0} = \{\mathcal{L}(X_t^\varepsilon, Z_t^\varepsilon)\}_{t\geq 0}$ is solution to $(WS(\varepsilon))$. The process $(X^\varepsilon, Z^\varepsilon)$ can be seen as the evolution of the characteristics (position, size) of a typical particle. Naturally, the process $(X^\varepsilon, Z^\varepsilon)$ has to be a Markov process (non time-homogeneous), who belongs *a.s.* to the following path space:

Definition 2.5 *We denote by $\mathcal{P}_2(\mathbb{R}^p \times \mathbb{R}_+)$ the set of probability measures Q_0 on $\mathbb{R}^p \times \mathbb{R}_+$ admitting a second order moment: $\int (|x|^2 + z^2) Q_0(dx,dz) < \infty$. For Q_0 in $\mathcal{P}_2(\mathbb{R}^p \times \mathbb{R}_+)$, we denote by $\mathcal{T}_{Q_0}$ the space of functions from $[0,\infty)$ into $\mathbb{R}^p \times \mathcal{H}_{Q_0}$ defined by*

$$\mathcal{T}_{Q_0} = C([0,\infty), \mathbb{R}^p) \times \mathbb{D}^\uparrow([0,\infty), \mathcal{H}_{Q_0}) \tag{2.4}$$

where $\mathbb{D}^\uparrow([0,\infty), \mathcal{H}_{Q_0})$ is the space of $\mathcal{H}_{Q_0}$-valued increasing càdlàg functions on $[0,\infty)$. We also denote by $\mathcal{P}_2(\mathcal{T}_{Q_0})$ the set of probability measures Q on $\mathcal{T}_{Q_0}$ admitting a moment of order 2 in the strong sense, that is for any $T < \infty$,

$$\int_{\mathcal{T}_{Q_0}} \sup_{[0,T]} \left(|x(t)|^2 + [z(t)]^2 \right) dQ(x,z) < \infty. \tag{2.5}$$

Then, we restrict the problem $(WS(\varepsilon))$ to these spaces. We introduce now the martingale problem naturally associated to $(WS(\varepsilon))$.

Definition 2.6 *Let $\varepsilon > 0$ and $Q_0 \in \mathcal{P}_2(\mathbb{R}^p \times \mathbb{R}_+)$ be fixed. Consider the canonical process $(X_t, Z_t)_{t\geq 0}$ on $\mathcal{T}_{Q_0}$. Consider a probability measure $Q^\varepsilon \in \mathcal{P}_2(\mathcal{T}_{Q_0})$ and denote by Q^ε_t the law of (X_t, Z_t) under Q^ε. We say that Q^ε satisfies the martingale problem $(MP(\varepsilon))$ if the law of (X_0, Z_0) under Q^ε is Q_0 and if*
1. For all $T > 0$,

$$\sup_{t\in[0,T]} \left[\int_{\mathbb{R}^p\times\mathbb{R}_+} d(z) Q^\varepsilon_t(dx, dz) + \int_{\mathbb{R}^p\times\mathbb{R}_+} Q^\varepsilon_t(dx, dz) \int_{\mathbb{R}^p\times\mathbb{R}_+} Q^\varepsilon_t(dx', dz') K(z, z') \right] < \infty.$$

2. For any $\phi \in C^2_b(\mathbb{R}^p \times \mathbb{R}_+)$, the process

$$\begin{aligned} &\phi(X_t, Z_t) - \phi(X_0, Z_0) - \int_0^t \Delta_x \phi(X_s, Z_s) d(Z_s) ds \\ &- \int_0^t \int_{\mathbb{R}^p\times\mathbb{R}_+} [\phi(X_s, Z_s + z') - \phi(X_s, Z_s)] \frac{K(Z_s, z')}{z'} \frac{1\!\!1_{\{|X_s - x'| \leq \varepsilon\}}}{v_p \varepsilon^p} Q^\varepsilon_s(dx', dz') ds \end{aligned} \tag{2.6}$$

is a square integrable Q^ε-martingale.

By taking expectations in (2.6), we find the natural connection between $(MP(\varepsilon))$ and $(WS(\varepsilon))$. We express it in the following remark.

Remark 2.7 *Let $\varepsilon > 0$ and $Q_0 \in \mathcal{P}_2(\mathbb{R}^p \times \mathbb{R}_+)$ be fixed. Assume that $Q^\varepsilon \in \mathcal{P}_2(\mathcal{T}_{Q_0})$ satisfies the martingale problem $(MP(\varepsilon))$. Then the flow of the time-marginals $(Q^\varepsilon_t)_{t\geq 0}$ of Q^ε satisfies $(WS(\varepsilon))$.*

We may also give a pathwise representation of this martingale problem. To this aim, we need to introduce some more notations.

Definition 2.8 *We consider two probability spaces: $(\Omega, \mathcal{F}, \mathbb{P})$ is an abstract space and $([0,1], \mathcal{B}[0,1], d\alpha)$ is an auxiliary space (here $d\alpha$ denotes the Lebesgue measure). In order to avoid confusion, the elements on this second space will be called $\alpha-$elements.*

Definition 2.9 *Let $\varepsilon > 0$ and $Q_0 \in \mathcal{P}_2(\mathbb{R}^p \times \mathbb{R}_+)$ be fixed. We say that $((X_0, Z_0), (X^\varepsilon, Z^\varepsilon), (\tilde{X}^\varepsilon, \tilde{Z}^\varepsilon), N^\varepsilon, B)$ is a solution to $(SDE(\varepsilon))$ if the following conditions hold.*

1. *(X_0, Z_0) is a random variable with values in $\mathbb{R}^p \times \mathbb{R}_+$, of law Q_0. $(B_t)_{t\geq 0}$ is a $\mathbb{R}^p$-valued Brownian motion. The random measure $N^\varepsilon(ds, d\alpha, du)$ is a Poisson measure on $[0,\infty) \times [0,1] \times [0,\infty)$ with intensity measure $(1/v_p\varepsilon^p) ds d\alpha du$. These random elements are independent.*

2. *$(X^\varepsilon_t(\omega), Z^\varepsilon_t(\omega))_{t\geq 0}$ is a process belonging a.s. to $\mathcal{T}_{Q_0}$ and its law belongs to $\mathcal{P}_2(\mathcal{T}_{Q_0})$. $(\tilde{X}^\varepsilon_t(\alpha), \tilde{Z}^\varepsilon_t(\alpha))_{t\geq 0}$ is an α-process such that $\mathcal{L}(X^\varepsilon, Z^\varepsilon) = \mathcal{L}_\alpha(\tilde{X}^\varepsilon, \tilde{Z}^\varepsilon)$.*

3. *For all $T > 0$ $\sup_{t\in[0,T]} \Big(\mathbb{E}[d(Z^\varepsilon_t)] + \mathbb{E}\mathbb{E}_\alpha[K(Z^\varepsilon_t, \tilde{Z}^\varepsilon_t(\alpha))] \Big) < \infty.$*

4. The following system of S.D.E.s is satisfied:

$$\begin{cases} X_t^\varepsilon = X_0 + \int_0^t \sqrt{2d(Z_s^\varepsilon)}dB_s \\ Z_t^\varepsilon = Z_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{Z}_{s-}^\varepsilon(\alpha) \mathbb{1}_{\left\{u \le \frac{K(Z_{s-}^\varepsilon, \tilde{Z}_{s-}^\varepsilon(\alpha))}{\tilde{Z}_{s-}^\varepsilon(\alpha)}\right\}} \mathbb{1}_{\{|X_{s-}^\varepsilon - \tilde{X}_{s-}^\varepsilon(\alpha)| \le \varepsilon\}} N^\varepsilon(ds, d\alpha, du). \end{cases} \tag{$SDE(\varepsilon)$}$$

Let us now clarify the connection between $(MP(\varepsilon))$ and $(SDE(\varepsilon))$. Roughly speaking a solution Q^ε to $(MP(\varepsilon))$ is the law of a solution $(X^\varepsilon, Z^\varepsilon)$ to $(SDE(\varepsilon))$.

Remark 2.10 *Let $\varepsilon > 0$ and $Q_0 \in \mathcal{P}_2(\mathbb{R}^p \times \mathbb{R}_+)$ be fixed.*

1. *Assume that $((X_0, Z_0), (X^\varepsilon, Z^\varepsilon), (\tilde{X}^\varepsilon, \tilde{Z}^\varepsilon), N^\varepsilon, B)$ is a solution to $(SDE(\varepsilon))$. Then the law $Q^\varepsilon = \mathcal{L}(X^\varepsilon, Z^\varepsilon)$ is a solution to the martingale problem $(MP(\varepsilon))$.*

2. *Assume now that $Q^\varepsilon \in \mathcal{P}_2(\mathcal{T}_{Q_0})$ satisfies $(MP(\varepsilon))$. Let $(\tilde{X}^\varepsilon, \tilde{Z}^\varepsilon)$ be any $\mathcal{T}_{Q_0}$-valued α-process of law Q^ε. Denote by (X, Z) the canonical $\mathcal{T}_{Q_0}$-valued process. Then there exists, on an enlarged probability space (from the canonical one), a Poisson measure $N^\varepsilon(ds, d\alpha, du)$, a Brownian motion $(B_t)_{t\ge 0}$, and a $\mathbb{R}^p \times \mathbb{R}$-valued random variable (X_0, Z_0) of law Q_0 such that $((X_0, Z_0), (X, Z), (\tilde{X}^\varepsilon, \tilde{Z}^\varepsilon), N^\varepsilon, B)$ is a solution to $(SDE(\varepsilon))$.*

Once again we do not prove this remark, because it is completely standard. To prove 1, it suffices to apply Itô formula to $\phi(X_t, Z_t)$. The proof of 2 is more difficult, but it relies on now well-known representation theorems for jump processes. We refer to Deaconu-Fournier-Tanré [DFTab] for rigorous proofs of similar results (in the homogeneous case).

2.3 The interacting particle system

We now would like to introduce a numerical approximation scheme to approach the solution of $(SDE(\varepsilon))$. To this aim, we introduce an interacting particle system, which allows to linearize the nonlinear equation $(SDE(\varepsilon))$.

Definition 2.11 *Let $\varepsilon > 0$ and $Q_0 \in \mathcal{P}_2(\mathbb{R}^p \times \mathbb{R}_+)$ be fixed. The number $n \ge 1$ of particles is also fixed. For $i \in \{1, ..., n\}$, consider a family $(B_t^{i,n})_{t\ge 0}$ of i.i.d. $\mathbb{R}^p$-valued Brownian motions, and a family $(X_0^{i,n}, Z_0^{i,n})$ of i.i.d. $\mathbb{R}^p \times \mathbb{R}_+$-valued random variables of law Q_0. Finally, denote by $(N^{i,n,\varepsilon}(ds, dj, du))$, for $i \in \{1, ..., n\}$, a family of i.i.d. Poisson measures on $[0, \infty) \times \{1, ..., n\} \times [0, \infty)$, with intensity measures*

$$\frac{1}{v_p \varepsilon^p} ds \left(\frac{1}{n} \sum_{k=1}^n \delta_k(dj) \right) du. \tag{2.8}$$

All these random elements are supposed to be independent. A process $\{(X_t^{1,n,\varepsilon}, Z_t^{1,n,\varepsilon}), ..., (X_t^{n,n,\varepsilon}, Z_t^{n,n,\varepsilon})\}_{t\ge 0} \in (\mathcal{T}_{Q_0})^n$ is said to solve the interacting particle system $(PS(\varepsilon, n))$ if for any $t \ge 0$ and any $i \in \{1, ..., n\}$

$$\begin{cases} X_t^{i,n,\varepsilon} = X_0^{i,n} + \int_0^t \sqrt{2d(Z_s^{i,n,\varepsilon})} dB_s^{i,n} \\ Z_t^{i,n,\varepsilon} = Z_0^{i,n} + \int_0^t \int_j \int_0^\infty Z_{s-}^{j,n,\varepsilon} \mathbb{1}_{\left\{u \le \frac{K(Z_{s-}^{i,n,\varepsilon}, Z_{s-}^{j,n,\varepsilon})}{Z_{s-}^{j,n,\varepsilon}}\right\}} \mathbb{1}_{\{|X_{s-}^{i,n,\varepsilon} - X_{s-}^{j,n,\varepsilon}| \le \varepsilon\}} N^{i,n,\varepsilon}(ds, dj, du). \end{cases} \tag{$PS(\varepsilon,n)$}$$

For $\{(X_t^{1,n,\varepsilon}, Z_t^{1,n,\varepsilon}), ..., (X_t^{n,n,\varepsilon}, Z_t^{n,n,\varepsilon})\}_{t\geq 0} \in (\mathcal{T}_{Q_0})^n$ a solution to $(PS(\varepsilon, n))$, we denote by $\mu^{n,\varepsilon} = \frac{1}{n}\sum_{i=1}^n \delta_{(X^{i,n,\varepsilon}, Z^{i,n,\varepsilon})}$ the corresponding empirical measure, which is a $\mathcal{P}(\mathcal{T}_{Q_0})$-valued random variable.

Naturally we expect that for every fixed k, $\{(X^{i_1,n,\varepsilon}, Z^{i_1,n,\varepsilon}), \ldots, (X^{i_k,n,\varepsilon}, Z^{i_k,n,\varepsilon})\}$ become independent as n tends to infinity, and thus that for any i_0, $(X^{i_0,n,\varepsilon}, Z^{i_0,n,\varepsilon})$ converges, in a certain sense, to a solution $(X^\varepsilon, Z^\varepsilon)$ of $(SDE(\varepsilon))$ as n tends to infinity.

3 Main results

We collect in this section the results (and main steps of the proofs) of the paper.

3.1 Existence for the particle system $(PS(n, \varepsilon))$

We first give an existence result for $(PS(n, \varepsilon))$, which requires the following very weak assumption on the initial condition, the coagulation kernel K and the diffusion term d.

Assumption $(H1)$:

1. The initial condition Q_0 belongs to $\mathcal{P}_2(\mathbb{R}^p \times \mathbb{R}_+)$, and $Q_0(\mathbb{R}^p \times (0, \infty)) = 1$.
2. The coagulation kernel $K : \mathcal{H}_{Q_0} \times \mathcal{H}_{Q_0} \mapsto \mathbb{R}_+$ is measurable, symmetric, and there exists a constant C_K such that
$$K(z, z') \leq C_K(1 + z + z'), \quad \forall z, z' \in \mathcal{H}_{Q_0}.$$
3. The diffusion $d : \mathcal{H}_{Q_0} \mapsto \mathbb{R}_+$ is measurable, and there exists a constant C_d such that
$$\sqrt{2d(z)} \leq C_d(1 + z), \quad \forall z \in \mathcal{H}_{Q_0}.$$

Proposition 3.1 *Let $\varepsilon > 0$ and $n \geq 1$ be fixed. Assume $(H1)$. Then there exists a unique solution $\{(X_t^{1,n,\varepsilon}, Z_t^{1,n,\varepsilon}), ..., (X_t^{n,n,\varepsilon}, Z_t^{n,n,\varepsilon})\}_{t\geq 0} \in (\mathcal{T}_{Q_0})^n$ to (PS(n, ε)).*
Assume furthermore that for some $q \geq 1$, the initial condition Q_0 has a moment of order q. Then for any $T < \infty$, there exists a constant $C(T, \varepsilon)$ such that

$$\sup_{n\geq 1} \sup_{i\in\{1,\ldots,n\}} \mathbb{E}\left[\sup_{[0,T]} \left(|X_t^{i,n,\varepsilon}|^q + |Z_t^{i,n,\varepsilon}|^q\right)\right] \leq C(T, \varepsilon). \tag{3.1}$$

3.2 Convergence of $(PS(n, \varepsilon))$ to $(SDE(\varepsilon))$

For ε fixed we can obtain a compactness result for the particle system $(PS(n, \varepsilon))$, under the assumption $(H1)$ and a stronger hypothesis on Q_0:

Assumption $(H2)$: There exists $q_0 > 2$ such that the initial condition Q_0 admits a moment of order q_0.

Let us first of all precise the topologies we use.

Notation 3.2 *1. $C([0, \infty), \mathbb{R}^p)$ is endowed with the topology of the uniform convergence on every compact subset of $[0, \infty)$, while $\mathbb{D}^\uparrow([0, \infty), \mathcal{H}_{Q_0})$ is endowed with its Skorohod topology. $\mathcal{T}_{Q_0}$ is endowed with the corresponding product topology.*

2. $\mathcal{P}(\mathcal{T}_{Q_0})$ *and* $\mathcal{P}(\mathcal{P}(\mathcal{T}_{Q_0}))$ *are endowed with the corresponding weak topologies.*

We have the following compactness result.

Proposition 3.3 *Let* $\varepsilon > 0$ *be fixed, and assume* $(H1)$ *and* $(H2)$*.*

1. *Then* $\{\mathcal{L}(X^{1,n,\varepsilon}, Z^{1,n,\varepsilon})\}_n$ *is tight in the set* $\mathcal{P}(\mathcal{T}_{Q_0})$ *of probability measures on* $\mathcal{T}_{Q_0}$*.*
2. *Consider the empirical measure* $\mu^{n,\varepsilon}$ *associated with* $(PS(n,\varepsilon))$*. Then* $\{\mathcal{L}(\mu^{n,\varepsilon})\}_n$ *is tight in* $\mathcal{P}(\mathcal{P}(\mathcal{T}_{Q_0}))$*.*

We show also that a propagation of chaos result holds for the particle system. This means that the particles become asymptotically independent as their number n goes to infinity. This allows us to obtain that any limiting point of $\{\mu^{n,\varepsilon}\}_n$ is deterministic. To prove this result, we need to add some stronger hypotheses.

Assumption $(H3)$: There exists a minimum size, *i.e.* there exists $c_0 > 0$ such that $Q_0(\mathbb{R}^p \times [c_0, \infty)) = 1$.

This assumption is quite stringent in the continuous case, but it always holds in the discrete case, with $c_0 = 1$.

Proposition 3.4 *Let* $\varepsilon > 0$ *be fixed and assume* $(H1)$ *and* $(H3)$*. Then there is propagation of chaos in (local) variation norm for the particle system. This simply means that for any* $T > 0$ *and any* $k \geq 1$ *fixed*

$$\left|\mathcal{L}[(X^{1,n,\varepsilon}, Z^{1,n,\varepsilon}), ..., (X^{k,n,\varepsilon}, Z^{k,n,\varepsilon})] - [\mathcal{L}(X^{1,n,\varepsilon}, Z^{1,n,\varepsilon})]^{\otimes k}\right|_T \underset{n\to\infty}{\longrightarrow} 0 \tag{3.2}$$

where $|\ .\ |_T$ *is the variation norm on the set of measures on* $[C([0,T],\mathbb{R}^p)\times\mathbb{D}^{\uparrow}([0,T],\mathcal{H}_{Q_0})]^k$*.*

Then we deduce the following consequence.

Corollary 3.5 *Let* $\varepsilon > 0$ *be fixed and assume* $(H1)$ *to* $(H3)$*. Consider any limiting point* μ^ε *of* $\{\mu^{n,\varepsilon}\}_n$*, say that* μ^ε *is the limit (in law) of* $\{\mu^{n_k,\varepsilon}\}_k$*. Then* μ^ε *is a deterministic probability measure on* $\mathcal{T}_{Q_0}$*. This implies that the convergence of* $\mu^{n_k,\varepsilon}$ *to* μ^ε *holds also in probability. Furthermore, the* $\mathcal{T}_{Q_0}$*-valued random variable* $(X^{1,n_k,\varepsilon}, Z^{1,n_k,\varepsilon})$ *converges in law to a* $\mathcal{T}_{Q_0}$*-valued random variable of law* μ^ε*.*

The next step is to prove that on the "boundary" of the indicator function $\mathbb{1}_{\{|x-x'|\leq\varepsilon\}}$, there is no problem.

Assumption $(H4)$: The diffusion part d is continuous and strictly positive on $\mathcal{H}_{Q_0}$. The coagulation kernel K is continuous on $\mathcal{H}_{Q_0} \times \mathcal{H}_{Q_0}$.

Proposition 3.6 *Let* $\varepsilon > 0$ *be fixed and assume* $(H1)$ *to* $(H4)$*. Consider any limiting point* μ^ε *of* $\{\mu^{n,\varepsilon}\}_n$*, say that* μ^ε *is the limit of* $\{\mu^{n_k,\varepsilon}\}_k$*. Then* μ^ε *is the law of a* $\mathcal{T}_{Q_0}$*-valued random variable* $(X^\varepsilon, Z^\varepsilon)$*. For any* $t > 0$*, the law of* X^ε_t *has a density with respect to the Lebesgue measure on* $\mathbb{R}^p$*.*

We will finally be able to conclude the convergence of $(PS(n,\varepsilon))$ to $(SDE(\varepsilon))$.

Theorem 3.7 *Let* $\varepsilon > 0$ *be fixed and assume* $(H1)$ *to* $(H4)$*. Then any limiting point* $\mu^\varepsilon \in \mathcal{P}(\mathcal{T}_{Q_0})$ *of* $\{\mu^{n,\varepsilon}\}_n$ *is a solution to the martingale problem* $(MP(\varepsilon))$*.*

The following corollary is obvious from the previous theorem.

Corollary 3.8 *Let* $\varepsilon > 0$ *be fixed and assume* $(H1)$ *to* $(H4)$*. Then there is existence for* $(MP(\varepsilon))$*, for* $(SDE(\varepsilon))$ *and for* $(WS(\varepsilon))$*.*

3.3 Convergence of $(WS(\varepsilon))$ to (WS)

Our aim is now to prove that the family of solutions to $(WS(\varepsilon))$ converges, up to extraction, to a solution of (WS). Some of the ideas below have been communicated to us by James R. Norris, and we refer to his forthcoming paper [Nor01]. We begin with a lemma, which allows to rewrite $(WS(\varepsilon))$ in terms of semi-group.

Lemma 3.9 *Assume $(H1)$ to $(H4)$. Consider a family of solutions $\{((X_0,Z_0),(X^\varepsilon,Z^\varepsilon),(\tilde{X},\tilde{Z}),N^\varepsilon,B)\}_{\varepsilon>0}$ to $(SDE(\varepsilon))$, and denote by $Q_t^\varepsilon(dx,dz)$ the law of $(X_t^\varepsilon,Z_t^\varepsilon)$. Consider also the p-dimensional heat kernel*

$$q(t,x,y)=\frac{1}{(2\pi t)^{p/2}}e^{-\frac{|x-y|^2}{2t}}. \tag{3.3}$$

Then for all φ in $C_c(\mathbb{R}^p\times(0,\infty))$, for all $t\geq 0$ and all $\varepsilon>0$,

$$\begin{aligned}\int_{\mathbb{R}^p\times\mathbb{R}_+}\varphi(x,z)Q_t^\varepsilon(dx,dz)=\int_{\mathbb{R}^p\times\mathbb{R}_+}Q_0(dx,dz)\int_{\mathbb{R}^p}dyq(td(z),x,y)\varphi(y,z)\\ +\int_0^t ds\int_{\mathbb{R}^p\times\mathbb{R}_+}Q_s^\varepsilon(dx,dz)\int_{\mathbb{R}^p\times\mathbb{R}_+}Q_s^\varepsilon(dx',dz')\frac{K(z,z')}{z'}\frac{1\!\!1_{\{|x-x'|\leq\varepsilon\}}}{v_p\varepsilon^p}\\ \int_{\mathbb{R}^p}dy\Big\{q((t-s)d(z+z'),x,y)\varphi(y,z+z')-q((t-s)d(z),x,y)\varphi(y,z)\Big\}.\end{aligned}\tag{3.4}$$

To obtain the compactness result in ε, we add the following assumption.
Assumption $(H5)$:

1. The diffusion coefficient d is decreasing, and there exist some constants $0<\underline{d}<\overline{d}$ such that for all $z\in\mathcal{H}_{Q_0}$, $\underline{d}\leq d(z)\leq\overline{d}$.
2. There exists a constant C_0 and a probability measure ν_0 on $(0,\infty)$ such that $Q_0(dx,dz)\leq C_0dx\nu_0(dz)$.

Notice that the assumption "d is decreasing" is physically natural, since it says that larger a particle is, slower it moves. We carry on with a second lemma.

Lemma 3.10 *Assume $(H1)$ to $(H5)$. Consider a family of solutions $\{((X_0,Z_0),(X^\varepsilon,Z^\varepsilon),(\tilde{X},\tilde{Z}),N^\varepsilon,B)\}_{\varepsilon>0}$ to $(SDE(\varepsilon))$. Then for each $\varepsilon>0$ and each $t\geq 0$, the law of X_t^ε has a density $\delta^\varepsilon(t,.)$ with respect to the Lebesgue measure on $\mathbb{R}^p$, and this density is uniformly bounded: for all $x\in\mathbb{R}^p$, $\delta^\varepsilon(t,x)\leq C_0\left(\frac{\overline{d}}{\underline{d}}\right)^{p/2}$, where C_0, $\underline{d}$ and $\overline{d}$ are defined in $(H5)$.*

This lemma allows us to obtain a compactness result.

Proposition 3.11 *Assume $(H1)$ to $(H5)$. Consider a family of solutions $\{((X_0,Z_0),(X^\varepsilon,Z^\varepsilon),(\tilde{X},\tilde{Z}),N^\varepsilon,B)\}_{\varepsilon>0}$ to $(SDE(\varepsilon))$. Denote by Q^ε the law of $(X^\varepsilon,Z^\varepsilon)$. Then the family $\{Q^\varepsilon\}_{\varepsilon>0}$ is tight in $\mathcal{P}(\mathcal{T}_{Q_0})$.*

To prove that the limit points are solutions to (WS), we have to obtain a sort of "space-equicontinuity" of the family $Q_t^\varepsilon(dx,dz)=\mathcal{L}(X_t^\varepsilon,Z_t^\varepsilon)$, for $t>0$ fixed. This will be expressed in the next lemma.

Lemma 3.12 *Assume* $(H1)$ *to* $(H5)$. *Consider a family of solutions* $\{((X_0, Z_0), (X^\varepsilon, Z^\varepsilon), (\tilde{X}, \tilde{Z}), N^\varepsilon, B)\}_{\varepsilon>0}$ *to* $(SDE(\varepsilon))$. *Denote by* $Q_t^\varepsilon(dx, dz)$ *the law of* $(X_t^\varepsilon, Z_t^\varepsilon)$. *From Lemma (3.10) we know that we may write* $Q_t^\varepsilon(dx, dz) = \gamma_t^\varepsilon(x, dz)dx$. *Consider a function* $\alpha \in C_c((0,\infty))$ *and set* $f_t^\varepsilon(x) = \int_{\mathbb{R}_+} \alpha(z)\gamma_t^\varepsilon(x, dz)$. *Then* f_t^ε *is differentiable on* $\mathbb{R}^p$ *for any* $t > 0$ *and* $\varepsilon > 0$. *Furthermore, for any* $T > 0$, *there exists a constant* A_T *(depending on the function* α*) such that for all* $0 < t \le T$, *all* $x \in \mathbb{R}^p$ *and all* $\varepsilon > 0$, $|\nabla_x f_t^\varepsilon(x)| \le \frac{A_T}{\sqrt{t}}$.

We are finally able to obtain convergence as ε goes to 0.

Theorem 3.13 *Assume* $(H1)$ *to* $(H5)$. *Consider a family of solutions* $\{((X_0, Z_0), (X^\varepsilon, Z^\varepsilon), (\tilde{X}, \tilde{Z}), N^\varepsilon, B)\}_{\varepsilon>0}$ *to* $(SDE(\varepsilon))$. *Consider a sequence* $Q^{\varepsilon_k} = \mathcal{L}(X^{\varepsilon_k}, Z^{\varepsilon_k})$ *converging weakly to some* Q *in* $\mathcal{P}(\mathcal{T}_{Q_0})$. *Denote by* $Q_t \in \mathcal{P}(\mathbb{R}^p \times \mathbb{R}_+)$ *the time marginal of* Q. *Then*
(i) For all $t \ge 0$, *we may write* $Q_t(dx, dz) = \gamma_t(x, dz)dx$.
(ii) $\{Q_t\}_{t\ge0}$ *is a solution to* (WS).
(iii) $\{Q_t\}_{t\ge0}$ *is spatially regular, in the sense that for all* $\alpha \in C_c((0,\infty))$, *the map* $x \mapsto \int \alpha(z)\gamma_t(x, dz)$ *has a bounded derivative.*

First of all let us mention that our aim in this work is to construct a probabilistic approach and a numerical approximation scheme for the solution of the diffusive coagulation equation. We are not searching for new existence results. Keeping this in mind, let us now compare our existence results with those of Laurençot-Mishler, [LM01a], [LM01b]. Their results are clearly globally better than ours. In the discrete case, we obtain the same type of weak solution than [LM01a]. In the continuous case, we obtain “measure-solutions”, while “function-solutions” are built in [LM01b]. We assume more conditions about the diffusion coefficient d and the initial condition. However, we allow the standard kernel $K(z, z') = z + z'$, which is not the case in [LM01a], [LM01b], who assume that $K(z, z')/z'$ tends to 0 when z' tends to infinity, for each z. Notice finally that in the continuous case, a monotonicity condition about the kernel K is assumed in [LM01b].

4 Proofs

In this section, we give the proofs of the previously announced results.

4.1 Existence for the particle system $(PS(n, \varepsilon))$

We begin with the existence result for the particle system.
Proof of Proposition 3.1 We will only sketch the proof, since it is rather standard. Let us consider $\varepsilon > 0$ and $n \ge 1$ to be fixed. We consider also, as in Definition 2.11, a set of random objects $(B_t^{i,n})_{t\ge0}$, $(X_0^{i,n}, Z_0^{i,n})$, and $N^{i,n,\varepsilon}(ds, dj, du)$, for $i \in \{1, ..., n\}$.
First we can easily obtain existence and uniqueness for a particle system with cutoff $(PS(n, \varepsilon)(M))$. We define for $M > 0$ fixed, $K^M(z, z') = K(z \wedge M, z' \wedge M)$. K^M is clearly bounded. We also set $Z_0^{i,n,M} = Z_0^{i,n} \vee (1/M)$, for each i in $\{1, ..., n\}$, in order to avoid problems for masses near 0.
We denote by $(PS(\varepsilon, n)(M))$ the particle system defined as $(PS(\varepsilon, n))$ after replacing the kernel K by K^M and $Z_0^{i,n}$ by $Z_0^{i,n,M}$ for each i. Then it is clear that one may replace, in $(PS(n, \varepsilon)(M))$, the Poisson measures $N^{i,n,\varepsilon}$ by their restrictions $N^{i,n,\varepsilon}|_{[0,\infty)\times\{1,...,n\}\times[0,C_M]}$,

where $C_M = MC_K(1+2M)$. Existence and uniqueness are completely standard for $(PS(n,\varepsilon)(M))$, see *e.g.* Ikeda-Watanabe, [IW89]. Furthermore, it is quite clear that the unique solution
$((X^{1,n,\varepsilon,M}, Z^{1,n,\varepsilon,M}), ..., (X^{n,n,\varepsilon,M}, Z^{n,n,\varepsilon,M}))$ to $(PS(n,\varepsilon)(M))$ belongs *a.s.* to $(\mathcal{T}_{Q_0})^n$.
We can easily check that for any $T<\infty$, there exists a constant $C(T,\varepsilon)$ depending only on T and ε such that for any $M \geq 1$,

$$\sup_{i\in\{1,...,n\}} \mathbb{E}\left[\sup_{[0,T]} Z_s^{i,n,\varepsilon,M}\right] \leq C(T,\varepsilon). \tag{4.1}$$

Consider now, for each M, the stopping time $\tau_M = \inf\{t>0, \sup_{i\in\{1,...,n\}} |Z_t^{1,n,\varepsilon,M}| > M\}$. By (4.1) we deduce easily that : $\tau_M \to \infty$ *a.s.* as $M\to\infty$.
Consider also the subset Ω_M of Ω defined by $\Omega_M = \left\{\omega; \inf_{i\in\{1,...,n\}} Z_0^{i,n} \geq 1/M\right\}$. Then we know from $(H1)-1$ that $\mathbb{P}(\Omega_M)$ tends to 1 as M increases to infinity.
The way to build a solution to $(PS(n,\varepsilon))$ is now clear. For $\omega\in\Omega$ fixed and $T<\infty$ we choose M large enough, in order to obtain $\omega\in\Omega_M$ and $\tau_M(\omega)\geq T$, and set $X_t^{i,n,\varepsilon}(\omega) = X_t^{i,n,\varepsilon,M}(\omega)$ and $Z_t^{i,n,\varepsilon}(\omega) = Z_t^{i,n,\varepsilon,M}(\omega)$ for all $i\in\{1,...,n\}$ and all $t\in[0,T]$. Classical arguments show that this defines a unique solution to $(PS(n,\varepsilon))$.
We finally check (3.1). Assume thus that for some $q\geq 1$, $\mathbb{E}(Z_0^q + |X_0|^q) < \infty$. Since the particle system is symmetric, and since $Z^{1,n,\varepsilon}$ is nondecreasing, we first notice that $\sup_{i\in\{1,...,n\}} \mathbb{E}\left[\sup_{[0,T]}\{|Z_s^{i,n,\varepsilon}|^q + |X_s^{i,n,\varepsilon}|^q\}\right] \leq \mathbb{E}\left[|Z_T^{1,n,\varepsilon}|^q + \sup_{[0,T]}|X_s^{1,n,\varepsilon}|^q\right]$.
Then, a simple computation using $(H1)$, the Hölder inequality and the symmetry of the system shows that

$$\begin{aligned}
\mathbb{E}\left[|Z_t^{1,n,\varepsilon}|^q\right] &\leq \mathbb{E}(Z_0^q) + \int_0^t \frac{1}{n}\sum_{j=1}^n \mathbb{E}\left[\left\{|Z_s^{1,n,\varepsilon} + Z_s^{j,n,\varepsilon}|^q - |Z_s^{1,n,\varepsilon}|^q\right\} \frac{K(Z_s^{1,n,\varepsilon}, Z_s^{j,n,\varepsilon})}{Z_s^{j,n,\varepsilon}}\right] \frac{ds}{v_p\varepsilon^p} \\
&\leq \mathbb{E}(Z_0^q) + C_{q,\varepsilon}\int_0^t \frac{1}{n}\sum_{j=1}^n \mathbb{E}\left[\left\{|Z_s^{1,n,\varepsilon}|^{q-1} + |Z_s^{j,n,\varepsilon}|^{q-1}\right\}\left\{1 + Z_s^{1,n,\varepsilon} + Z_s^{j,n,\varepsilon}\right\}\right] ds \\
&\leq \mathbb{E}(Z_0^q) + C_{q,\varepsilon}\int_0^t \left[1 + \mathbb{E}\left[|Z_s^{1,n,\varepsilon}|^q\right]\right] ds
\end{aligned} \tag{4.2}$$

where the constant $C_{q,\varepsilon}$ depends only on q and ε. The Gronwall Lemma allows to conclude that $\mathbb{E}\left[|Z_T^{1,n,\varepsilon}|^q\right] \leq C_{q,\varepsilon,T}$.
Next, using the Burkholder-Davis-Gundy and Hölder inequalities and $(H1)$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{[0,T]}|X_s^{1,n,\varepsilon}|^q\right] &\leq C_q\mathbb{E}(|X_0|^q) + C_q\mathbb{E}\left[\left(\int_0^T 2d(Z_s^{1,n,\varepsilon})ds\right)^{q/2}\right] \\
&\leq C_q + C_q\int_0^T \mathbb{E}\left(1 + |Z_s^{1,n,\varepsilon}|^q\right) ds \leq C_{q,\varepsilon,T}
\end{aligned} \tag{4.3}$$

thanks to the previous estimate. This ends the proof. □

4.2 Convergence of $(PS(n,\varepsilon))$ to $(SDE(\varepsilon))$

We now check the tightness result in n.
Proof of Proposition 3.3 We notice first that 1 is equivalent to 2, due to Méléard

[Mél96], Lemma 4.5. It is thus sufficient to prove 1.
Let us also remark that proving 1 is equivalent to prove that $(\mathcal{L}(X^{1,n,\varepsilon}))_{n\geq 1}$ is tight in $C([0,\infty),\mathbb{R}^p)$ and $(\mathcal{L}(Z^{1,n,\varepsilon}))_{n\geq 1}$ is tight in $\mathbb{D}([0,\infty),\mathbb{R}^p)$.
The tightness result of $(\mathcal{L}(X_t^{1,n,\varepsilon}))_n$ is a classical consequence of the fact that $\sup_n \sup_{s\leq T} \mathbb{E}[\sqrt{(d(Z_s^{1,n,\varepsilon}))^{q_0}}] \leq C_{T,\varepsilon}$ with $q_0 > 2$, which follows from hypothesis $(H1)-3$, $(H2)$ and from (3.1).
Consider now the size component $Z^{i,n,\varepsilon}$. We will apply the Aldous criterion (see Jacod-Shiryaev [JS87]). Consider the set $\mathcal{A}(T,\delta)$ of couples (S,S') of stopping times satisfying *a.s.* $S \leq S' \leq (S+\delta)\wedge T$. We have to check that the conditions below are satisfied.
(i) $\left(\mathcal{L}\left(\sup_{[0,T]} |Z_t^{1,n,\varepsilon}|\right)\right)_{n\geq 1}$ is tight in $\mathcal{P}(\mathbb{R}_+)$.
(ii) For any $T>0$ and any $\eta>0$

$$\lim_{\delta\to 0}\sup_n \sup_{(S,S')\in\mathcal{A}(T,\delta)} \mathbb{P}\left[|Z_{S'}^{1,n,\varepsilon} - Z_S^{1,n,\varepsilon}| > \eta\right] = 0. \tag{4.4}$$

Let us notice that (i) is clearly satisfied thanks to (3.1) with $q=1$. In order to prove (ii) we use $(H1)-2$ and (3.1) with $q=1$. We obtain, using symmetry arguments, that

$$\begin{aligned}
&\mathbb{P}\left[|Z_{S'}^{1,n,\varepsilon} - Z_S^{1,n,\varepsilon}| > \eta\right] \leq \frac{1}{\eta}\mathbb{E}\left[|Z_{S'}^{1,n,\varepsilon} - Z_S^{1,n,\varepsilon}|\right]\\
&\leq \frac{1}{\eta}\mathbb{E}\Big[\int_S^{S'}\int_j\int_0^\infty Z_{s-}^{j,n,\varepsilon}\mathbb{1}_{\left\{u\leq\frac{K(Z_{s-}^{1,n,\varepsilon},Z_{s-}^{j,n,\varepsilon})}{Z_{s-}^{j,n,\varepsilon}}\right\}}\mathbb{1}_{\{|X_{s-}^{1,n,\varepsilon}-X_{s-}^{j,n,\varepsilon}|\leq\varepsilon\}}N^{1,n,\varepsilon}(ds,dj,du)\Big]\\
&\leq \frac{1}{\eta}\mathbb{E}\left[\int_S^{(S+\delta)\wedge T} C_K(1+2Z_s^{1,n,\varepsilon})\frac{ds}{v_p\varepsilon^p}\right] \leq A(\varepsilon)\frac{\delta}{\eta}\,\mathbb{E}\left[\sup_{[0,T]}(1+Z_s^{1,n,\varepsilon})\right] \leq \frac{A(T,\varepsilon)}{\eta}\,\delta
\end{aligned} \tag{4.5}$$

where $A(T,\varepsilon)$ depends only on T and ε. This justifies (ii), and concludes the proof. □

The next proof concerns the propagation of chaos result, for the particle system, as the number n of particles grows to infinity, for ε fixed.
Proof of Proposition 3.4 Graham-Méléard proved in [GM97] a similar result for the case of a particle system associated with a non homogeneous Boltzmann equation. Although we can not apply their result, one can follow their proof line by line in the case where the total rate of jump per particle is finite. We thus first consider a case with cutoff in Step 1. Step 2 is devoted the convergence (in total variation) of the particle system with cutoff to the one without cutoff. Finally, we conclude in Step 3.
We consider the particle system $((X^{1,n,\varepsilon},Z^{1,n,\varepsilon}),...,(X^{n,n,\varepsilon},Z^{n,n,\varepsilon}))$ associated with the initial conditions $((X_0^{1,n},Z_0^{1,n})$,...,$(X_0^{n,n},Z_0^{n,n}))$, with Poisson measures $N^{1,n,\varepsilon}$, ..., $N^{n,n,\varepsilon}$ and with Brownian Motions $B^{1,n}$, ..., $B^{n,n}$.
Step 1 For any $M<\infty$, we denote by K^M the truncated coagulation kernel $K^M(z,z') = K(z\wedge M, z')$. Then we denote by $\{(X^{i,n,\varepsilon,M},Z^{i,n,\varepsilon,M})\}_{i\in\{1,...,n\}}$ the particle system associated with the kernel K^M and with the same initial conditions, Poisson measures, and Brownian motions as $\{(X^{i,n,\varepsilon},Z^{i,n,\varepsilon})\}_{i\in\{1,...,n\}}$.
Then it is clear that the maximum rate of jump of each $Z^{i,n,\varepsilon,M}$ is bounded from above by

$$\Lambda_{M,\varepsilon} = \frac{1}{v_p\varepsilon^p}\sup_{z,z'\in\mathcal{H}_{Q_0}}\frac{K(z\wedge M,z')}{z'} \leq \frac{1}{v_p\varepsilon^p}C_K\left(1+\frac{1+M}{c_0}\right) \tag{4.6}$$

where C_K and c_0 are defined respectively in $(H1)$ and $(H3)$.
Then we can prove, by following line by line the proof of Graham-Méléard [GM97], that for any n and any $k \leq n$,

$$|\mathcal{L}[(X^{1,n,\varepsilon,M}, Z^{1,n,\varepsilon,M}), ..., (X^{k,n,\varepsilon,M}, Z^{k,n,\varepsilon,M})] - \mathcal{L}(X^{1,n,\varepsilon,M}, Z^{1,n,\varepsilon,M})^{\otimes k}|_T \leq k(k-1)\frac{\Lambda_{M,\varepsilon}T + (\Lambda_{M,\varepsilon}T)^2}{n-1}. \tag{4.7}$$

Step 2 Consider now, for $M > 0$, $i \in \{1, ..., n\}$ the stopping time
$T_i^{n,\varepsilon,M} = \inf\ \{t \geq 0\ ;\ Z_t^{i,n,\varepsilon} \geq M\}$. An uniqueness argument shows that for any i and M,

$$(X_t^{i,n,\varepsilon}, Z_t^{i,n,\varepsilon}) = (X_t^{i,n,\varepsilon,M}, Z_t^{i,n,\varepsilon,M}),\ \text{ for all } t \in [0, T_i^{n,\varepsilon,M}). \tag{4.8}$$

Indeed, assume that for some t, $(X_t^{i,n,\varepsilon}, Z_t^{i,n,\varepsilon}) \neq (X_t^{i,n,\varepsilon,M}, Z_t^{i,n,\varepsilon,M})$. Then
- either $Z_t^{i,n,\varepsilon} > M$, and hence $t \geq T_i^{n,\varepsilon,M}$,
- or there was a particle j whose size was greater than M which has coagulated, before t, on the particle i. But then, we obviously deduce that $Z_t^{i,n,\varepsilon} > M$, and hence $t \geq T_i^{n,\varepsilon,M}$,
- or there was a particle j_1, whose size was greater than M, which has coagulated on a particle of j_2, which then has coagulated on i, all of this happened on $[0, t]$. Once again in this case, it is clear that $Z_t^{i,n,\varepsilon} > M$, and so $t \geq T_i^{n,\varepsilon,M}$,
- etc...

Hence (4.8) holds. We deduce that for any $l \in \{1, ..., n\}$, using only the definition of the total variation norm,

$$\begin{aligned}\Big|\mathcal{L}[(X^{1,n,\varepsilon,M}, Z^{1,n,\varepsilon,M}), \ldots, (X^{l,n,\varepsilon,M}, Z^{l,n,\varepsilon,M})] - \mathcal{L}[(X^{1,n,\varepsilon}, Z^{1,n,\varepsilon}), \ldots, (X^{l,n,\varepsilon}, Z^{l,n,\varepsilon})]\Big|_T \\ \leq 2P(T_1^{n,\varepsilon,M} \wedge ... \wedge T_l^{n,\varepsilon,M} \leq T) \leq 2l\mathbb{P}(T_1^{n,\varepsilon,M} \leq T)\end{aligned} \tag{4.9}$$

where the last inequality is obtained by using a symmetry argument. One easily checks, by applying (3.1) with $q = 1$, that

$$P(T_1^{n,\varepsilon,M} \leq T) \leq P\Big(\sup_{[0,T]} |Z_t^{1,n,\varepsilon}| \geq M\Big) \leq \frac{C(T, \varepsilon)}{M}, \tag{4.10}$$

the constant $C(T, \varepsilon)$ being independent of n and M.
Step 3 Let finally k be fixed. Combining (4.7), (4.9) with $l = k$ and after with $l = 1$ and (4.10), we obtain that for any $M > 1$,

$$\begin{aligned}\Big|\mathcal{L}[(X^{1,n,\varepsilon}, Z^{1,n,\varepsilon}), ..., (X^{k,n,\varepsilon}, Z^{k,n,\varepsilon})] - \mathcal{L}(X^{1,n,\varepsilon}, Z^{1,n,\varepsilon})^{\otimes k}\Big|_T \\ \leq \frac{k(k-1)}{n-1}(\Lambda_{M,\varepsilon}T + (\Lambda_{M,\varepsilon}T)^2) + C(T, \varepsilon)\frac{4k}{M}.\end{aligned} \tag{4.11}$$

For a suitable choice of M with respect to n, we conclude that the propagation of chaos result (3.2) holds. □

We carry on with the easy consequence that all limiting point of $\mu^{n,\varepsilon}$ is deterministic.
Proof of Corollary 3.5 Let ε be fixed, and let $\{\mu^{n_k,\varepsilon}\}_k$ be a subsequence of $\{\mu^{n,\varepsilon}\}_n$ converging in law to some μ^ε. We have to prove that μ^ε is deterministic. For $T > 0$, denote by $\mathcal{T}_{Q_0}(T)$ the set of functions in $\mathcal{T}_{Q_0}$ restricted to the time interval $[0, T]$. It

suffices to prove that for any continuous bounded function $\phi : \mathcal{T}_{Q_0}(T) \mapsto \mathbb{R}$, the variable $\int \phi d\mu^\varepsilon$ is deterministic. We obtain this by showing that

$$\text{Var} \int \phi d\mu^\varepsilon = \lim_k \mathbb{E}\left[\left(\int \phi d\mu^{n_k,\varepsilon}\right)^2\right] - \lim_k \mathbb{E}\left[\int \phi d\mu^{n_k,\varepsilon}\right]^2 = 0, \qquad (4.12)$$

which is classically a consequence of the propagation of chaos result.
We now check the second assertion of the corollary, that is $\mathcal{L}(X^{1,n_k,\varepsilon}, Z^{1,n_k,\varepsilon})$ goes to μ^ε as k goes to infinity. We have to prove that for any continuous bounded ϕ from $\mathcal{T}_{Q_0}$ into $\mathbb{R}$, $\lim_k \mathbb{E}\left[\phi(X^{1,n_k,\varepsilon}, Z^{1,n_k,\varepsilon})\right] = \int \phi d\mu^\varepsilon$. This is obvious, since $\mathbb{E}\left[\phi(X^{1,n_k,\varepsilon}, Z^{1,n_k,\varepsilon})\right] = \mathbb{E}\left[\int \phi d\mu^{n_k,\varepsilon}\right]$, also the map $\mu \mapsto \int \phi d\mu$ is continuous and bounded on $\mathcal{P}(\mathcal{T}_{Q_0})$, and since $\mathbb{E}\left[\int \phi d\mu^\varepsilon\right] = \int \phi d\mu^\varepsilon$. The proof is now complete. □

We carry on with the result on the regularity of the space marginals of μ^ε.
Proof of Proposition 3.6 Let ε be fixed, and denote by μ^ε the limit in law of a convergent subsequence $\{\mu^{n_k,\varepsilon}\}_{k\geq 1}$ of $\{\mu^{n,\varepsilon}\}_{n\geq 1}$. We know from Corollary 3.5 that $\mu^\varepsilon \in \mathcal{P}(\mathcal{T}_{Q_0})$ is deterministic, and that $(X^{1,n_k,\varepsilon}, Z^{1,n_k,\varepsilon})$ goes in law to some $(X^\varepsilon, Z^\varepsilon)$ of law μ^ε. We have to prove that the law of X^ε_t has a density for all $t > 0$. This will be done in several steps. First of all, we define the natural filtration $\mathcal{F}_t = \sigma(X^\varepsilon_s, Z^\varepsilon_s,\ s \leq t)$. In the first step, we will express X^ε as a stochastic integral *w.r.t.* a Brownian motion. In the second step, we will prove that the corresponding integrand is piecewise constant, which will allow to conclude in step 3.
Step 1 We first prove that there exists a $\{\mathcal{F}_t\}_t$-Brownian Motion B, independent of X_0, such that

$$X^\varepsilon_t = X_0 + \int_0^t \sqrt{2d(Z^\varepsilon_s)} dB_s. \qquad (4.13)$$

Using standard representation theorems, it suffices to show that for any $\phi \in C^2_c(\mathbb{R}^p)$, the process

$$M^\phi_t = \phi(X^\varepsilon_t) - \phi(X^\varepsilon_0) - \int_0^t d(Z^\varepsilon_s)\Delta\phi(X^\varepsilon_s) ds \qquad (4.14)$$

is an $\{\mathcal{F}_t\}_t$-martingale. We have thus to prove that for any $k \in \mathbb{N}^*$, any $g_1, ..., g_k$ in $C_b(\mathbb{R}^p \times \mathbb{R}_+)$ and any $0 \leq s_1 < ... < s_k < s < t$,

$$\mathbb{E}\Big[g_1(X^\varepsilon_{s_1}, Z^\varepsilon_{s_1}) \dots g_k(X^\varepsilon_{s_k}, Z^\varepsilon_{s_k})\{\phi(X^\varepsilon_t) - \phi(X^\varepsilon_s) - \int_s^t d(Z^\varepsilon_u)\Delta\phi(X^\varepsilon_u) du\}\Big] = 0. \qquad (4.15)$$

Using the explicit expression of $(PS(n, \varepsilon))$, it is clear that this equality holds by replacing everywhere X^ε and Z^ε by $X^{1,n_k,\varepsilon}$ and $Z^{1,n_k,\varepsilon}$. We can make k go to infinity by using classical arguments (see the proof of the Theorem 3.7 bellow for a similar problem). This concludes Step 1.
Step 2 We now check that the càdlàg process Z^ε is *a.s.* constant between its jumps, with an *a.s.* finite number of jumps on each finite time interval.
It is clear that for each k, $Z^{1,n_k,\varepsilon}$ belongs *a.s.* to the following set:

$$\begin{aligned}\mathcal{S} = \ & \Big\{z \in \mathbb{D}^\uparrow([0,\infty), \mathcal{H}_{Q_0}) / \exists\ 0 < t_1 < ... < t_k < ...,\ \lim_k t_k = \infty, \\ & \quad \exists\ \alpha_1, ..., \alpha_k, ... \in [c_0, \infty),\ z(t) = \alpha_0 + \textstyle\sum_{i=1}^\infty \alpha_i \mathbb{1}_{\{t \geq t_i\}}\Big\},\end{aligned} \qquad (4.16)$$

where c_0 is defined in $(H3)$. This set is closed in $\mathbb{D}^{\uparrow}([0,\infty),\mathcal{H}_{Q_0})$ for the Skorohod topology. Hence Z^ε belongs *a.s.* to $\mathcal{S}$. This ends the proof of Step 2.

Step 3 We are finally able to conclude. First denote by $0 = T_0 < T_1 < ...$ the successive times of jumps of Z^ε. These are of course $\{\mathcal{F}_t\}_t$-stopping times. Then we notice that due to (4.13) and since Z^ε is constant between its jumps, we have, for $t > 0$,

$$X_t^\varepsilon = \sum_{i\geq 0} \mathbb{1}_{[T_i,T_{i+1})}(t)\left[X_{T_i}^\varepsilon + \sqrt{2d(Z_{T_i}^\varepsilon)}(B_t - B_{T_i})\right]. \tag{4.17}$$

Since Z^ε is quasi-càg (thanks to the Aldous criterion), we can replace $\mathbb{1}_{[T_i,T_{i+1})}$ by $\mathbb{1}_{(T_i,T_{i+1})}$. The conclusion follows easily. □

We are finally able to prove that μ^ε satisfies $(MP(\varepsilon))$.

Proof of Theorem 3.7 Denote by $\mu^\varepsilon \in \mathcal{P}(\mathcal{T}_{Q_0})$ the deterministic limit in law of a convergent sub-sequence $\{\mu^{n_k,\varepsilon}\}_{k\geq 1}$ of $\{\mu^{n,\varepsilon}\}_{n\geq 1}$. We have to check that μ^ε satisfies $(MP(\varepsilon))$. Consider $0 \leq s_1 < \ldots < s_l < s < t$, $g_1,\ldots,g_l \in C_b(\mathbb{R}^p \times \mathbb{R}_+)$ and $\phi \in C_b^2(\mathbb{R}^p \times \mathbb{R}_+)$. Consider also $F : \mathcal{T}_{Q_0} \times \mathcal{T}_{Q_0} \mapsto \mathbb{R}$, defined by

$$\begin{aligned} F((x,z),(x',z')) &= g_1(x(s_1),z(s_1)) \times \ldots \times g_l(x(s_l),z(s_l)) \\ &\times \Bigg\{ \phi(x(t),z(t)) - \phi(x(s),z(s)) - \int_s^t \Delta_x\phi(x(u),z(u))d(z(u))du \\ &- \int_s^t [\phi(x(u),z(u)+z'(u)) - \phi(x(u),z(u))]\,\frac{K(z(u),z'(u))}{z'(u)}\,\mathbb{1}_{\{|x(u)-x'(u)|<\varepsilon\}}\frac{1}{v_p\varepsilon^p}du \Bigg\}. \end{aligned} \tag{4.18}$$

In order to obtain that μ^ε satisfies $(MP(\varepsilon))$ it is sufficient to prove that for all F of the form given in (4.18),

$$< \mu^\varepsilon \otimes \mu^\varepsilon, F >= 0. \tag{4.19}$$

The proof is divided in several steps. We will first prove in Step 1 below that for any $k \geq 1$,

$$\mathbb{E}\left[< \mu^{n_k,\varepsilon} \otimes \mu^{n_k,\varepsilon}, F >\right] = 0. \tag{4.20}$$

Secondly, in Step 2 we will check that for any positive constant M,

$$\lim_{k\to\infty} \mathbb{E}\left[< \mu^{n_k,\varepsilon} \otimes \mu^{n_k,\varepsilon}, F \wedge M \vee (-M) >\right] =< \mu^\varepsilon \otimes \mu^\varepsilon, F \wedge M \vee (-M) > . \tag{4.21}$$

We finally will obtain in Step 3 that

$$\lim_{M\to\infty} \sup_k \mathbb{E}\left[< \mu^{n_k,\varepsilon} \otimes \mu^{n_k,\varepsilon}, |F|\mathbb{1}_{\{|F|>M\}} >\right] = 0. \tag{4.22}$$

Gathering together (4.20), (4.21) and (4.22) leads to (4.19).

Step 1 A simple computation, using the explicit expressions of F and $\mu^{n_k,\varepsilon}$ leads to

$$< \mu^{n_k,\varepsilon} \otimes \mu^{n_k,\varepsilon}, F >= \frac{1}{n_k}\sum_{i=1}^{n_k} g_1(X_{s_1}^{i,n_k,\varepsilon}, Z_{s_1}^{i,n_k,\varepsilon}) \ldots g_l(X_{s_l}^{i,n_k,\varepsilon}, Z_{s_l}^{i,n_k,\varepsilon})[M_t^{i,n_k,\varepsilon}(\phi) - M_s^{i,n_k,\varepsilon}(\phi)] \tag{4.23}$$

where

$$M_t^{i,n_k,\varepsilon}(\phi) = \phi(X_t^{i,n_k,\varepsilon}, Z_t^{i,n_k,\varepsilon}) - \phi(X_0^{i,n_k,\varepsilon}, Z_0^{i,n_k,\varepsilon}) - \int_0^t \Delta_x \phi(X_u^{i,n_k,\varepsilon}, Z_u^{i,n_k,\varepsilon}) d(Z_u^{i,n_k,\varepsilon}) du$$
$$- \int_0^t \int_0^\infty \frac{1}{n_k} \sum_{j=1}^{n_k} \left[\phi(X_u^{i,n_k,\varepsilon}, Z_u^{i,n_k,\varepsilon} + Z_u^{j,n_k,\varepsilon}) - \phi(X_u^{i,n_k,\varepsilon}, Z_u^{i,n_k,\varepsilon}) \right]$$
$$\mathbb{1}_{\left\{ z \le \frac{K(Z_u^{i,n_k,\varepsilon}, Z_u^{j,n_k,\varepsilon})}{Z_u^{j,n_k,\varepsilon}} \right\}} \mathbb{1}_{\{|X_u^{i,n_k,\varepsilon} - X_u^{j,n_k,\varepsilon}| < \varepsilon\}} \frac{1}{v_p \varepsilon^p} dz du. \tag{4.24}$$

By applying Itô formula to $(PS(n_k,\varepsilon))$ in order to express $\phi(X_t^{i,n_k,\varepsilon}, Z_t^{i,n_k,\varepsilon})$, we obtain that $M^{i,n_k,\varepsilon}(\phi)$ is a martingale starting from 0, for every i. It is then clear from (4.23) that $\mathbb{E}(< \mu^{n_k,\varepsilon} \otimes \mu^{n_k,\varepsilon}, F >) = 0$ for any k.

Step 2 We now have to check (4.21). We know that $\mu^{n_k,\varepsilon}$ goes in law to μ^ε, and that μ^ε is deterministic. This implies that for any bounded function ψ from $\mathcal{P}(\mathcal{T}_{Q_0})$ into $\mathbb{R}$, which is continuous at μ^ε, there is convergence of $\mathbb{E}[\psi(\mu^{n_k,\varepsilon})]$ to $\psi(\mu^\varepsilon)$. We thus first check that the map $\psi(\mu) =< \mu \otimes \mu, F >$ is continuous at μ^ε. Using the explicit expression of F, we see that the map ψ is continuous at any point of the subset $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$ of $\mathcal{P}(\mathcal{T}_{Q_0})$, where

$$\begin{aligned} \mathcal{C}_1 &= \left\{ Q \in \mathcal{P}(\mathcal{T}_{Q_0}) \big/ \forall u \in (0,\infty), \textstyle\int_{\mathcal{T}_{Q_0}{}^2} \mathbb{1}_{\{|x(u)-x'(u)|<\varepsilon\}} Q(dx,dz) Q(dx',dz') = 0 \right\}, \\ \mathcal{C}_2 &= \left\{ Q \in \mathcal{P}(\mathcal{T}_{Q_0}) \big/ Q\left(\{\Delta z(s_1) = \ldots = \Delta z(s_l) = \Delta z(s) = \Delta z(t) = 0\}\right) = 1 \right\}. \end{aligned} \tag{4.25}$$

We shall prove that μ^ε belongs to both sets $\mathcal{C}_1$ and $\mathcal{C}_2$. Proposition 3.6 allows to obtain that μ^ε belongs to $\mathcal{C}_1$. On the other hand, we know from Corollary 3.5 that μ^ε is the weak limit of $\{\mathcal{L}(X^{1,n_k,\varepsilon}, Z^{1,n_k,\varepsilon})\}_k$, and we checked in the proof of Proposition 3.3 that the sequence $\{\mathcal{L}(Z^{1,n_k,\varepsilon})\}_k$ satisfies the Aldous criterion. This shows that its limit (*i.e.* the "size" marginal of μ^ε) is the law of a quasi-càg process, which implies directly that μ^ε belongs to $\mathcal{C}_2$. Now it is clear that for any constant $M > 0$, $\psi \wedge M \vee (-M)$ is bounded and continuous at μ^ε. We get (4.21).

Step 3 We are now interested in (4.22). First notice, by using (4.18), $(H1)$ and the fact that ϕ belongs to C_b^2, that there exists a constant $A(\varepsilon)$ such that for any (x,z), $(x',z') \in \mathcal{T}_{Q_0}$

$$|F((x,z),(x',z'))| \le A(\varepsilon)\ [1 + z(t) + z'(t)]\,. \tag{4.26}$$

Then, computing explicitly $\mathbb{E}\left[< \mu^{n_k,\varepsilon} \otimes \mu^{n_k,\varepsilon}, |F| \mathbb{1}_{\{|F|>M\}} >\right]$ and applying (3.1) with $q = 2$ allow to conclude that (4.22) holds. This ends the proof. □

4.3 Convergence of $(WS(\varepsilon))$ to (WS)

The idea of this convergence is to use the formulation of the problem in terms of semi-group and take benefit on the properties of the heat kernel. We omit the proof of Lemma 3.9, because it is completely standard.

Proof of Lemma 3.10 Applying (3.4) with the test function $\varphi(x,z) = h(x) d^{p/2}(z)$, for some nonnegative $h \in C_c(\mathbb{R}^p)$, we obtain that

$$\int_{\mathbb{R}^p \times \mathbb{R}_+} h(x) d^{\frac{p}{2}}(z) Q_t^\varepsilon(dx,dz) \le \int_{\mathbb{R}^p \times \mathbb{R}_+} Q_0(dx,dz) \int_{\mathbb{R}^p} dy q(td(z),x,y) h(y) d^{\frac{p}{2}}(z). \tag{4.27}$$

Indeed, the last term in (3.4) becomes negative with such a test function: since d is decreasing and h is nonnegative, we can easily check, using the explicit expression of the p-dimensional heat kernel, that for any s, t, x, y, z, z'

$$h(y)\{d^{p/2}(z+z')q((t-s)d(z+z'),x,y) - d^{p/2}(z)q((t-s)d(z),x,y)\} \leq 0. \tag{4.28}$$

Using finally $(H5)$ and the fact that $\int q(s,x,y)dx = 1$, we deduce from (4.27) that for any nonnegative $h \in C_c(\mathbb{R}^p)$,

$$\int_{\mathbb{R}^p\times\mathbb{R}_+} h(x)Q_t^\varepsilon(dx,dz) \leq \frac{1}{\underline{d}^{p/2}}\int_{\mathbb{R}^p\times\mathbb{R}_+} h(x)d^{p/2}(z)Q_t^\varepsilon(dx,dz) \leq C_0\left(\frac{\overline{d}}{\underline{d}}\right)^{p/2}\int_{\mathbb{R}^p} h(y)dy. \tag{4.29}$$

This concludes the proof. □

This key lemma allows us to prove the compactness in ε.

Proof of Proposition 3.11 First $\{\mathcal{L}(X^\varepsilon)\}_{\varepsilon>0}$ is tight in $C([0,\infty),\mathbb{R}^p)$ because d is bounded. Thus, we have only to check that $\{\mathcal{L}(Z^\varepsilon)\}_{\varepsilon>0}$ is tight in $\mathbb{D}([0,\infty),\mathbb{R}_+)$, by using the Aldous criterion (see Jacod-Shiryaev, [JS87]). We have to show (i) and (ii) below.
(i) For any $T>0$, $\sup_{\varepsilon>0}\mathbb{E}[\sup_{[0,T]}|Z_t^\varepsilon|] < \infty$.
(ii) For any $\eta > 0$ and any $T < T_0$

$$\lim_{\delta\to 0}\sup_{\varepsilon>0}\sup_{(S,S')\in\mathcal{A}(\delta,T)} P\left[|Z_{S'}^\varepsilon - Z_S^\varepsilon| > \eta\right] = 0, \tag{4.30}$$

where $\mathcal{A}(\delta,T)$ is the set of the couples (S,S') of stopping times satisfying *a.s.* $0 \leq S \leq S' \leq (S+\delta)\wedge T$.
We begin with (i). First, we have *a.s.* $\sup_{[0,T]}|Z_t^\varepsilon| = Z_T^\varepsilon$. Using the expression of Z^ε and $(H1)$, we obtain

$$\begin{aligned}\mathbb{E}[Z_t^\varepsilon] &= \mathbb{E}[Z_0] + \int_0^t \mathbb{E}\mathbb{E}_\alpha\left[K(Z_s^\varepsilon,\tilde{Z}_s^\varepsilon(\alpha))\mathbb{1}_{\{|X_s^\varepsilon-\tilde{X}_s^\varepsilon(\alpha)|<\varepsilon\}}\right]\frac{ds}{v_p\varepsilon^p}\\ &\leq \mathbb{E}[Z_0] + C_K\int_0^t \mathbb{E}\mathbb{E}_\alpha\left[\left(1+Z_s^\varepsilon+\tilde{Z}_s^\varepsilon(\alpha)\right)\mathbb{1}_{\{|X_s^\varepsilon-\tilde{X}_s^\varepsilon(\alpha)|<\varepsilon\}}\right]\frac{ds}{v_p\varepsilon^p}.\end{aligned} \tag{4.31}$$

For symmetrical reasons, we get

$$\mathbb{E}[Z_t^\varepsilon] \leq \mathbb{E}[Z_0] + 2C_K\int_0^t \mathbb{E}\left[(1+Z_s^\varepsilon)\sup_{x\in\mathbb{R}^p}P_\alpha\left(|\tilde{X}_s^\varepsilon(\alpha)-x|<\varepsilon\right)\right]\frac{ds}{v_p\varepsilon^p}. \tag{4.32}$$

Thanks to Lemma 3.10, it is clear that for all $s \geq 0$,

$$\sup_{x\in\mathbb{R}^p}\frac{1}{v_p\varepsilon^p}P_\alpha\left(|\tilde{X}_s^\varepsilon(\alpha)-x|<\varepsilon\right) \leq C_0\left(\frac{\overline{d}}{\underline{d}}\right)^{p/2}. \tag{4.33}$$

Hence, there exists a constant D such that

$$\mathbb{E}[Z_t^\varepsilon] \leq \mathbb{E}[Z_0] + D\int_0^t (1+\mathbb{E}[Z_s^\varepsilon])\,ds. \tag{4.34}$$

Gronwall's Lemma allows to conclude the proof of (i).
We still have to check (ii). We denote by J_t^ε the number of jumps of Z^ε on $[0,t]$:

$$J_t^\varepsilon = \int_0^t\int_0^1\int_0^\infty \mathbb{1}_{\left\{z\leq\frac{K(Z_{s-}^\varepsilon,\tilde{Z}_{s-}^\varepsilon(\alpha))}{\tilde{Z}_{s-}^\varepsilon(\alpha)}\right\}}\mathbb{1}_{\left\{|X_{s-}^\varepsilon-\tilde{X}_{s-}^\varepsilon(\alpha)|\leq\varepsilon\right\}}N^\varepsilon(ds,d\alpha,dz). \tag{4.35}$$

Since Z^ε is constant between is jumps, one obtains that for any $\varepsilon > 0$, $\eta > 0$, $\delta > 0$, $T < \infty$ and for any $(S, S') \in \mathcal{A}(\delta, T)$,

$$\begin{aligned} P\left[|Z^\varepsilon_{S'} - Z^\varepsilon_S| > \eta\right] &\le P\left[|J^\varepsilon_{S'} - J^\varepsilon_S| \ge 1\right] \le \mathbb{E}\left[|J^\varepsilon_{S'} - J^\varepsilon_S|\right] \\ &\le \mathbb{E}\left[\int_{(S,S']} \mathbb{E}_\alpha\left(\frac{K(Z^\varepsilon_u, \tilde{Z}^\varepsilon_u(\alpha))}{\tilde{Z}^\varepsilon_u(\alpha)} 1\!\!1_{\left\{\left|X^\varepsilon_u - \tilde{X}^\varepsilon_u(\alpha)\right| \le \varepsilon\right\}}\right) \frac{ds}{v_p \varepsilon^p}\right]. \end{aligned} \tag{4.36}$$

Thanks to $(H3)$, we also know that $Z_0 \ge c_0$ *a.s.*, which implies that $\mathcal{H}_{Q_0} \subset [c_0, \infty)$. Using $(H1)$, we deduce that there exists a constant D such that for any z, z' in $\mathcal{H}_{Q_0}$, $K(z, z')/z' \le D(1 + z)$. We thus obtain, using successively (4.33) and (i) that

$$\begin{aligned} P\left[|Z^\varepsilon_{S'} - Z^\varepsilon_S| > \eta\right] &\le \mathbb{E}\left[\int_S^{S+\delta} D(1 + Z^\varepsilon_s) \sup_{x \in \mathbb{R}^p} P_\alpha\left(\left|\tilde{X}^\varepsilon_s - x\right| \le \varepsilon\right) \frac{ds}{v_p \varepsilon^p}\right] \\ &\le D' \, \mathbb{E}\left[\int_S^{S+\delta} (1 + Z^\varepsilon_s) ds\right] \le A(T)\, \delta \end{aligned} \tag{4.37}$$

the constant $A(T)$ being independent of $\delta \in [0, 1]$, $\eta > 0$, $\varepsilon > 0$ and $(S, S') \in \mathcal{A}(\delta, T)$. Hence (ii) follows easily. This concludes the present proof. □

We now prove the technical Lemma concerning the regularity in x of $Q^\varepsilon_t(dx, dz)$.

Proof of Lemma 3.12 Let $\varepsilon > 0$, $T > 0$ and $t \in (0, T]$ be fixed. Applying (3.4) for test functions of the form $\varphi(x, z) = h(x)\alpha(z)$, gives an integral expression for f^ε_t that we differentiate by using the Lebesgue theorem:

$$\begin{aligned} \nabla_y f^\varepsilon_t(y) = &\int_{\mathbb{R}^p \times \mathbb{R}_+} Q_0(dx, dz) \nabla_y q(td(z), x, y)\alpha(z) \\ &+ \int_0^t ds \int_{\mathbb{R}^p \times \mathbb{R}_+} Q^\varepsilon_s(dx, dz) \int_{\mathbb{R}^p \times \mathbb{R}_+} Q^\varepsilon_s(dx', dz') \frac{K(z, z')}{z'} \frac{1\!\!1_{\{|x - x'| \le \varepsilon\}}}{v_p \varepsilon^p} \\ &\left\{\nabla_y q((t - s)d(z + z'), x, y)\alpha(z + z') - \nabla_y q((t - s)d(z), x, y)\alpha(z)\right\}. \end{aligned} \tag{4.38}$$

Thanks to $(H5)$, for any $z \in \mathcal{H}_{Q_0}$, any $x, y \in \mathbb{R}^p$ and any $u > 0$

$$|\nabla_y q(ud(z), x, y)| \le |\nabla_y q(u\overline{d}, x, y)| \left(\frac{\overline{d}}{\underline{d}}\right)^{1+p/2}. \tag{4.39}$$

Since furthermore α is bounded and has a compact support in $(0, \infty)$, one easily obtains, using $(H5)$, the existence of a constant D such that

$$\begin{aligned} |\nabla_y f^\varepsilon_t(y)| \le &D \int_{\mathbb{R}^p} |\nabla_y q(t\overline{d}, x, y)| dx \\ &+ D \int_0^t ds \int_{\mathbb{R}^p \times \mathbb{R}_+} Q^\varepsilon_s(dx, dz) \int_{\mathbb{R}^p \times \mathbb{R}_+} Q^\varepsilon_s(dx', dz') \frac{1\!\!1_{\{|x - x'| \le \varepsilon\}}}{v_p \varepsilon^p} |\nabla_y q((t - s)\overline{d}, x, y)|. \end{aligned} \tag{4.40}$$

By Lemma 3.10 it is clear that for any $x \in \mathbb{R}^p$ and any $s \ge 0$

$$\int_{\mathbb{R}^p \times \mathbb{R}_+} \frac{1\!\!1_{\{|x - x'| \le \varepsilon\}}}{v_p \varepsilon^p} Q^\varepsilon_s(dx', dz') \le C_0 \left(\frac{\overline{d}}{\underline{d}}\right)^{p/2} \tag{4.41}$$

and that, in an obvious sense, $\int_{\mathbb{R}_+} Q_s^\varepsilon(dx,dz) \le C_0 \left(\frac{\bar{d}}{\underline{d}}\right)^{p/2} dx$. We obtain thus the existence of a constant D such that

$$|\nabla_y f_t^\varepsilon(y)| \le D \int_{\mathbb{R}^p} |\nabla_y q(t\bar{d}, x, y)| dx + D \int_0^t ds \int_{\mathbb{R}^p} |\nabla_y q((t-s)\bar{d}, x, y)| dx. \tag{4.42}$$

To conclude, we notice that for any $u > 0$ and any $y \in \mathbb{R}^p$, $\int_{\mathbb{R}^p} |\nabla_y q(u,x,y)|dx \le \sqrt{\frac{p}{u}}$. Hence, $|\nabla_y f_t^\varepsilon(y)| \le D\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)$, the constant D not depending on y, t, nor ε. This concludes the proof. □

We finally check that the limiting points of $(WS(\varepsilon))$ satisfy (WS).

Proof of Theorem 3.13 First of all we deduce from the fact that $\{\mathcal{L}(Z_t^\varepsilon)\}_{\varepsilon>0}$ satisfies the Aldous criterion that for each $t \ge 0$, $Q_t^{\varepsilon_k}$ goes weakly to Q_t in $\mathcal{P}(\mathbb{R}^p \times \mathbb{R}_+)$: this is not *a priori* obvious, since the projections $z \mapsto z(t)$ are not continuous from $\mathbb{D}([0,\infty), \mathcal{H}_{Q_0})$, but the Aldous criterion says that any limit point is quasi-càg, which suffices.

Then we notice that points (i) and (iii) follows immediately from lemmas 3.10 and 3.12. Let us prove (ii). We know that for each k, $\{Q_t^{\varepsilon_k}\}_{t\ge0}$ is a solution to $(WS(\varepsilon_k))$, see Definition 2.4. We have to make k go to infinity. Let us consider $\phi \in C_c^2(\mathbb{R}^p \times (0,\infty))$ and $t \ge 0$ to be fixed. It is obvious from the definition of the weak convergence that

$$\int_{\mathbb{R}^p\times\mathbb{R}_+} \phi(x,z) Q_t^{\varepsilon_k}(dx,dz) \underset{k\to\infty}{\longrightarrow} \int_{\mathbb{R}^p\times\mathbb{R}_+} \phi(x,z) Q_t(dx,dz). \tag{4.43}$$

Furthermore, since d is continuous, for each $s \ge 0$, we have

$$\int_{\mathbb{R}^p\times\mathbb{R}_+} d(z)\Delta_x\phi(x,z) Q_s^{\varepsilon_k}(dx,dz) \underset{k\to\infty}{\longrightarrow} \int_{\mathbb{R}^p\times\mathbb{R}_+} d(z)\Delta_x\phi(x,z) Q_s(dx,dz) \tag{4.44}$$

which implies, thanks to the Lebesgue Theorem, that

$$\lim_{k\to\infty} \int_0^t ds \int_{\mathbb{R}^p\times\mathbb{R}_+} d(z)\Delta_x\phi(x,z) Q_s^{\varepsilon_k}(dx,dz) = \int_0^t ds \int_{\mathbb{R}^p\times\mathbb{R}_+} d(z)\Delta_x\phi(x,z) Q_s(dx,dz). \tag{4.45}$$

We finally have to check that

$$\begin{aligned} &\int_0^t ds \int_{(\mathbb{R}^p\times\mathbb{R}_+)^2} Q_s^{\varepsilon_k}(dx,dz) Q_s^{\varepsilon_k}(dx',dz') \frac{K(z,z')}{z'} \frac{\mathbb{1}_{\{|x-x'|\le\varepsilon_k\}}}{v_p\varepsilon_k^p} [\psi(x,z+z') - \psi(x,z)] \\ &\underset{k\to\infty}{\longrightarrow} \int_0^t ds \int_{\mathbb{R}^p\times\mathbb{R}_+} Q_s(dx,dz) \int_{\mathbb{R}_+} \gamma_s(x,dz') \frac{K(z,z')}{z'} [\psi(x,z+z') - \psi(x,z)]. \end{aligned} \tag{4.46}$$

Using again Lebesgue Theorem (for the measure ds), $(H1)$ and the fact that ϕ belongs to $C_c^2(\mathbb{R}^p \times (0,\infty))$, it clearly suffices to prove that for any $\psi \in C_c^2(\mathbb{R}^p)$, any α and β in $C_c^2((0,\infty))$ and any $s > 0$, $\lim_{k\to\infty} I_k = I$, with the notations

$$I_k = \int_{\mathbb{R}^p} \psi(x) f^{\varepsilon_k}(x) T^{\varepsilon_k} g^{\varepsilon_k}(x) dx, \quad I = \int_{\mathbb{R}^p} \psi(x) f(x) g(x) dx \tag{4.47}$$

and

$$\begin{aligned} f^{\varepsilon_k}(x) &= \int_{\mathbb{R}_+} \alpha(z) \gamma_s^{\varepsilon_k}(x,dz), \quad f(x) = \int_{\mathbb{R}_+} \alpha(z)\gamma_s(x,dz) \\ g^{\varepsilon_k}(x) &= \int_{\mathbb{R}_+} \beta(z) \gamma_s^{\varepsilon_k}(x,dz), \quad g(x) = \int_{\mathbb{R}_+} \beta(z)\gamma_s(x,dz) \end{aligned} \tag{4.48}$$

where γ is defined by (i) (see the statement of the Theorem) and where the mollifier operator T^ε is defined, for any function h on $\mathbb{R}^p$, as $T^\varepsilon h(x) = \frac{1}{v_p\varepsilon^p}\int_{\mathbb{R}^p} 1\!\!1_{\{|x-x'|\le\varepsilon\}}h(x')dx'$. Since $Q_t^{\varepsilon_k}$ goes weakly to Q_t we deduce that $f^{\varepsilon_k}(x)dx$ goes weakly to $f(x)dx$. By using Lemma 3.12, it is easily checked that $f^{\varepsilon_k}(x)$ converges to $f(x)$ for every x, and of course, $g^{\varepsilon_k}(x)$ converges also to $g(x)$ for every x. We finally obtain using Lemmas 3.10 and 3.12, and the fact that T^{ε_k} is symmetric that there exists a constant D such that:

$$\begin{aligned}
|I_k - I| &\le \int_{\mathbb{R}^p} |\psi(x)||f^{\varepsilon_k}(x) - f(x)|\,|T^{\varepsilon_k}g^{\varepsilon_k}(x)|dx \\
&\quad + \left|\int_{\mathbb{R}^p} \psi(x)f(x)T^{\varepsilon_k}(g - g^{\varepsilon_k})(x)dx\right| + \int_{\mathbb{R}^p} |\psi(x)f(x)||T^{\varepsilon_k}g(x) - g(x)|dx \\
&\le D\int_{\mathbb{R}^p} |\psi(x)|\,|f^{\varepsilon_k}(x) - f(x)|dx + \int_{\mathbb{R}^p} |T^{\varepsilon_k}(\psi f)(x)|\,|g^{\varepsilon_k}(x) - g(x)|dx \\
&\quad + D\int_{\mathbb{R}^p} |\psi(x)|\,||\nabla g||_\infty \varepsilon_k dx \\
&\le D\int_{\text{supp }\psi} \{|f^{\varepsilon_k}(x) - f(x)| + |g^{\varepsilon_k}(x) - g(x)| + \varepsilon_k\}dx
\end{aligned} \tag{4.49}$$

which goes to 0, thanks to the Lebesgue Theorem. Hence (4.46) holds, and $\{Q_s\}_{s\ge0}$ satisfies (WS). This ends the proof. □

5 Numerical study

We present in this part the numerical approximation naturally connected with the probabilistic approach for the diffusive coagulation equation. First, we describe the simulation algorithm for our particle system. After we present some numerical results.

5.1 The simulation algorithm

Let the dimension $p \in \mathbb{N}^*$, the number of particles $n \in \mathbb{N}^*$, the initial distribution $Q_0 \in \mathcal{P}(\mathbb{R}^p \times \mathbb{R}_+)$ and the delocalisation parameter $\varepsilon > 0$ be fixed. Recall that v_p is given in Definition 2.4. The aim of this section is to simulate the solution $\{(X_t^{1,n,\varepsilon}, Z_t^{1,n,\varepsilon}), ..., (X_t^{n,n,\varepsilon}, Z_t^{n,n,\varepsilon})\}_{t\ge0}$ of $(PS(n,\varepsilon))$, under $(H1)$. The algorithm writes:

Step 0 Simulate $(X_0^{1,n,\varepsilon}, Z_0^{1,n,\varepsilon})$, ..., $(X_0^{n,n,\varepsilon}, Z_0^{n,n,\varepsilon})$ independent and of law Q_0.

Step 1 Compute $m_1 = \sup_{i,j} \frac{K(Z_0^{i,n,\varepsilon}, Z_0^{j,n,\varepsilon})}{Z_0^{j,n,\varepsilon}}$. Then simulate S_1 of exponential law with parameter $n \times m_1/(v_p\varepsilon^p)$, and set $T_1 = S_1$. Simulate B_1^1, ..., B_1^n independent $\mathbb{R}^p$-valued Brownian motions on $[0, S_1]$, and set for each l

$$\begin{cases} X_t^{l,n,\varepsilon} = X_0^{l,n,\varepsilon} + \sqrt{2d(Z_0^{l,n,\varepsilon})}B_1^l(t) & \forall\, t \in [0, T_1] \\ Z_t^{l,n,\varepsilon} = Z_0^{l,n,\varepsilon} & \forall\, t \in [0, T_1). \end{cases} \tag{5.1}$$

After, choose (i_1, j_1) uniformly among $\{1, ..., n\}^2$, and simulate U_1 according to a uniform law over $[0, m_1]$.

If $|X_{T_1}^{i_1,n,\varepsilon} - X_{T_1}^{j_1,n,\varepsilon}| \le \varepsilon$ and $U_1 \le K(Z_0^{i_1,n,\varepsilon}, Z_0^{j_1,n,\varepsilon})/Z_0^{j_1,n,\varepsilon}$, then set

$$\begin{cases} Z_{T_1}^{i_1,n,\varepsilon} = Z_0^{i_1,n,\varepsilon} + Z_0^{j_1,n,\varepsilon}, \text{ and} \\ Z_{T_1}^{l,n,\varepsilon} = Z_0^{l,n,\varepsilon} \text{ for all } l \ne i_1. \end{cases} \tag{5.2}$$

The first line express the coagulation of the j_1th particle to the i_1th particle. Else, simply set $Z_{T_1}^{l,n,\varepsilon} = Z_0^{l,n,\varepsilon}$ for all l. The computation continues this way.

Remark 5.1 *Assume that one needs to simulate only a finite number of values $\mu_{t_1}^{n,\varepsilon}$, ..., $\mu_{t_k}^{n,\varepsilon}$ of $\{\mu_t^{n,\varepsilon}\}_{t\geq 0}$, as is always the case in practise. Then one may use a trick based on the following remark: at each "fictive or effective coagulation", there is no need to move the locations of all the particles, but only those that have to be tested. Hence, instead of moving the location of all the particles at each fictive and effective change of state, it suffices to move the location of only two particles at each fictive coagulation, and the location of all the particles when the coagulation is effective. Since the proportion of effective coagulations is very small, the use of this remark reduces strongly the calculating time of the algorithm.*

5.2 How to choose n and ε ?

We would now like to have an idea about how to choose n and ε. To this aim, we consider a very simple and poor test case, which is unfortunately the only explicit computation we are able to handle. We consider the three dimensional case $p = 3$, a constant diffusion coefficient $d(z) = 1/2$, the coagulation kernel $K(z, z') = 1$ and the initial condition Q_0 of the form

$$Q_0(dx, dz) = \frac{1}{\sigma^3 2\pi\sqrt{2\pi}} e^{-|x|^2/2\sigma^2} dx \otimes \delta_1(dz). \tag{5.3}$$

Denote, in this case, by $n(t, x, k)$ the solution to (SC). We are able to prove that

$$a_t := \int_{\mathbb{R}^3} \sum_{k\geq 1} k^2 n(t, x, k) dx = 1 + \left(1/\sigma - 1/\sqrt{\sigma^2 + t}\right) / 4\pi\sqrt{\pi}. \tag{5.4}$$

We would like to compare a_t with $\bar{Z}_t^{n,\varepsilon} = \int_{\mathcal{T}_{Q_0}} z(t)\mu^{n,\varepsilon}(dx, dz) = \frac{1}{n}\sum_{i=1}^n Z_t^{i,n,\varepsilon}$.
Figure 1 deals with σ^2 very small ($\sigma^2 = 0.2$), thus the particles are initially quite concentrated at 0. In Figure 1a, we compare the true value a_t, for $t \in [0, 10]$, with $\bar{Z}_t^{n,\varepsilon}$, for $n = 5000$ and different values of ε. Figure 1b treats the case $n = 50000$.

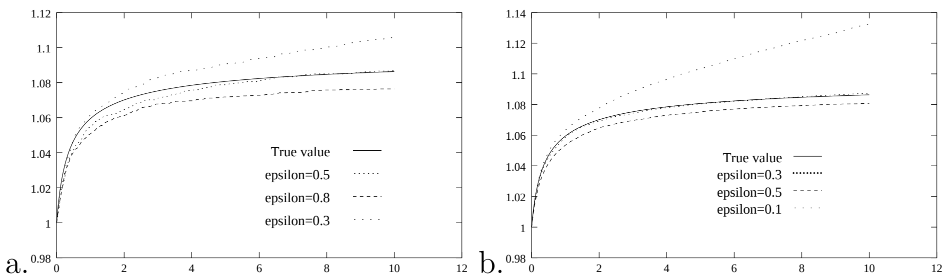

Figure 1: a. $\sigma^2 = 0.2$, $n = 5000$ and b. $\sigma^2 = 0.2$, $n = 50000$.

Figure 2 deals with larger σ^2 ($\sigma^2 = 5$), thus the particles are initially quite well-distributed. On Figure 2a, we compare the true value a_t, for $t \in [0, 10]$, with $\bar{Z}_t^{n,\varepsilon}$, for $n = 50000$, and different values of ε. Figure 2b treats the case $n = 250000$.

First notice that choosing ε as small as possible is never judicious. For each n fixed, there is an "optimal" ε, which tends to 0 when n increases to infinity, but which is never

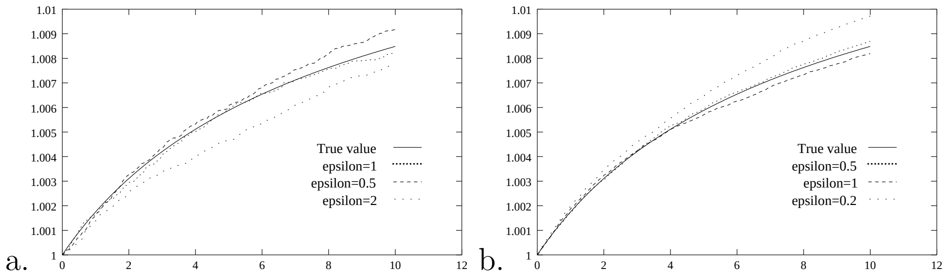

Figure 2: a. $\sigma^2 = 5$, $n = 50000$ and b. $\sigma^2 = 5$, $n = 250000$.

0. This can be easily explained: in the present paper, we first make n tend to infinity in $(PS(n,\varepsilon))$, and after we let ε go to 0. It is clear that making first ε tend to 0 in $(PS(n,\varepsilon))$ and after n tend to infinity would not lead to the coagulation diffusive equation (WS).
Let us also mention that the simulation is faster in the case where $\sigma^2 = 0.5$ than $\sigma^2 = 0.2$ (with the same ε and n). Indeed, this comes from the fact that we use Remark 5.1: there are less effective coagulations when the initial system is well-distributed ($\sigma^2 = 5$) than when it is concentrated ($\sigma^2 = 0.2$).
To give an idea of the calculating time, simulating the particle system on $t \in [0, 10]$ with $\sigma^2 = 0.2$, $n = 5000$, and $\varepsilon = 0.5$ takes the same time as $\sigma^2 = 5$, $n = 12500$, $\varepsilon = 0.5$, a few seconds in each case.
Notice that for n fixed, the curves are more regular when $\sigma^2 = 0.2$ than when $\sigma^2 = 5$. This might imply that they are more precise (more "deterministic"). This is also natural: when σ^2 is small, there are more effective coagulations, which allows the "Law of Large Numbers" act strongly: comparing Figures 1b and 2a, which concern both $n = 50000$ particles, we see that when $\sigma^2 = 0.2$, the curves are much more regular.
Conversely, the best ε seems to be smaller, at n fixed, when $\sigma^2 = 0.2$ than when $\sigma^2 = 5$. This may also be explained: the variations of the space density are bigger when $\sigma^2 = 0.2$ than when $\sigma^2 = 5$, hence the mollification is a better approximation when $\sigma^2 = 5$.
Finally, Figure 3 shows $\bar{Z}_t^{n,\varepsilon} - a_t$, for $t \in [0, 10]$, in the case where $\sigma^2 = 5$, $n = 250000$, and $\varepsilon = 0.5$. We can observe the Brownian behaviour of the stochastic process $\bar{Z}_t^{n,\varepsilon} - a_t$ which might illustrate a central limit theorem, see [DFT01] for a similar result in the homogeneous case.

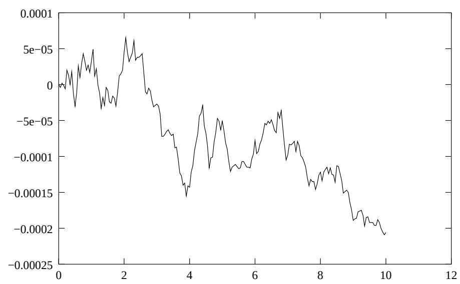

Figure 3: $\sigma^2 = 5$, $n = 250000$, $\varepsilon = 0.5$.

Acknowledgements We wish to thank James Norris for a very fruitful discussion.

References

[Ald99] D.J. Aldous. Deterministic and Stochastic Models for Coalescence (Aggregation, Coagulation): A Review of the Mean-Field Theory for Probabilists. *Bernoulli*, 5:3–48, 1999.

[DFT01] M. Deaconu, N. Fournier, and E. Tanré. Study of a stochastic particle syste, associated with the Smoluchowski coagulation equation. *Prépublication de l'Institut Élie Cartan, Nancy*, 15, 2001.

[DFTab] M. Deaconu, N. Fournier, and E. Tanré. A pure jump Markov process associated with Smoluchowski's coagulation equation. *Prépublication de l'Institut Élie Cartan de Nancy*, 6, 2001 (to appear in Ann. Probab.).

[EW00] A. Eibeck and W. Wagner. An efficient stochastic algorithm for studying coagulation dynamics and gelation phenomena. *SIAM J. Sci. Comput.*, 22(6):921–948, 2000.

[GM97] C. Graham and S. Méléard. Stochastic particle approximations for generalized Boltzmann models and convergence estimates. *Ann. Probab.*, 25(1):115–132, 1997.

[IW89] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes.* North-Holland Publishing Co., Amsterdam, second edition, 1989.

[Jeo98] I. Jeon. Existence of gelling solutions for coagulation-fragmentation equations. *Comm. Math. Phys.*, 194:541–567, 1998.

[Jou01] B. Jourdain. Nonlinear processes associated with the discrete Smoluchowski coagulation fragmentation equation. *Prépublication du CERMICS*, 207, 2001.

[JS87] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes.* Springer, 1987.

[LM01a] P. Laurençot and S. Mischler. Global existence for the discrete diffusive coagulation-fragmentation equations in l^1. *Prépublication de l'École Normale Supérieure*, 1, 2001.

[LM01b] P. Laurençot and S. Mischler. The continuous coagulation-fragmentation equations with diffusion. *Prépublication de l'École Normale Supérieure*, 6, 2001.

[Lus78] A. A. Lushnikov. Some new aspects of coagulation theory. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosfer. I Okeana*, 14(10):738–743, 1978.

[Mar68] A. H. Marcus. Stochastic coalescence. *Technometrics*, 10(1):133–148, 1968.

[Mél96] S. Méléard. Asymptotic behaviour of some interacting particle systems, McKean-Vlasov and Boltzmann models. In *Cours du CIME 95, Probabilistic models for nonlinear pde's, LNM*, volume 1627. Springer, 1996.

[Nor99] J.R. Norris. Smoluchowski's coagulation equation: uniqueness, non-uniqueness and hydrodynamic limit for the stochastic coalescent. *Ann. Appl. Probab.*, 9,:78–109, 1999.

[Nor00] J.R. Norris. Cluster Coagulation. *Commun. Math. Phys.*, 209:407–435, 2000.

[Nor01] J.R. Norris. Deterministic models of coagulation with diffusion. *Private Communication*, 2001.

[Tan78] H. Tanaka. Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 46:67–105, 1978.