Malliavin Calculus for Parabolic S.P.D.E.s with Jumps.

Nicolas FOURNIER[1]

October, 5, 1999

Abstract

We study a parabolic S.P.D.E. driven by a white noise and a compensated Poisson measure. We first define the solutions in a weak sense, and we prove the existence and the uniqueness of a weak solution. Then we use the Malliavin calculus in order to show that under some non-degeneracy assumptions, the law of the weak solution admits a density with respect to the Lebesgue measure. To this aim, we introduce two derivative operators associated with the white noise and the Poisson measure. The one associated with the Poisson measure is studied in details.

0 Introduction.

Let $T > 0$ be a positive time, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a probability space, and let L be a positive real number. We consider $[0,T]\times[0,L]$ and $[0,T]\times[0,L]\times I\!R$ endowed with their Borel σ-fields. Let $W(dx,dt)$ be a space-time white noise on $[0,L]\times[0,T]$ based on $dxdt$ (see e.g. J.B. Walsh [9], p 269), and let N be a Poisson measure on $[0,T]\times[0,L]\times I\!R$, independent of W, with intensity measure $\nu(dt,dx,dz) = dtdxq(dz)$, where q is a positive σ-finite measure on $I\!R$. The compensated Poisson measure is denoted by $\tilde{N} = N - \nu$. Our purpose is to study the following one-dimensional stochastic partial differential equation on $[0,L]\times[0,T]$:

$$\left[\frac{\partial V}{\partial t}(x,t) - \frac{\partial^2 V}{\partial x^2}(x,t)\right] dxdt = g(V(x,t))dxdt + f(V(x,t))W(dx,dt) + \int_{I\!R} h(V(x,t),z)\tilde{N}(dt,dx,dz) \qquad (0.1)$$

with Neumann boundary conditions

$$\forall\, t > 0, \quad \frac{\partial V}{\partial x}(0,t) = \frac{\partial V}{\partial x}(L,t) = 0 \qquad (0.2)$$

and with deterministic initial condition $V(x,0) = \mathcal{V}_0(x)$.

In this paper, we first prove the existence and uniqueness of a weak solution $\{V(x,t)\}$ to (0.1). Then we show, in the case where $q(dz)$ admits a sufficiently regular density, and under some non-degeneracy conditions, that the law of $V(x,t)$ is absolutely continuous with respect to the Lebesgue measure as soon

[1]Laboratoire de Probabilités, UMR 7599, Université Paris VI, 4, Place Jussieu, Tour 56, 3^o étage, F-75252 Paris Cédex 05, e-mail : fournier@proba.jussieu.fr.

as $t > 0$.
Parabolic S.P.D.E.s driven by a white noise, i.e. equation (0.1) with $h \equiv 0$, have been introduced by Walsh, [8] and [9]. In [9], he defines his weak solutions, then he proves a theorem of existence, uniqueness and regularity. Since, various properties of Walsh's equation have been investigated. In particular the Malliavin calculus has been developped by Pardoux, Zhang, [6], and Bally, Pardoux, [1].
But Walsh builds his equation in order to model a discontinuous neurophysiological phenomenous. In [8], he explains that the white noise W approximates a Poisson point process. This approximation is realistic because there are many jumps, and the jumps are very small, but in any case, the observed phenomenous is discontinuous. However, S.P.D.E.s with jumps are much less known. In the case where $f \equiv 0$, Saint Loubert Bié has studied in [7] the existence, uniqueness, regularity, and stochastic variational calculus. We prove here a result of existence and uniqueness, because we define in a slighlty different way the weak solutions, and because Saint Loubert Bié does not study exactly the same equation. Furthermore, his result about the absolute continuity does not extend to the present case.
The Malliavin calculus for jump processes we will build extends the work of Bichteler, Gravereaux, Jacod, [2], who study diffusion processes with jumps. We can not apply directly their methods, essentially because the weak solution of (0.1) is not a semi-martingale. Bichteler et al., [2], use a "scalar product of derivation", which does not allow to obtain satisfying results in the present case (see Saint Loubert Bié, [7]). Thus we have to introduce a real "derivative operator", which gives more information.
Our method is also inspired by the paper of Bally, Pardoux, [1] who prove the existence of a smooth density in the case where $h \equiv 0$.
The present work is organized as follows. In Section 1, we define the solutions of (0.1) in a weak sense, which is easy in the continuous case but slightly more difficult here, because there are "predictability" problems. Then we state our main results. An existence and uniqueness result is proved in Section 2. We study the existence of a density for the law of the weak solution in Section 3. Finally, an appendix lies at the end of the paper.

1 Statement of the main results.

In the whole work, we assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ is the canonical product probability space associated with W and N. In particular,

$$\mathcal{F}_t = \sigma\{W(A)\ ;\ A \in \mathcal{B}([0,L]\times[0,t])\} \vee \sigma\{N(B)\ ;\ B \in \mathcal{B}([0,t]\times[0,L]\times I\!R)\} \tag{1.1}$$

Definition 1.1 *Consider a process $Y = \{Y(y,s)\}_{[0,L]\times[0,T]}$. We will say that Y is*

- predictable *if it is $Pred \otimes \mathcal{B}([0,L])$-measurable, where $Pred$ is the predictable σ-field on $\Omega \times [0,T]$.*
- bounded in L^2 *if* $\sup_{[0,L]\times[0,T]} E(Y^2(y,s)) < \infty$.
- *a* version *of $X = \{X(y,s)\}_{[0,L]\times[0,T]}$ if for all $y \in [0,L]$, all $s \in [0,T]$, a.s., $Y(y,s) = X(y,s)$.*
- *a* weak version *of $X = \{X(y,s)\}_{[0,L]\times[0,T]}$ if $dP(\omega)dyds$-a.e., $Y(y,s)(\omega) = X(y,s)(\omega)$.*
- of class $\mathcal{PV}$ *if it is bounded in L^2 and if it is a weak version of a predictable process.*

We now define the stochastic integrals we will use.

Definition 1.2 *Let Y be a process that admits a predictable weak version Y_-. Let Φ be a measurable function such that*

$$\int_0^T \int_0^L \int_{I\!R} E\left(\phi^2(Y(y,s),s,y,z)\right) q(dz)dyds < \infty \tag{1.2}$$

Then we set

$$\int_0^T \int_0^L \int_{I\!R} \phi(Y(y,s),s,y,z)\tilde{N}(ds,dy,dz) = \int_0^T \int_0^L \int_{I\!R} \phi(Y_-(y,s),s,y,z)\tilde{N}(ds,dy,dz) \tag{1.3}$$

The obtained random variable does not depend on the choice of the predictable version, up to a $P(d\omega)$-negligible set. We define in the same way the stochastic integral against the white noise.

Using the classical stochastic integration theory (see Jacod, Shiryaev, [4] p 71-74, and Walsh, [9] p 292-298), we deduce, since $Y_- = Y\ dPdyds$-a.e., that :

$$E\left[\left(\int_0^T \int_0^L \int_{I\!R} \Phi(Y(y,s),s,y,z)\tilde{N}(ds,dy,dz)\right)^2\right] = \int_0^T \int_0^L \int_{I\!R} E\left(\Phi^2(Y(y,s),s,y,z)\right) q(dz)dyds \tag{1.4}$$

$$E\left[\left(\int_0^T \int_0^L \Phi(Y(y,s),s,y)W(dy,ds)\right)^2\right] = \int_0^T \int_0^L E\left(\Phi^2(Y(y,s),s,y)\right) dyds \tag{1.5}$$

$$E\left[\left(\int_0^T \int_0^L \Phi(Y(y,s),s,y)dyds\right)^2\right] \le TL\int_0^T \int_0^L E\left(\Phi^2(Y(y,s),s,y)\right) dyds \tag{1.6}$$

We now would like to define the weak solutions of (0.1). First, we suppose the following conditions, which in particular allow all the integrals below to be well-defined.
Assumption (H) : f and g satisfy some global lipschitz conditions on $I\!R$, h is measurable on $I\!R \times I\!R$, and there exists a positive function $\eta \in L^2(I\!R, q)$ such that for all $x, y, z \in I\!R$,

$$|h(0,z)| \le \eta(z) \quad \text{and} \quad |h(x,z) - h(y,z)| \le \eta(z)|x-y| \tag{1.7}$$

Assumption (D) : $\mathcal{V}_0$ is deterministic, $\mathcal{B}([0,L])$-measurable, and bounded.

Following Walsh, [8], [9], or Saint Loubert Bié, [7], we define the weak solutions of (0.1) by using an evolution equation. Let $G_t(x,y)$ be the Green kernel of the deterministic system :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad ; \qquad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \tag{1.8}$$

This means that $G_t(x,y)$ is the solution of the system with initial condition a Dirac mass at y. It is well-known that

$$G_t(x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n\in Z} \left[\exp\left(\frac{-(y-x-2nL)^2}{4t}\right) + \exp\left(\frac{-(y+x-2nL)^2}{4t}\right)\right] \tag{1.9}$$

All the properties of G that we will use can be found in the Appendix. Now we can define the weak solutions of equation (0.1).

Definition 1.3 *Assume (H) and (D). A process V of class $\mathcal{PV}$ is said to be a weak solution of (0.1) if for all x in $[0,L]$, all $t > 0$, a.s.*

$$\begin{aligned} V(x,t) &= \int_0^L G_t(x,y)\mathcal{V}_0(y)dy + \int_0^t \int_0^L G_{t-s}(x,y)f(V(y,s))W(dy,ds) \\ &+ \int_0^t \int_0^L G_{t-s}(x,y)g(V(y,s))dyds + \int_0^t \int_0^L \int_{I\!R} G_{t-s}(x,y)h(V(y,s),z)\tilde{N}(ds,dy,dz) \end{aligned} \tag{1.10}$$

where we have used Definition 1.2

Let us finally state our first result.

Theorem 1.4 *Assume (H) and (D). Equation (0.1) admits a unique solution $V \in \mathcal{PV}$ in the sense of Definition 1.3. The uniqueness holds in the sense that if $V' \in \mathcal{PV}$ is another weak solution, then V and V' are two versions of the same process, i.e. for each x, t, a.s., $V(x,t) = V'(x,t)$.*

It is not standard to work with predictable weak versions. In the continuous case, no such problem appear, and the classical diffusion processes with jumps are a.s. càdlàg. But here, the paths of a weak solution cannot be càdlàg in time. Indeed, this is even impossible in the much simpler case where $\mathcal{V}_0 = 1$, $f = g = 0$, $h(x,z) = 1$, where $q(I\!R) < \infty$, and where the Poisson measure is not compensated. In such a case, the Poisson measure is finite, thus it can be written as $N = \sum_{i=1}^{\mu} \delta_{\{T_i, X_i, Z_i\}}$, and hence the weak solution of (0.1) is given by

$$V(x,t) = 1 + \sum_{i=1}^{\mu} G_{t-T_i}(x, X_i) 1_{\{t > T_i\}}$$

In this case, we see that for each $\omega \in \Omega$ satisfying $\mu(\omega) \geq 1$, the map $t \mapsto V(X_1(\omega), t)(\omega)$ explodes when t decreases to $T_1(\omega)$.

We are now interested in the Malliavin calculus. We thus suppose some more conditions. First, the intensity measure of N has to be sufficiently "regular".

Assumption (M) : N has the intensity measure $\nu(ds, dy, dz) = \varphi(z) 1_O(z) ds dy dz$, where O is an open subset of $I\!R$, and φ is a strictly positive C^1 function on O.

The functions f, g, h also have to be regular enough.

Assumption (H') : f and g are C^1 functions on $I\!R$, and their derivatives are bounded. The function $h(x,z)$ on $I\!R \times O$ admits the continuous partial derivatives h'_z, h'_x, and $h''_{zx} = h''_{xz}$. There exist a constant K and a function $\eta \in L^2(O, \varphi(z)dz)$ such that for all $x \in I\!R$, all $z \in O$,

$$|h'_z(0,z)| + |h''_{xz}(x,z)| \leq K \quad ; \quad |h(0,z)| + |h'_x(x,z)| \leq \eta(z) \tag{1.11}$$

Notice that (H') is stronger than (H).
Let ρ be a strictly positive C^1 function on O such that ρ and ρ' are bounded, and such that

$$\rho \in L^1(O, \varphi(z)dz) \tag{1.12}$$

This "weight function" can be chosen according to the parameters of (0.1). The next condition is technical.

Assumption (S) : there exists a family of C^1 positive functions K_ϵ on O, with compact support (in O), bounded by 1, and such that

$$\forall\, z \in O, K_\epsilon(z) \longrightarrow_{\epsilon \to 0} 1 \quad ; \quad \int_O \left(K'_\epsilon(z)\right)^2 \eta^2(z) \rho(z) \varphi(z) dz \longrightarrow_{\epsilon \to 0} 0 \tag{1.13}$$

We finally suppose one of the following non-degeneracy conditions :

Assumption (EW) : for all x in $I\!R$, $f(x) \neq 0$
or
Assumption $(EP1)$: $f = 0$, and there exists $\hat{\eta} \in L^1(O, \varphi(z)dz)$ such that $0 \leq h'_x(x,z) \leq \hat{\eta}(z)$. For all x in $I\!R$,

$$\int_O 1_{\{h'_z(x,z) \neq 0\}} \varphi(z) dz = \infty \tag{1.14}$$

or
Assumption $(EP2)$: we set $\mathcal{H} = \{z \in O \ /\forall x \in I\!R,\ h'_z(x,z) \neq 0\}$. There exist some constants $C_0 > 0$, $r_0 \in]\frac{3}{4}, 1[$, and $\gamma_0 \geq 0$ such that for all $\gamma \geq \gamma_0$,

$$\int_{\mathcal{H}} \left(1 - e^{-\gamma\rho(z)}\right) \varphi(z)dz \geq C_0 \times \gamma^{r_0} \tag{1.15}$$

Our second main result is the next theorem.

Theorem 1.5 *Assume (M), (D), (H'), and (S). Let V be the unique weak solution of (0.1) in the sense of Definition 1.3, and let $(x,t) \in [0,L] \times]0,T]$. Then under one of the assumptions (EW), $(EP1)$ or $(EP2)$, the law of $V(x,t)$ admits a density with respect to the Lebesgue measure on $I\!R$.*

We will use two derivative operators. The first one, associated with the white noise, is classical (see Nualart [5]). The second operator, associated with the Poisson measure, is inspired from Bichteler, Gravereaux, and Jacod, Chapter IV in [2]. They study the Malliavin calculus for diffusion processes with jumps, in the case where the intensity measure of the Poisson measure is $1_O(z)dsdz$. Furthermore, they do not use any derivative operator : they work with a "scalar product of derivation", which gives less information. Using this method, we could probably prove Theorem 1.5 only under $(EP1)$.
Our theorem gives in fact two results : the law of $V(x,t)$ admits a density either thanks to W or thanks to N. It seems to be very difficult to state a "joint" non-degeneracy condition (see Subsection 3.5).
Assumption (EW) looks reasonnable : although Pardoux and Zhang prove this Theorem under a really less stringent assumption when $h = 0$ in [6] (it suffices that $\exists y \in [0,L]$ such that $f(\mathcal{V}_0(y)) \neq 0$), they use the continuity of their solution. The first condition in $(EP1)$ ($f = 0$, $h'_x \geq 0$, $h'_x \leq \hat{\eta}$) is very stringent, but the second one might be optimal : Bichteler et al. also have to assume this kind of condition. Finally, $(EP2)$ is much more general, but it is an uniform non degeneracy assumption.

St Loubert Bié proves in [7] the existence of a density under the assumption $f = 0$, an hypothesis less stringent than (M), an assumption quite similar to (H'), and under $(h1)$ or $(h2)$ below (the notations are adapted to our context) :
$(h1)$: $h'_x = 0$ and $\int_O 1_{\{h'_z(z)=0\}}\varphi(z)dz = \infty$
or
$(h2)$: $\eta \in L^1(O, \varphi(z)dz)$, $h'_x \geq 0$, and something like $(EP1)$, but depending on the solution process V.

Condition $(h1)$ is very restrictive, and $(h2)$ is not very tractable : one has to know the behaviour of the weak solution. Saint Loubert Bié uses in both case the positivity of N (as in the proof of Theorem 1.5 under $(EP1)$). But since the white noise is signed, this method can not be extended to the case where $f \not\equiv 0$. That is why in this work, the most interesting assumption is probably $(EP2)$.

Let us finally give examples about assumptions (S) and $(EP2)$.

Remark 1.6 *Assume that $O = I\!R$. Then (S) is satisfied for any φ, η, and any choice of ρ.*

Proof : it suffices to choose a family of C^1 positive functions of the form

$$K_\epsilon(z) = \begin{cases} 1 & \text{if} \quad |z| < 1/\epsilon \\ 0 & \text{if} \quad |z| > 1/\epsilon + 2 \end{cases}$$

such that $|K'_\epsilon(z)| \leq 1_{\{|z| \in [1/\epsilon, 1/\epsilon+2\}}$. Using the Lebesgue Theorem and the fact that $\rho\eta^2 \in L^1(I\!R, \varphi(z)dz)$, (1.13) is immediate.

Example 1 : assume that $O = I\!R$, and that φ is a C^1 function on $I\!R$ satisfying, for some $K > a > 0$, $K > \varphi > a$. We consider a function $h(x,z) = c(x)\eta(z)$, where c is a strictly positive C^1 function on $I\!R$

of which the derivative is bounded. η has to be C^1 on $I\!R$, to belong to $L^2(I\!R, \varphi(z)dz)$, and η' must be bounded. If for some $b \in I\!R$, $[b, \infty[\subset \{\eta' \neq 0\}$, then (M), (H'), (S), and $(EP2)$ are satisfied :
thanks to Remark 1.6, it suffices to check $(EP2)$. Choosing $\rho(z) \geq z^{\frac{-7}{6}} 1_{\{z \geq b \vee 1\}}$, we see that (1.15) is satisfied, since $[b, \infty[\subset \mathcal{H}$, and using

$$\forall\, x \in [0,1], \quad 1 - e^{-x} \geq x/2 \tag{1.16}$$

Example 2 : assume that $O =]0,1[$, and that $\varphi(z) = 1/z^r$, for some $r > 7/4$. We consider a function $h(x,z) = c(x)\eta(z)$, where c is a strictly positive C^1 function on $I\!R$ of which the derivative is bounded, and where $\eta(z) = z^\alpha$, for some $\alpha > 1 \vee \frac{r-1}{2} \vee \frac{7-r}{6}$. Then (M), (H'), (S), and $(EP2)$ are satisfied :
(M) is met, and (H') holds, since $\alpha > 1 \vee \frac{r-1}{2}$. It is clearly possible to choose $\rho(z)$ of the form

$$\rho(z) = \begin{cases} z^\beta & \text{if} \quad z \leq 1/4 \\ (1-z)^\delta & \text{if} \quad z \geq 3/4 \end{cases} \tag{1.17}$$

with $\beta > 1 \vee (r-1)$ and $\delta \geq 1$. Using (1.16), the facts that $\mathcal{H} =]0,1[$ and that $\rho(z) \geq z^\beta 1_{]0,1/4[}(z)$, we see that $(EP2)$ is satisfied if $\beta < \frac{4}{3}(r-1)$.
We now choose a family K_ϵ of C^1 positive functions on $]0,1[$, bounded by 1, and satisfying

$$K_\epsilon(z) = \begin{cases} 0 & \text{if} \quad z < \epsilon/2 \\ 1 & \text{if} \quad \epsilon < z < 1-\epsilon \\ 0 & \text{if} \quad 1-\epsilon/2 < z < 1 \end{cases} \quad ; \quad |K'_\epsilon(z)| \leq \frac{4}{\epsilon} 1_{]\epsilon/2,\epsilon[\cup]1-\epsilon,1-\epsilon/2[}(z)$$

An explicit computation shows that (S) is satisfied if $\beta > r + 1 - 2\alpha$ and if $\delta > 1$.
Since $\alpha > \frac{7-r}{6}$ and $r > 7/4$, it is possible to choose β in $](r-1), \frac{4}{3}(r-1)[\cap]1, \infty[\cap]r+1-2\alpha, \infty[$, and the conclusion follows.

Let us finally remark that (S) is satisfied for any O, η, ϕ, if ρ is of class C_b^2, and if $\rho(z) + |\rho'(z)| \longrightarrow_{z \to \partial O} 0$. Bichteler, Gravereaux and Jacod assume this kind of condition about ρ in [2].

2 Existence and uniqueness.

In this short section, we sketch the proof of Theorem 1.4. We begin with a fundamental lemma.

Lemma 2.1 *Assume (H). Let Y be a process of class $\mathcal{PV}$. Then the processes*

$$U(x,t) = \int_0^t \int_0^L \int_{I\!R} G_{t-s}(x,y) h(Y(y,s),z) \tilde{N}(ds,dy,dz) \tag{2.1}$$

$$X(x,t) = \int_0^t \int_0^L G_{t-s}(x,y) f(Y(y,s)) W(dy,ds) \tag{2.2}$$

$$Z(x,t) = \int_0^t \int_0^L G_{t-s}(x,y) g(Y(y,s)) dyds \tag{2.3}$$

belong to $\mathcal{PV}$.

<u>Proof</u> : let us for example prove the lemma for U. First notice that U is bounded in L^2 thanks to (1.4), (H), the fact that Y is bounded in L^2, and the Appendix (4.3). We still have to prove that U admits a predictable weak version. We know from Walsh, [9] p 323, that

$$\sup_{x \in [0,L]} \int_0^T \int_0^L \left(G_t(x,y) - \sum_{k=0}^N \phi_k(x)\phi_k(y) e^{-\lambda_k t} \right)^2 dydt \longrightarrow^{N \to \infty} 0 \tag{2.4}$$

where $\phi_0 = \frac{1}{\sqrt{L}}$, $\phi_k(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{k\pi x}{L}\right)$, and $\lambda_k = \left(\frac{\pi}{L}\right)^2 k^2$. We thus can approximate $U(x,t)$ by

$$U^N(x,t) = \sum_{k=0}^{N} \phi_k(x) e^{-\lambda_k t} \int_0^t \int_0^L \int_{I\!R} \phi_k(y) e^{\lambda_k s} h(V(y,s),z) \tilde{N}(ds,dy,dz)$$

which clearly admits a predictable version since for each k, $\phi_k(x)e^{-\lambda_k t}$ is deterministic and the process

$$t \mapsto \int_0^t \int_0^L \int_{I\!R} \phi_k(y) e^{\lambda_k s} h(V(y,s),z) \tilde{N}(ds,dy,dz)$$

is a càdlàg martingale. Using (1.4), (H), and (2.4), one easily checks that, when N goes to infinity,

$$\sup_{x,t} E\left(\left(U(x,t) - U^N(x,t)\right)^2\right) \longrightarrow 0 \tag{2.5}$$

Since for each N, U^N admits a predictable version, and since there exists a subsequence of U^N going $dPdxdt$-a.e. to U, we deduce that U admits a predictable weak version.

Let us remark that even if Y is a predictable process, the process U defined by (2.1) is not *a priori* predictable, but only admits a predictable weak version.

Proof of Theorem 1.4 : the uniqueness easily follows from Gronwall's Lemma applied to the function $\phi(t) = \sup_x E((V(x,t) - V'(x,t))^2)$. Let us prove the existence. To this aim, we first build the following Picard approximations.

$$\begin{aligned} V^0(x,t) &= \int_0^L G_t(x,y)\mathcal{V}_0(y)dy \\ V^{n+1}(x,t) &= V^0(x,t) + \int_0^t \int_0^L G_{t-s}(x,y) f(V^n(y,s)) W(dy,ds) \\ &\quad + \int_0^t \int_0^L G_{t-s}(x,y) g(V^n(y,s)) dyds \\ &\quad + \int_0^t \int_0^L \int_{I\!R} G_{t-s}(x,y) h(V^n(y,s),z) \tilde{N}(ds,dy,dz) \end{aligned} \tag{2.6}$$

Due to (D) and the appendix (4.2), V^0 is deterministic and bounded. We deduce from Lemma 2.1 that for each n, V^n is well-defined and is of class $\mathcal{PV}$. A simple computation using (1.4), (1.5), (1.6), assumption (H), and the appendix, (4.2), shows that for each $n \geq 1$,

$$E\left((V^{n+1}(x,t) - V^n(x,t))^2\right) \leq K \int_0^t \frac{ds}{\sqrt{t-s}} \sup_x E\left((V^n(x,s) - V^{n-1}(x,s))^2\right) \tag{2.7}$$

We now set $\phi_n(t) = \sup_x E\left((V^{n+1}(x,t) - V^n(x,t))^2\right)$. We obtain, iterating once (2.7), and using the fact that $\int_0^t \frac{ds}{\sqrt{t-s}} \int_0^s \frac{du}{\sqrt{s-u}} \leq 4$,

$$\phi_n(t) \leq K \int_0^t \phi_{n-1}(s) \frac{ds}{\sqrt{t-s}} \leq K \int_0^t \phi_{n-2}(s) ds \tag{2.8}$$

Since ϕ_0 is bounded (because of (D)), we deduce from the first inequality in (2.7) that ϕ_1 is also bounded. Thus Picard's Lemma allows to conclude that

$$\sum_n \sup_{[0,T]} \left(\phi_{2n}(t)\right)^{1/2} < \infty \quad ; \quad \sum_n \sup_{[0,T]} \left(\phi_{2n+1}(t)\right)^{1/2} < \infty \tag{2.9}$$

and hence,

$$\sum_n \sup_{x,t} E\left((V^{n+1}(x,t) - V^n(x,t))^2\right) < \infty \tag{2.10}$$

This clearly implies the existence of a process V bounded in L^2 such that, when n tends to infinity,

$$\sup_{[0,L]\times[0,T]} E\left((V^n(x,t) - V(x,t))^2\right) \longrightarrow 0 \tag{2.11}$$

This process belongs to $\mathcal{PV}$: for each n, there exists a predictable process V^n_- which is a weak version of V^n, and it is clear that V^n_- goes to V $dPdxdt$-a.e.
Finally, making n go to infinity in (2.6), (by using (2.11)), we see that V satisfies (1.10). The proof is finished.

3 The Malliavin calculus.

The aim of this main section is to prove Theorem 1.5. In Subsection 3.1, we will define some derivative operators. In Subsection 3.2, we will state the main properties of these operators, and derive a criterion of absolute continuity. We will say how to "differentiate" stochastic integrals in Subsection 3.3, and then "differentiate" the weak solution of (0.1) in Subsection 3.4. We will conclude in Subsection 3.5.

In the whole section, we will assume at least (M), (D), (H'), and (S).

3.1 The derivative operators.

We denote by $C^p_c(I\!R^d)$ (resp. $C^p_b(I\!R^d)$) the set of C^p functions on $I\!R^d$ with compact support (resp. of which the derivatives of order 1 to p are bounded). As said previously, we will define two derivative operators.

We begin with the derivative operator associated with the Poisson measure. We first denote by $\mathcal{CL}$ the set of measurable functions $l(s,y,z)$ on $[0,T]\times[0,L]\times O$, with compact support, C^2 on O (in z), such that l, l'_z, and l''_{zz} are bounded on $[0,T]\times[0,L]\times O$. Then we define the domain

$$\mathcal{S}^N = \left\{X = F(N(g_1),...,N(g_d)) + a \;\Big/ d \geq 1,\; F \in C^2_c(I\!R^d),\; g_i \in \mathcal{CL},\; a \in I\!R\right\} \tag{3.1}$$

where $N(g_i)$ stands for $\int_0^T \int_O^L \int_O g_i(s,x,z)N(ds,dx,dz)$. If $X \in \mathcal{S}^N$, if $\alpha \in [0,L]$, $\tau \in [0,T]$, and $\zeta \in O$, we set

$$D^N_{\alpha,\tau,\zeta}X = \sum_{i=1}^d \partial_i F(N(g_1),...,N(g_d))(g_i)'_z(\alpha,\tau,\zeta) \tag{3.2}$$

In order to "close" $(\mathcal{S}^N, D^N)$, we introduce another operator. For $X \in \mathcal{S}^N$,

$$\begin{aligned} L^N X &= \frac{1}{2}\sum_{i=1}^d \partial_i F(N(g_1),...,N(g_d))N\left(\rho.(g_i)''_{zz} + \left(\rho\frac{\varphi'}{\varphi} + \rho'\right).(g_i)'_z\right) \\ &+ \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j F(N(g_1),...,N(g_d))N\left(\rho.(g_i)'_z.(g_j)'_z\right) \end{aligned} \tag{3.3}$$

We finally define a scalar product. If $S_{\alpha,\tau,\zeta}(\omega)$ and $T_{\alpha,\tau,\zeta}(\omega)$ are in $L^2(P(d\omega)\rho(\zeta)N(\omega,d\tau,d\alpha,d\zeta))$, we set

$$\langle S,T\rangle_{\rho N} = \int_0^T \int_0^L \int_O S_{\alpha,\tau,\zeta}T_{\alpha,\tau,\zeta}\;\rho(\zeta)N(d\tau,d\alpha,d\zeta) \quad \text{and} \quad \langle S\rangle_{\rho N} = \langle S,S\rangle_{\rho N}$$

Then one gets easily, for all X and Y in $\mathcal{S}^N$,

$$\left\langle D^N X, D^N Y\right\rangle_{\rho N} = L^N XY - XL^N Y - YL^N X \tag{3.4}$$

By adapting Bichteler et al., [2] Proposition 9-3, p 113., we check in the lemma below that L^N is well-defined : if $X = F(N(f_1),...,N(f_d)) = G(N(g_1),...,N(g_q))$, then using one expression or the other will give the same $L^N X$.

Lemma 3.1 *If $X = F(N(f_1),...,N(f_k)) \in \mathcal{S}^N$, and if $X \equiv 0$, then $L^N X \equiv 0$, and thus $\left\langle D^N X\right\rangle_{\rho N} \equiv 0$.*

Proof : we assume that Ω is the set of the integer valued measures on $[0,T]\times[0,L]\times O$. Let $\omega\in\Omega$ and $(t,x,z)\in \text{supp }\omega$ be fixed. We set $\omega' = \omega - \delta_{(t,x,z)}$, and for $\lambda\in\Lambda$, $\omega^\lambda = \omega' + \delta_{(t,x,z+\lambda)}$, where Λ is a neighbourhood of 0 in $I\!R$ such that $z+\Lambda\subset O$. Then ω' and ω^λ are in Ω. We set

$$X_{t,x,z}(\lambda) = X(\omega^\lambda) = F(\omega'(f_1) + f_1(t,x,z+\lambda),...,\omega'(f_k)+f_k(t,x,z+\lambda))$$

Then $X_{t,x,z}$ vanishes identically, and is C^2 in λ. We deduce that

$$\frac{1}{\varphi(z)}\frac{\partial}{\partial\lambda}\left(\rho(z+\lambda)\varphi(z+\lambda)\times\frac{\partial}{\partial\lambda}X_{t,x,z}(\lambda)\right)\Big|_{\lambda=0} = 0$$

Writing this explicitely, and summing the obtained expression on all the points $(t,x,z)\in\text{supp }\omega$, we get $L^N X(\omega) = 0$.

We thus see that for each $\omega\in\Omega$, $D^N X(\omega)$ is well-defined up to a $N(\omega)$-negligible set : we could replace $D^N_{\alpha,\tau,\zeta}X(\omega)$ with anything when $N(\omega,\{\alpha,\tau,\zeta\}) = 0$. In order to understand this notion of derivative, assume that Ω is the canonical space associated with N. Then every $\omega\in\Omega$ is a counting measure on $[0,T]\times[0,L]\times O$, and

$$1_{\{\omega(\{\alpha,\tau,\zeta\}=1\}}D^N_{\alpha,\tau,\zeta}X(\omega) = 1_{\{\omega(\{\alpha,\tau,\zeta\}=1\}}\frac{\partial}{\partial\lambda}X(\omega-\delta_{(\alpha,\tau,\zeta)}+\delta_{(\alpha,\tau,\zeta+\lambda)})\Big|_{\lambda=0} \tag{3.5}$$

The following lemma can be proved as Proposition 9-3, d), p 113 in [2].

Lemma 3.2 *If X and Y are in $\mathcal{S}^N$, then*

$$E(XL^N Y) = E(YL^N X) = -\frac{1}{2}E\left(\left\langle D^N X, D^N Y\right\rangle_{\rho N}\right) \tag{3.6}$$

We deduce from (3.6) the following lemma, which shows that D^N is closable.

Lemma 3.3 *Let Z_n be a sequence of $\mathcal{S}^N$ which goes to 0 in L^2. Assume that there exists $S_{\alpha,\tau,\zeta}(\omega)\in L^2(P(d\omega)\rho(\zeta)N(\omega,d\tau,d\alpha,d\zeta))$ such that $E\left(\left\langle D^N Z_n - S\right\rangle_{\rho N}\right)$ goes to 0. Then $E(\langle S\rangle_{\rho N}) = 0$.*

Proof : Let X be in $\mathcal{S}^N$. The Cauchy-Schwarz inequality yields

$$E\left(\left\langle S, D^N X\right\rangle_{\rho N}\right) = \lim_n E\left(\left\langle D^N Z_n, D^N X\right\rangle_{\rho N}\right)$$

But, thanks to Lemma 3.2, $E\left(\left\langle D^N Z_n, D^N X\right\rangle_{\rho N}\right) = -2E(Z_n L^N X)$. Since Z_n goes to 0 in L^2, we deduce that $E\left(\left\langle S, D^N X\right\rangle_{\rho N}\right) = 0$. Then we apply this with $X = Z_k$, and we let k go to infinity.

We now define the derivative operator associated with the white noise. First we define the domain of the "smooth variables" :

$$\mathcal{S}^W = \Big\{X = F(W(f_1), ..., W(f_k)) + a \Big/ k \geq 1,\ F \in C_c^2(\mathbb{R}^k),\ f_i \in L^2([0,L]\times[0,T]),\ a \in \mathbb{R}\Big\} \tag{3.7}$$

Here, $W(f_i) = \int_0^T \int_0^L f_i(s,x)W(dx,ds)$. If X is in $\mathcal{S}^N$, if $\alpha \in [0,L]$ and $\tau \in [0,T]$, we set :

$$D^W_{\alpha,\tau} X = \sum_{i=1}^k \partial_i F(W(f_1), ..., W(f_k)) f_i(\alpha,\tau) \tag{3.8}$$

If $S_{\alpha,\tau}(\omega)$ and $T_{\alpha,\tau}(\omega)$ are in $L^2(P(d\omega)d\alpha d\tau)$, we set

$$\langle S, T\rangle_{leb} = \int_0^T \int_0^L S_{\alpha,\tau} T_{\alpha,\tau} d\alpha d\tau \quad \text{and} \quad \langle S\rangle_{leb} = \langle S, S\rangle_{leb} \tag{3.9}$$

The following lemma can be found in Nualart, [5], p 26.

Lemma 3.4 *Let Z_n be a sequence of $\mathcal{S}^W$ that goes to 0 in L^2. Assume that there exists $S_{\alpha,\tau}(\omega) \in L^2(P(d\omega)d\alpha d\tau)$ such that $E\left(\left\langle D^W Z_n - S\right\rangle_{leb}\right)$ goes to 0. Then $E(\langle S\rangle_{leb}) = 0$.*

Now we can build the operators on the product space. The smooth variables domain is

$$\mathcal{S} = \Big\{Y = H(W(f_1), , ..., W(f_d), N(g_1), ..., N(g_k)) + a \Big/ H \in C_c^2(\mathbb{R}^{d+k}),\ k + d \geq 1, \\ f_i \in L^2([0, L \times [0,T]),\ g_j \in \mathcal{CL},\ a \in \mathbb{R}\Big\} \tag{3.10}$$

If Y belongs to $\mathcal{S}$, we define $D^N Y$ and $L^N Y$ (resp. $D^W Y$) as previously, considering the variables $W(f_1), ..., W(f_d)$ (resp. $N(g_1), ..., N(g_k)$) as constants :

$$D^W_{\alpha,\tau} Y = \sum_{i=1}^d \partial_i H(W(f_1), , ..., W(f_d), N(g_1), ..., N(g_k)) f_i(\alpha,\tau) \tag{3.11}$$

$$D^N_{\alpha,\tau,\zeta} Y = \sum_{i=d+1}^{d+k} \partial_i H(W(f_1), , ..., W(f_d), N(g_1), ..., N(g_k)) (g_i)'_z(\alpha,\tau,\zeta) \tag{3.12}$$

The scalar products are denoted as previously, and we see that if X and Z are in $\mathcal{S}$, then X, Z, and $\left\langle D^W X, D^W Z\right\rangle_{leb}$ are bounded ; and $\left\langle D^N X, D^N Z\right\rangle_{\rho N}$ belongs to $\cap_{p<\infty} L^p$.
If Z belongs to $\mathcal{S}$, we set

$$|||Z|||_2 = \left[E\left(Z^2\right) + E\left(\left\langle D^W Z\right\rangle_{leb}\right) + E\left(\left\langle D^N Z\right\rangle_{\rho N}\right)\right]^{\frac{1}{2}} \tag{3.13}$$

We denote by $\mathcal{D}_2$ the closure of $\mathcal{S}$ for this norm. Because of Lemmas 3.3 and 3.4, the operators $D^W_{\alpha,\tau}$ and $D^N_{\alpha,\tau,\zeta}$ can be extended to the space $\mathcal{D}_2$.

Remark 3.5 *We have extended D^W and D^N to $\mathcal{D}_2$, and the weak solution of (0.1) will belong to this space. But no integration by parts formula (like (3.6)) hold on $\mathcal{D}_2$, because L^N can not be extended to this space. Nevertheless, the "differentiability" of our solution will allow us to prove Theorem 1.5.*

Notation 3.6 *We will denote by $\{T_n\}_{n\geq 0}$ a sequence of stopping times, by $\{(X_n, Z_n)\}_{n\geq 0}$ a sequence of $[0,L]\times O$-valued random variables, such that for each n, (X_n, T_n) is $\mathcal{F}_{T_n}$-measurable, and such that*

$$N(dt, dx, dz) = \sum_{n=0}^{\infty} \delta_{(T_n, X_n, Z_n)}(dt, dx, dz) \tag{3.14}$$

Remark 3.7 *1. The way we have closed $(\mathcal{S}^N, D^N)$ shows that if $X \in \mathcal{D}_2$, and if $Y = X$ a.s., then $Y \in \mathcal{D}_2$, and a.s.,*

$$\left\langle D^W X - D^W Y \right\rangle_{leb} = 0 \quad ; \quad \left\langle D^N X - D^N Y \right\rangle_{\rho N} = 0 \tag{3.15}$$

2. Let $S_{\alpha,\tau,\zeta}(\omega)$ and $S'_{\alpha,\tau,\zeta}(\omega)$ belong to $L^2(P(d\omega)\rho(\zeta)N(\omega, d\tau, d\alpha, d\zeta))$. In the whole sequel, the notation "$S_{\alpha,\tau,\zeta} = S'_{\alpha,\tau,\zeta}$" or "$S = S'$" will mean

$$a.s., \quad \langle S - S' \rangle_{\rho N} = 0 \tag{3.16}$$

which is the same as

$$a.s., \quad \forall\, n \in I\!N, \quad S_{X_n,T_n,Z_n} = S'_{X_n,T_n,Z_n} \tag{3.17}$$

3. Let X be a random variable, eventually defined a.s. (X may be a stochastic integral,...). In order to prove that $X \in \mathcal{D}_2$ and that, for some $S_{\alpha,\tau}(\omega) \in L^2(P(d\omega)d\alpha d\tau)$, some $T_{\alpha,\tau,\zeta}(\omega) \in L^2(P(d\omega)\rho(\zeta)N(\omega, d\tau, d\alpha, d\zeta))$,

$$D^W_{\alpha,\tau} X = S_{\alpha,\tau} \quad \text{and} \quad D^N_{\alpha,\tau,\zeta} X = T_{\alpha,\tau,\zeta} \tag{3.18}$$

it suffices to find a sequence $\{X_n\}$ in $\mathcal{S}$ (or in $\mathcal{D}_2$) such that, when n goes to infinity,

$$E\left((X - X_n)^2\right) + E\left(\left\langle S - D^W X_n \right\rangle_{leb}\right) + E\left(\left\langle T - D^N X_n \right\rangle_{\rho N}\right) \longrightarrow 0 \tag{3.19}$$

3.2 Properties of the derivative operators.

We now give the usual properties of our derivative operators. We omit the proofs of the two first ones, because the results are well known in the Gaussian context (we refer to Nualart [5]), and the adaptations are easy.

Proposition 3.8 *$\mathcal{D}_2$, endowed with the following scalar product, is Hilbert :*

$$\langle Y, Z \rangle_{\mathcal{D}_2} = E(YZ) + E\left(\left\langle D^W Y, D^W Z \right\rangle_{leb}\right) + E\left(\left\langle D^N Y, D^N Z \right\rangle_{\rho N}\right) \tag{3.20}$$

Proposition 3.9 *1. Let Y be in $\mathcal{D}_2$ and let F be in $C^1_b(I\!R)$. Then $Z = F(Y)$ belongs to $\mathcal{D}_2$, $D^W Z = F'(Y) D^W Y$, and $D^N Z = F'(Y) D^N Y$.*

2. If f_0 is in $L^2([0, L] \times [0, T])$, then $W(f_0)$ belongs to $\mathcal{D}_2$, $D^W W(f_0) = f_0$, and $D^N W(f_0) = 0$.

3. If g_0 is a measurable function on $[0, T] \times [0, L] \times O$, of class C^1 on O, with compact support, such that g_0 and $(g_0)'_z$ are bounded, then $N(g_0)$ and $\tilde{N}(g_0)$ are in $\mathcal{D}_2$, $D^W N(g_0) = D^W \tilde{N}(g_0) = 0$, and $D^N N(g_0) = D^N \tilde{N}(g_0) = (g_0)'_z$.

We carry on with a proposition which deals with the conditional expectations.

Proposition 3.10 *1. Let Z be in $\mathcal{S}$. Consider the càdlàg martingale $Z_s = E(Z|\mathcal{F}_s)$. Then, for each $s \in [0, T]$, Z_s belongs to $\mathcal{D}_2$, and for all α, τ, a.s.,*

$$D^W_{\alpha,\tau} Z_s = E\left(D^W_{\alpha,\tau} Z \middle| \mathcal{F}_s\right) 1_{\{\tau \le s\}} \tag{3.21}$$

and for all n, a.s.,

$$D^N_{X_n,T_n,Z_n} Z_s = E\left(D^N_{X_n,T_n,Z_n} Z \middle| \mathcal{F}_s\right) 1_{\{T_n \le s\}} \tag{3.22}$$

Furthermore, $|||Z_s|||_2 \le |||Z|||_2$.

2. Let Y be a $\mathcal{F}_s$-measurable element of $\mathcal{D}_2$. Then for each α,τ, each n,

$$D^W_{\alpha,\tau}Y = D^W_{\alpha,\tau}Y 1_{\{\tau\le s\}} \quad and \quad D^N_{X_n,T_n,Z_n}Y = D^N_{X_n,T_n,Z_n}Y 1_{\{T_n\le s\}} \tag{3.23}$$

and these random variables are $\mathcal{F}_s$-measurable.

3. Let X and Z be in $\mathcal{D}_2$. Assume that X and Z are independent. Then XZ belongs to $\mathcal{D}_2$, and

$$D^W XZ = XD^W Z + ZD^W X \quad ; \quad D^N XZ = XD^N Z + ZD^N X. \tag{3.24}$$

Proof : 1. Let $s\in[0,T]$ and $Z = F(W(f_1),...,W(f_m),N(g_1),...,N(g_d)) \in \mathcal{S}$. We set

$$\bar{f}_i(\alpha,\tau) = f_i(\alpha,\tau)1_{\{\tau\le s\}} \quad ; \quad \hat{f}_i(\alpha,\tau) = f_i(\alpha,\tau)1_{\{\tau> s\}}$$

$$\bar{g}_i(\alpha,\tau,\zeta) = g_i(\alpha,\tau,\zeta)1_{\{\tau\le s\}} \quad ; \quad \hat{g}_i(\alpha,\tau,\zeta) = g_i(\alpha,\tau,\zeta)1_{\{\tau> s\}}$$

Then for each i, $W(\bar{f}_i)$ and $N(\bar{g}_i)$ are $\mathcal{F}_s$-measurable, and $W(\hat{f}_i)$, $N(\hat{g}_i)$ are independent of $\mathcal{F}_s$. Thus, if μ denotes the law of $(W(\hat{f}_1),...,N(\hat{g}_d))$ and if $H(X_1,...,Y_d) = \int_{\mathbb{R}^{m+d}} F(X_1+x_1,...,Y_d+y_d))\mu(dx_1,...,dy_d)$, then

$$Z_s = E\left(Z|\mathcal{F}_s\right) = H(W(\bar{f}_1),...,N(\bar{g}_d)) \tag{3.25}$$

Since F is of class C^1_c, it is easy to check that H is in C^1_b. It is not difficult to deduce that Z_s belongs to $\mathcal{D}_2$, and that for all α,τ, all n,

$$D^W_{\alpha,\tau}Z_s = \sum_{i=1}^{m}\partial_i H(W(\bar{f}_1),...,N(\bar{g}_d))\bar{f}_i(\alpha,\tau)$$

$$D^N_{X_n,T_n,Z_n}Z_s = \sum_{i=m+1}^{m+d}\partial_i H(W(\bar{f}_1),...,N(\bar{g}_d))(\bar{g}_i)'_z(X_n,T_n,Z_n)$$

But $\partial_i H(W(\bar{f}_1),...,N(\bar{g}_d)) = E\left(\partial_i F(W(f_1),...,N(g_d)) \mid \mathcal{F}_s\right)$. We easily deduce (3.21), and on the other hand,

$$\begin{aligned} D^N_{X_n,T_n,Z_n}Z_s &= \sum_{i=m+1}^{m+d} E\left[\partial_i F(W(f_1),...,N(g_d)) \mid \mathcal{F}_s\right](g_i)'_z(X_n,T_n,Z_n)1_{\{T_n\le s\}} \\ &= E\left(\sum_{i=m+1}^{m+d}\partial_i F(W(f_1),...,N(g_d))(g_i)'_z(X_n,T_n,Z_n)1_{\{T_n\le s\}} \,\middle|\, \mathcal{F}_s\right) \\ &= E\left(D^N_{X_n,T_n,Z_n}Z \middle| \mathcal{F}_s\right)1_{\{T_n\le s\}} \end{aligned} \tag{3.26}$$

and (3.22) follows. The inequality $|||Z_s|||_2 \le |||Z|||_2$ is a consequence of (3.21), (3.22) and of Jensen's inequality : for example,

$$\begin{aligned} E\left(\left\langle D^N Z_s\right\rangle_{\rho N}\right) &= E\left(\sum_n E\left(D^N_{X_n,T_n,Z_n}Z\middle|\mathcal{F}_s\right)^2 1_{\{T_n\le s\}}\rho(Z_n)\right) \\ &\le E\left(\sum_n E\left(\left(D^N_{X_n,T_n,Z_n}Z\right)^2\middle|\mathcal{F}_s\right)1_{\{T_n\le s\}}\rho(Z_n)\right) \\ &= E\left(\left\{\sum_n\left(D^N_{X_n,T_n,Z_n}Z\right)^2 1_{\{T_n\le s\}}\rho(Z_n)\middle|\mathcal{F}_s\right\}\right) \\ &\le E\left(\left\langle D^N Z\right\rangle_{\rho N}\right) \end{aligned} \tag{3.27}$$

2. is a straightforward consequence of 1. : let Z^k be a sequence of $\mathcal{S}$ going to Y in $\mathcal{D}_2$. Then it is clear that $Z_s^k = E(Z^k|\mathcal{F}_s)$ goes to Y in L^2. Furthermore, we know from 1. that for each k, Z_s^k belongs to $\mathcal{D}_2$, and

$$|||Z_s^{k+1} - Z_s^k|||_2 \leq |||Z^{k+1} - Z^k|||_2$$

Thus Z_s^k is Cauchy in $\mathcal{D}_2$, and thus tends to Y in $\mathcal{D}_2$. One concludes easily by using 1.
3. Let $\mathcal{G}$ (resp. $\mathcal{A}$) be the σ-field generated by X (resp. Z). Let X'^k (resp. Z'^k) be a sequence of $\mathcal{S}$ going to X (resp. Z) in $\mathcal{D}_2$. Using the same arguments as in 1., one can check that $X^k = E(X'^k|\mathcal{G})$ (resp. $X^k = E(X'^k|\mathcal{G})$, still belongs to $\mathcal{D}_2$ and converges to X (resp. Z) in $\mathcal{D}_2$. Furthermore, the variables X, X^k, $\left\langle D^N X\right\rangle_{leb}$, $\left\langle D^N X^k\right\rangle_{leb}$, $\left\langle D^N (X - X^k)\right\rangle_{leb}$, and $\left\langle D^N X\right\rangle_{\rho N}$, $\left\langle D^N X^k\right\rangle_{\rho N}$, $\left\langle D^N (X - X^k)\right\rangle_{\rho N}$ are $\mathcal{G}$-measurable. The same list of random variables, replacing X and X^k with Z and Z^k are $\mathcal{A}$-measurable. On the other hand, it is easy to check that for each k, $X^k Z^k \in \mathcal{D}_2$, and that $D^W X^k Z^k = X^k D^W Z^k + Z^k D^W X^k$ and $D^N X^k Z^k = X^k D^N Z^k + Z^k D^N X^k$. The convergence of $X^k Z^k$ to XZ in $\mathcal{D}_2$ is easily proved, using repeatedly the same independence argument.

We finally state the absolute continuity criterion that we will use. This is adapted from Nualart [5], p 87.

Theorem 3.11 *Assume that Z belongs to $\mathcal{D}_2$, and set $\sigma = \left\langle D^W Z\right\rangle_{leb} + \left\langle D^N Z\right\rangle_{\rho N}$. Then, if $\sigma > 0$ a.s., the law of Z admits a density with respect to the Lebesgue measure on $I\!R$.*

Proof : Suppose first that $|Z| < 1$. Let ϕ be a Lebesgue-negligible function from $]-1,1[$ to $[0,1]$. We have to show that $\phi(Z) = 0$ a.s. Let $\Psi(y) = \int_{-1}^{y} \phi(x)dx$. Following Nualart, [5] p 87, one can show that $\Psi(Z) \in \mathcal{D}_2$, that $D^W \Psi(Z) = \phi(Z) D^W Z$, and that $D^N \Psi(Z) = \phi(Z) D^N Z$.
On the other hand, $\Psi(Z) = 0$. So its derivatives vanish, and the uniqueness of the derivatives yields that a.s.,

$$\left\langle \phi(Z) D^W Z\right\rangle_{leb} + \left\langle \phi(Z) D^N Z\right\rangle_{\rho N} = 0$$

It follows that $\phi^2(Z)\sigma = 0$ a.s., and thus that $\phi(Z)$ vanishes almost surely. The first step is finished.
If Z is not bounded any more, it suffices to apply what precedes with $\Phi(Z)$, where Φ is a bijective C_b^1 function from $I\!R$ to $]-1,1[$, with a strictly positive derivative.

3.3 Derivation and stochastic integrals.

In the evolution equation (1.10), one can see three random integrals. In order to apply Theorem 3.11, we have to compute their derivatives. We begin with a remark which might avoid confusions.

Remark 3.12 *1. Let Y and Y' be two weak versions of the same process. Assume that for each y, each s, $Y(y,s) \in \mathcal{D}_2$. Then for almost all y, s,*

$$Y'(y,s) \in \mathcal{D}_2 \quad ; \quad \left\langle D^W Y(y,s) - D^W Y'(y,s)\right\rangle_{leb} + \left\langle D^N Y(y,s) - D^N Y'(y,s)\right\rangle_{\rho N} = 0 \ \text{ a.s.}$$

2. Let Y be a process such that for each y, s, $Y(y,s) \in \mathcal{D}_2$. Assume that for each α, τ, $D^W_{\alpha,\tau} Y(y,s) = S_{\alpha,\tau}(y,s)$ $dPdyds$-a.e. (i.e. $S_{\alpha,\tau}$ is a weak version of $D^W_{\alpha,\tau} Y$), and that for each n, $D^N_{X_n,T_n,Z_n} Y(y,s) = S'_{X_n,T_n,Z_n}(y,s)$ $dPdyds$-a.e. (i.e. S'_{X_n,T_n,Z_n} is a weak version of $D^N_{X_n,T_n,Z_n} Y$). Then for almost all y, s, a.s.,

$$\left\langle D^W Y(y,s) - S(y,s)\right\rangle_{leb} + \left\langle D^N Y(y,s) - S'(y,s)\right\rangle_{\rho N} = 0$$

Proof : 1. This is immediate, since for almost all y, s, a.s., $Y(y,s) = Y'(y,s)$.
2. For example, thanks to Fubini's Theorem,

$$\int_0^T \int_0^L E\left(\left\langle D^W Y(y,s) - S(y,s)\right\rangle_{leb}\right) dyds$$

$$= \int_0^T \int_0^L d\alpha d\tau E\left(\int_0^T \int_0^L \left(D^W_{\alpha,\tau} Y(y,s) - S_{\alpha,\tau}(y,s)\right)^2 dyds\right) = 0$$

Let us now define a class of processes of which the integrals will belong to $\mathcal{D}_2$.

Definition 3.13 *Let Y be a process of class $\mathcal{PV}$. We will say that Y is $\mathcal{D}_2$-$\mathcal{PV}$ if the following conditions hold :*

- *For every y, s, $Y(y,s)$ belongs to $\mathcal{D}_2$, and $\sup_{y,s} |||Y(y,s)|||_2 < \infty$.*
- *For each α, τ fixed, the process $D^W_{\alpha,\tau} Y(y,s)$ admits a predictable weak version (and vanishes when $\tau > s$). The map $\omega, \alpha, \tau, y, s \longrightarrow D^W_{\alpha,\tau} Y(y,s)(\omega)$ is globally measurable.*
- *For each $n \geq 0$, the process $D^N_{X_n,T_n,Z_n} Y(y,s)$ admits a predictable weak version (and vanishes when $T_n > s$).*

Remark 3.14 *Let $Z \in \mathcal{S}$. Consider the càdlàg martingale $Z_s = E(Z|\mathcal{F}_s)$. This process is in $\mathcal{D}_2$-$\mathcal{PV}$, and for all s, $|||Z_s|||_2 \leq |||Z|||_2$.*

This is a straightforward consequence of Proposition 3.10-1. The following Remark is a well-known fact about Hilbert spaces, and will allow to "separate" the variables y, s and ω in the stochastic integrals.

Remark 3.15 *Let $\{Z^k\}_{k\geq 0}$ be an orthonormal (for $\langle\,\rangle_{\mathcal{D}_2}$) basis of $\mathcal{S}$. Then every element Y of $\mathcal{D}_2$ can be written as*

$$Y = \sum_{k\geq 0} \lambda_k Z^k \quad \textit{with} \quad \sum_{k\geq 0} \lambda_k^2 < \infty \quad \textit{where} \quad \lambda_k = \left\langle Y, Z^k \right\rangle_{\mathcal{D}_2} \tag{3.28}$$

Let us apply this to a $\mathcal{D}_2$-$\mathcal{PV}$ process.

Lemma 3.16 *Let Y belong to $\mathcal{D}_2$-$\mathcal{PV}$.*

1. *Thanks to Remark 3.15, we can write Y as*

$$Y(y,s) = \sum_{k\geq 0} \phi_k(y,s) Z^k \quad \textit{with} \quad \sup_{y,s} \sum_{k\geq 0} \phi_k^2(y,s) < \infty \tag{3.29}$$

where $\phi_k(y,s) = \left\langle Y(y,s), Z^k \right\rangle_{\mathcal{D}_2}$ is $\mathcal{B}([0,L]\times[0,T])$-measurable.

2. *If $Z^k_s = E(Z^k|\mathcal{F}_s)$ (see Remark 3.14), then for every N,*

$$|||Y(y,s) - \sum_{k=0}^N \phi_k(y,s) Z^k_s|||_2 \leq |||Y(y,s) - \sum_{k=0}^N \phi_k(y,s) Z^k|||_2 \tag{3.30}$$

Proof : 1. The only problem is to prove that ϕ_k is measurable, which follows from the properties of a $\mathcal{D}_2$-$\mathcal{PV}$ process, since

$$\begin{aligned} \phi_k(y,s) &= E\left(Y(y,s)Z^k\right) + E\left(\int_0^T \int_0^L D^W_{\alpha,\tau} Y(y,s) D^W_{\alpha,\tau} Z^k d\alpha d\tau\right) \\ &\quad + \sum_n E\left(D^N_{X_n,T_n,Z_n} Y(y,s) D^N_{X_n,T_n,Z_n} Z^k\right) \end{aligned}$$

2. can be proved in the same way as Proposition 3.10.

We will also need the following definition.

Definition 3.17 *Let $S_{\alpha,\tau,\zeta}(\omega, y, s)$ be function on $\Omega \times ([0, L] \times [0, T] \times O) \times ([0, L] \times [0, T])$. We will say that S belongs to the class $\mathcal{DN}$ if $\sup_{y,s} E(\langle S(y, s)\rangle_{\rho N}) < \infty$ and if for each $n \geq 0$, the process $S_{X_n,T_n,Z_n}(y, s)$ admits a predictable weak version, and vanishes when $T_n > s$.*

These conditions, which are satisfied by the derivative related to N of a $\mathcal{D}_2$-$\mathcal{PV}$ process, will allow to prove the technical (but fundamental) lemma 3.18. Let us notice that if S is such a function, then $\langle S(y, s)\rangle_{\rho N}$ admits a predictable weak version (this is immediate by using the Notation 3.6). The following lemma takes the place of the L^2-isometry which is constantly used in the Gaussian case. The functions f, g, h are the parameters of (0.1), and satisfy (H').

Lemma 3.18 *Let Y be a process of class $\mathcal{PV}$, and let S be in $\mathcal{DN}$. We set, for $\tau \leq t$:*

$$\begin{aligned}
T^a_{\alpha,\tau,\zeta}(x, t) &= \int_0^t \int_0^L G_{t-s}(x, y) f'(Y(y, s)) S_{\alpha,\tau,\zeta}(y, s) W(dy, ds) \\
T^b_{\alpha,\tau,\zeta}(x, t) &= \int_0^t \int_0^L G_{t-s}(x, y) g'(Y(y, s)) S_{\alpha,\tau,\zeta}(y, s) dyds \\
T^c_{\alpha,\tau,\zeta}(x, t) &= \int_0^t \int_0^L \int_O G_{t-s}(x, y) h'_x(Y(y, s), z) S_{\alpha,\tau,\zeta}(y, s) \tilde{N}(ds, dy, dz)
\end{aligned} \tag{3.31}$$

and $T^a_{\alpha,\tau,\zeta}(x, t) = T^b_{\alpha,\tau,\zeta}(x, t) = T^c_{\alpha,\tau,\zeta}(x, t) = 0$ if $\tau > t$. These functions belong to $\mathcal{DN}$, and

$$E\left[\langle T^a(x, t)\rangle_{\rho N}\right] = \int_0^t \int_0^L G^2_{t-s}(x, y) E\left[\{f'(Y(y, s))\}^2 \langle S(y, s)\rangle_{\rho N}\right] dyds \tag{3.32}$$

$$E\left[\left\langle T^b(x, t)\right\rangle_{\rho N}\right] \leq TL \int_0^t \int_0^L G^2_{t-s}(x, y) E\left[\{g'(Y(y, s))\}^2 \langle S(y, s)\rangle_{\rho N}\right] dyds \tag{3.33}$$

$$E\left[\langle T^c(x, t)\rangle_{\rho N}\right] = \int_0^t \int_0^L \int_O G^2_{t-s}(x, y) E\left[\{h'_x(Y(y, s), z)\}^2 \langle S(y, s)\rangle_{\rho N}\right] \varphi(z) dzdyds \tag{3.34}$$

Notice here again that the integrals in (3.31) are not well-defined for each α, τ, ζ. One more time, we mean here that α, τ, ζ have to be replaced by X_n, T_n, Z_n.

<u>Proof</u> : we first notice that if (3.32), (3.33), and (3.34) hold, the lemma will follow easily, by using an easy adaptation of Lemma 2.1. Let us for example check (3.34). Using Notation 3.6, and applying Fubini's Theorem (everything is positive), we obtain

$$E\left(\langle T^c(x, t)\rangle_{\rho N}\right) = E\left(\sum_{n\geq 0} \rho(Z_n)\left[T^c_{X_n,T_n,Z_n}(x, t)\right]^2\right) = \sum_{n\geq 0} E\left(\rho(Z_n)\left[T^c_{X_n,T_n,Z_n}(x, t)\right]^2\right)$$

$$= \sum_{n\geq 0} E\left[\left(\int_0^t \int_0^L \int_O G_{t-s}(x, y) h'_x(Y(y, s), z)\sqrt{\rho(Z_n)} S_{X_n,T_n,Z_n}(y, s)\tilde{N}(ds, dy, dz)\right)^2\right]$$

But we know that $\sqrt{\rho(Z_n)} S_{X_n,T_n,Z_n}(y, s) = \sqrt{\rho(Z_n)} S_{X_n,T_n,Z_n}(y, s) 1_{\{T_n \leq s\}}$ belongs to $\mathcal{PV}$. We thus can apply (1.4) :

$$E\left(\langle T^c(x, t)\rangle_{\rho N}\right) = \sum_{n\geq 0} \int_0^t \int_0^L \int_O G^2_{t-s}(x, y) E\left(\{h'_x(Y(y, s), z)\}^2 \rho(Z_n) S^2_{X_n,T_n,Z_n}(y, s)\right) \varphi(z) dzdyds$$

We conclude by using again Fubini's Theorem.

Now we can state the main proposition of this section :

Proposition 3.19 *Let Y belong to $\mathcal{D}_2$-$\mathcal{PV}$. Let us consider the following processes*

$$\begin{aligned} U_1(x,t) &= \int_0^t \int_0^L G_{t-s}(x,y) f(Y(y,s)) W(dy,ds) \\ U_2(x,t) &= \int_0^t \int_0^L G_{t-s}(x,y) g(Y(y,s)) dyds \\ U_3(x,t) &= \int_0^t \int_0^L \int_{I\!R} G_{t-s}(x,y) h(Y(y,s),z) \tilde{N}(ds,dy,dz) \end{aligned}$$

Then U_1, U_2, U_3 are in $\mathcal{D}_2$-$\mathcal{PV}$. If Y_- is a predictable weak version of Y, then :

$$\begin{aligned} D^W_{\alpha,\tau} U_1(x,t) &= G_{t-\tau}(x,\alpha) f(Y_-(\alpha,\tau)) 1_{\{\tau\le t\}} + \int_0^t \int_0^L G_{t-s}(x,y) f'(Y(y,s)) D^W_{\alpha,\tau} Y(y,s) W(dy,ds) \\ D^N_{\alpha,\tau,\zeta} U_1(x,t) &= \int_0^t \int_0^L G_{t-s}(x,y) f'(Y(y,s)) D^N_{\alpha,\tau,\zeta} Y(y,s) W(dy,ds) \\ D^W_{\alpha,\tau} U_2(x,t) &= \int_0^t \int_0^L G_{t-s}(x,y) g'(Y(y,s)) D^W_{\alpha,\tau} Y(y,s) dyds \\ D^N_{\alpha,\tau,\zeta} U_2(x,t) &= \int_0^t \int_0^L G_{t-s}(x,y) g'(Y(y,s)) D^N_{\alpha,\tau,\zeta} Y(y,s) dyds \\ D^W_{\alpha,\tau} U_3(x,t) &= \int_0^t \int_0^L \int_{I\!R} G_{t-s}(x,y) h'_x(Y(y,s),z) D^W_{\alpha,\tau} Y(y,s) \tilde{N}(ds,dy,dz) \\ D^N_{\alpha,\tau,\zeta} U_3(x,t) &= G_{t-\tau}(x,\alpha) h'_z(Y_-(\alpha,\tau),\zeta) 1_{\{\tau\le t\}} \\ &\quad + \int_0^t \int_0^L \int_{I\!R} G_{t-s}(x,y) h'_x(Y(y,s),z) D^N_{\alpha,\tau,\zeta} Y(y,s) \tilde{N}(ds,dy,dz) \end{aligned}$$

Notice that the obtained derivatives do not depend on the choice for the predictable weak version Y_- of Y. Indeed, if Y^- is another predictable weak version of Y, then it is clear that for each x, t, a.s.,

$$\begin{aligned} &\int_0^t \int_0^L \left\{ G_{t-\tau}(x,\alpha) f(Y_-(\alpha,\tau)) - G_{t-\tau}(x,\alpha) f(Y^-(\alpha,\tau)) \right\}^2 d\alpha d\tau \\ &+ \int_0^t \int_0^L \int_{I\!R} \left\{ G_{t-\tau}(x,\alpha) h'_z(Y_-(\alpha,\tau),\zeta) - G_{t-\tau}(x,\alpha) h'_z(Y^-(\alpha,\tau),\zeta) \right\}^2 \rho(\zeta) N(d\tau,d\alpha,d\zeta) = 0 \end{aligned}$$

We will only prove the proposition for $U_3(x,t)$, because the other cases are simpler and proved similarly. We begin with a lemma :

Lemma 3.20 *Let ϕ be a measurable function on $[0,T]\times[0,L]\times O$, of class C^1 on O (in z), with ϕ'_z bounded, and such that $|\phi(s,y,z)| \le K\eta(z)$ (where $\eta \in L^2(O,\varphi(z)dz)$ is defined in (H')). Let $(t,x) \in]0,T] \times [0,L]$ be fixed, and*

$$G(s,y,z) = G_{t-s}(x,y)\phi(s,y,z) 1_{\{s\le t\}} \tag{3.35}$$

Then $\tilde{N}(G) = \int_0^T \int_0^L \int_O G(s,y,z) \tilde{N}(ds,dy,dz)$ belongs to $\mathcal{D}_2$, and its derivatives are given by :

$$D^W_{\alpha,\tau} \tilde{N}(G) = 0 \quad \textit{and} \quad D^N_{\alpha,\tau,\zeta} \tilde{N}(G) = G_{t-\tau}(x,\alpha) \phi'_z(\tau,\alpha,\zeta) 1_{\{\tau\le t\}} \tag{3.36}$$

<u>Proof</u> : this lemma is an easy extension of Proposition 3.9-3. Let $\{\mathcal{T}_\epsilon\}$ be a family of C^∞ functions on $I\!R$, such that $|\mathcal{T}'_\epsilon| \le 1$ such that

$$\mathcal{T}_\epsilon(u) = \begin{cases} u & \text{if } |u| \le 1/\epsilon \\ 1 + 1/\epsilon & \text{if } u \ge 2 + 1/\epsilon \\ -1 - 1/\epsilon & \text{if } u \le -2 - 1/\epsilon \end{cases} \qquad \text{and} \qquad |\mathcal{T}_\epsilon(u)| \le |u|$$

On the other hand, using assumption (S), we consider a family $\{K_\epsilon\}$ of C^1 positive functions on O, bounded by 1, with compact support (in O), and satisfying

$$\forall\, z \in O, K_\epsilon(z) \longrightarrow_{\epsilon\to 0} 1 \quad ; \quad \int_O \left(K'_\epsilon(z)\right)^2 \eta^2(z)\rho(z)\varphi(z)dz \longrightarrow_{\epsilon\to 0} 0 \tag{3.37}$$

We set

$$G_\epsilon(s,y,z) = \mathcal{T}_\epsilon(G_{t-s}(x,y))\mathcal{T}_\epsilon(\phi(s,y,z))K_\epsilon(z)1_{\{s\le t\}}$$

Then G_ϵ satisfies the conditions of Proposition 3.9-3 :

$$(G_\epsilon)'_z(s,y,z) = \mathcal{T}_\epsilon(G_{t-s}(x,y))1_{\{s\le t\}}\left[\mathcal{T}'_\epsilon(\phi(s,y,z))\phi'_z(s,y,z)K_\epsilon(z) + \mathcal{T}_\epsilon(\phi(s,y,z))K'_\epsilon(z)\right]$$

Thus, $\tilde{N}(G_\epsilon) \in \mathcal{D}_2$, and

$$D^W\tilde{N}(G_\epsilon) = 0 \quad ; \quad D^N\tilde{N}(G_\epsilon) = (G_\epsilon)'_z$$

One easily checks, by using the Lebesgue Theorem and (3.37), that $\tilde{N}(G_\epsilon)$ goes to $\tilde{N}(G)$ in L^2, and that

$$E\left(\langle G'_z - (G_\epsilon)'_z\rangle_{\rho N}\right)$$

tends to 0. This yields the result.

Proof of Proposition 3.19 for $U = U_3$

Step 1 : if z is fixed, $h(\ .\ ,z)$ is C^1_b on $I\!R$. Hence, using Proposition 3.9-1., for every (x,t,z) in $[0,L]\times[0,T]\times I\!R$, $h(Y(x,t),z) \in \mathcal{D}_2$, and we have

$$D^W h(Y(x,t),z) = h'_x(Y(x,t),z)D^W Y(x,t) \quad ; \quad D^N h(Y(x,t),z) = h'_x(Y(x,t),z)D^N Y(x,t)$$

Furthermore, due to (H'), we see that $\sup_{x,t} |||h(Y(x,t),z)|||_2 \le K\eta(z)$. Let us write $h(Y,z)$ in $\mathcal{D}_2$ (thanks to Lemma 3.16) as :

$$h(Y(x,t),z) = \sum_{k\ge 0}\phi_k(x,t,z)Z^k$$

We know that for all k, ϕ_k is measurable. Furthermore,

$$\sup_{x,t}\sum_k \phi_k^2(x,t,z) = \sup_{x,t}|||h(Y(x,t),z)|||_2^2 \le K\eta^2(z)$$

Using Lebesgue's Theorem and (H'), we see that

$$\begin{aligned}
\phi_k(y,s,z) &= \left\langle h(Y(y,s),z), Z^k\right\rangle_{\mathcal{D}_2} \\
&= E\left[h(Y(y,s),z)Z^k\right] + E\left[h'_x(Y(y,s),z)\left\langle D^W Y(y,s), D^W Z^k\right\rangle_{leb}\right] \\
&\quad + E\left[h'_x(Y(y,s),z)\left\langle D^N Y(y,s), D^N Z^k\right\rangle_{\rho N}\right]
\end{aligned} \tag{3.38}$$

is of class C^1 in z, and that its derivative is bounded. On the other hand, since $h'_z(\ .\ ,z)$ is of class C^1_b, Proposition 3.9-1. shows that $h'_z(Y(y,s),z)$ belongs to $\mathcal{D}_2$, allows to compute its derivatives, and to see that (due to (H')) :

$$\sup_{x,t,z}\sum_k \left((\phi_k)'_z(x,t,z)\right)^2 = \sup_{x,t,z}|||h'_z(Y(x,t),z)|||_2^2 < \infty$$

We see also that $h'_z(Y(y,s),z) = \sum_{k\ge 0}(\phi_k)'_z(y,s,z)Z^k$ in $\mathcal{D}_2$. Because of Remark 3.14, setting $Z^k_s = E(Z^k|\mathcal{F}_s)$ and

$$\Psi^N(y,s,z) = \sum_{k=0}^N \phi_k(y,s,z)Z^k_s \tag{3.39}$$

we know that :

$$\forall y,s,z \qquad |||\Psi^N(y,s,z)-h(Y(y,s),z)|||_2 \longrightarrow 0 \tag{3.40}$$

$$\sup_N \sup_{y,s} |||\Psi^N(y,s,z)-h(Y(y,s),z)|||_2^2 \leq K\eta^2(z) \tag{3.41}$$

$$\forall y,s,z \qquad |||(\Psi^N)'_z(y,s,z)-h'_z(Y(y,s),z)|||_2 \longrightarrow 0 \tag{3.42}$$

$$\sup_N \sup_{y,s,z} |||(\Psi^N)'_z(y,s,z)-h'_z(Y(y,s),z)|||_2^2 \leq \sup_{y,s,z} |||h'_z(Y(y,s),z)|||_2^2 < \infty \tag{3.43}$$

<u>Step 2</u> : let us first show that for each k,

$$\tilde{U}^k(x,t)=\int_0^t\int_0^L\int_O G_{t-s}(x,y)\phi_k(y,s,z)Z_s^k\tilde{N}(ds,dy,dz)$$

belongs to $\mathcal{D}_2$, and let us compute its derivatives. It is really useful to use the sequence Z_s^k, *because the processes $\phi_k(y,s,z)Z^k$ do not a priori admit predictable weak versions.* We use a Péano approximation for Z_s^k : if $0\leq s\leq T$, we set $s_n=\sup\left\{\frac{i}{n}T \Big/ \frac{i}{n}T<s\right\}\vee 0$. Then we consider

$$\begin{aligned}\tilde{U}_n^k(x,t) &= \int_0^t\int_0^L\int_O G_{t-s}(x,y)\phi_k(y,s,z)Z_{s_n}^k\tilde{N}(ds,dy,dz)\\ &= \sum_{i=0}^{n-1} Z^k_{\frac{i}{n}T}\times\int_{[0,t]\cap]\frac{i}{n}T,\frac{i+1}{n}T]}\int_0^L\int_O G_{t-s}(x,y)\phi_k(y,s,z)\tilde{N}(ds,dy,dz)\end{aligned} \tag{3.44}$$

Since $Z^k_{\frac{i}{n}T}$ belongs to $\mathcal{D}_2$ and is $\mathcal{F}_{\frac{i}{n}T}$-measurable, since ϕ_k satisfies the assumptions of Lemma 3.20, this Lemma and Proposition 3.10-3 allows us to say that $\tilde{U}_n^k(x,t)\in\mathcal{D}_2$, and

$$\begin{aligned}D^N_{\alpha,\tau,\zeta}\tilde{U}_n^k(x,t) &= \sum_{i=0}^{n-1}\int_{[0,t]\cap]\frac{i}{n}T,\frac{i+1}{n}T]}\int_0^L\int_O G_{t-s}(x,y)\phi_k(y,s,z)\tilde{N}(ds,dy,dz)\times D^N_{\alpha,\tau,\zeta}Z^k_{\frac{i}{n}T}\\ &\quad+\sum_{i=0}^{n-1} Z^k_{\frac{i}{n}T}\times G_{t-\tau}(x,\alpha)(\phi_k)'_z(\alpha,\tau,\zeta)1_{\{\tau\in]\frac{i}{n}T,\frac{i+1}{n}T]\cap[0,t]\}}\\ &= \int_0^t\int_0^L\int_O G_{t-s}(x,y)\phi_k(y,s,z)D^N_{\alpha,\tau,\zeta}Z^k_{s_n}\tilde{N}(ds,dy,dz)\\ &\quad+G_{t-\tau}(x,\alpha)(\phi_k)'_z(\alpha,\tau,\zeta)Z^k_{\tau_n}1_{\{\tau\leq t\}}\end{aligned} \tag{3.45}$$

and, by the same way,

$$D^W_{\alpha,\tau}\tilde{U}_n^k(x,t)=\int_0^t\int_0^L\int_O G_{t-s}(x,y)\phi_k(y,s,z)D^W_{\alpha,\tau}Z^k_{s_n}\tilde{N}(ds,dy,dz)$$

Thus we set

$$\begin{aligned}D^N_{\alpha,\tau,\zeta}\tilde{U}^k(x,t) &= \int_0^t\int_0^L\int_O G_{t-s}(x,y)\phi_k(y,s,z)D^N_{\alpha,\tau,\zeta}Z^k_s\tilde{N}(ds,dy,dz)\\ &\quad+G_{t-\tau}(x,\alpha)(\phi_k)'_z(\alpha,\tau,\zeta)Z^k_{\tau-}1_{\{\tau\leq t\}}\end{aligned} \tag{3.46}$$

and

$$D^W_{\alpha,\tau}\tilde{U}^k(x,t)=\int_0^t\int_0^L\int_O G_{t-s}(x,y)\phi_k(y,s,z)D^W_{\alpha,\tau}Z^k_s\tilde{N}(ds,dy,dz)$$

We still have to check the convergence in $\mathcal{D}_2$. First, because of (1.4), and since $|\phi_k|\leq K\eta\in L^2(O,\varphi(z)dz)$,

$$\begin{aligned}E\left[(\tilde{U}^k(x,t)-\tilde{U}_n^k(x,t))^2\right] &= \int_0^t\int_0^L\int_O G^2_{t-s}(x,y)\phi_k^2(y,s,z)E\left((Z^k_s-Z^k_{s_n})^2\right)\varphi(z)dzdyds\\ &\leq K\int_0^t\int_0^L G^2_{t-s}(x,y)E\left((Z^k_s-Z^k_{s_n})^2\right)dyds\end{aligned} \tag{3.47}$$

This goes to 0 by using (twice) the Lebesgue Theorem : for each s, $|Z^k_s - Z^k_{s_n}|$ goes to 0 a.s., and is smaller than $2\sup_\omega |Z^k(\omega)| < \infty$. Thus for each s, $E\left((Z^k_s - Z^k_{s_n})^2\right)$ goes to 0. This expectation is also bounded, and $G^2_{t-s}(x,y)$ belongs to $L^1(dyds)$: Lebesgue's Theorem (for $dyds$) yields the convergence.
On the other hand,

$$\begin{aligned}
&E\left[\left\langle D^N\tilde{U}^k(x,t) - D^N\tilde{U}^k_n(x,t)\right\rangle_{\rho N}\right]\\
&\le C\int_0^t\int_0^L\int_O G^2_{t-\tau}(x,\alpha)\left((\phi_k)'_z(\alpha,\tau,\zeta)\right)^2 E\left((Z^k_{\tau_n} - Z^k_{\tau-})^2\right)\rho(\zeta)\varphi(\zeta)d\zeta d\alpha d\tau\\
&+CE\Big[\int_0^t\int_0^L\int_O\left(\int_0^t\int_0^L\int_O G_{t-s}(x,y)\phi_k(y,s,z)\left(D^N_{\alpha,\tau,\zeta}Z^k_{s_n} - D^N_{\alpha,\tau,\zeta}Z^k_s\right)\tilde{N}(ds,dy,dz)\right)^2\\
&\qquad\qquad\rho(\zeta)N(d\tau,d\alpha,d\zeta)\Big]
\end{aligned}$$

The first part in this expression tends to 0 as above. Lemma 3.18 allows us to upperbound the second one with

$$\int_0^t\int_0^L\int_O G^2_{t-s}(x,y)\phi_k^2(y,s,z)E\left(\left\langle D^N Z^k_{s_n} - D^N Z^k_s\right\rangle_{\rho N}\right)\varphi(z)dzdyds$$

which goes to 0 by the same way (here $\left\langle D^N Z^k_{s_n} - D^N Z^k_s\right\rangle_{\rho N}$ is not upperbounded by a constant, but by the random variable

$$X^k = 4\sup_{[0,T]} E\left[\left\langle D^N Z^k\right\rangle_{\rho N}\;\middle|\;\mathcal{F}_s\right]$$

which belongs to $L^1(\Omega)$ due to Doob's inequality, since $\left\langle D^N Z^k\right\rangle_{\rho N} \in L^2(\Omega)$, because $Z^k \in \mathcal{S}$).
Finally, one can prove as well that $E\left[\left\langle D^W\tilde{U}^k(x,t) - D^W\tilde{U}^k_n(x,t)\right\rangle_{leb}\right]$ tends to 0.

Step 3 : we now approximate $U(x,t)$ with

$$U^N(x,t) = \sum_{k=0}^N \tilde{U}^k(x,t)$$

Using the first step, we know that $U^N(x,t)$ belongs to $\mathcal{D}_2$, and that ($\Psi^N(y,s,z)$ is defined by equation (3.39)) :

$$D^W_{\alpha,\tau}U^N(x,t) = \int_0^t\int_0^L\int_O G_{t-s}(x,y)D^W_{\alpha,\tau}\Psi^N(y,s,z)\tilde{N}(ds,dy,dz)$$

$$\begin{aligned}
D^N_{\alpha,\tau,\zeta}U^N(x,t) &= G_{t-\tau}(x,\alpha)(\Psi^N)'_z(\alpha,\tau,\zeta)1_{\{\tau\le t\}}\\
&+ \int_0^t\int_0^L\int_O G_{t-s}(x,y)D^N_{\alpha,\tau,\zeta}\Psi^N(y,s,z)\tilde{N}(ds,dy,dz)
\end{aligned}$$

We now denote by $D^W_{\alpha,\tau}U(x,t)$ and $D^N_{\alpha,\tau,\zeta}U(x,t)$ the expressions of the statement, even if we still do not know if these are really the derivatives of $U(x,t)$. First, using (1.4),

$$E\left[(U(x,t) - U^N(x,t))^2\right] \le \int_0^t\int_0^L\int_O G^2_{t-s}(x,y)E\left((h(Y(y,s),z) - \Psi^N(y,s))^2\right)\varphi(z)dydzds$$

This goes to 0 by the Lebesgue Theorem, and thanks to (3.40) and (3.41). Furthermore,

$$\begin{aligned}
&E\left[\left\langle D^N U(x,t) - D^N U^N(x,t)\right\rangle_{\rho N}\right]\\
&\le K\int_0^t\int_0^L\int_O G^2_{t-\tau}(x,\alpha)E\left((h'_z(Y(\alpha,\tau),\zeta) - (\Psi^N)'_z(\alpha,\tau,\zeta))^2\right)\rho(\zeta)\varphi(\zeta)d\zeta d\alpha d\tau\\
&+KE\Big[\int_0^T\int_0^L\int_O\Big(\int_0^t\int_0^L\int_O G_{t-s}(x,y)(D^N_{\alpha,\tau,\zeta}h(Y(y,s),z) - D^W_{\alpha,\tau}\Psi^N(y,s,z))\\
&\qquad\qquad\tilde{N}(ds,dy,dz)\Big)^2\rho(\zeta)N(d\tau,d\alpha,d\zeta)\Big]
\end{aligned}$$

The first term tends to 0 as above (because of (3.42) and (3.43)). We upperbound the second one, by using Lemma 3.18, with

$$\int_0^t \int_0^L \int_O G_{t-s}^2(x,y) E\left(\left\langle D^N \Psi^N(y,s,z) - D^N h(Y(y,s),z) \right\rangle_{\rho N} \right) \varphi(z) dz dy ds$$

which goes also to 0 thanks to the Lebesgue Theorem. The third convergence can be checked by the same way.

Step 4 : we still have to prove that $U(x,t)$ belongs to $\mathcal{D}_2$-$\mathcal{PV}$. First, U is predictable, as $D^W_{\alpha,\tau}U(x,t)$ if α, τ are fixed. The global measurability of $D^W_{\alpha,\tau}U(x,t)(\omega)$ is obvious, as the predictability of $D^W_{\alpha,\tau}U(x,t)$ (for α, τ fixed) and of $D^N_{X_n,T_n,Z_n}U(x,t)$ (for $n \geq 0$ fixed). U is classically bounded in L^2, and $E\left(\left\langle D^N U(x,t) \right\rangle_{\rho N} \right)$ is bounded by Lemma 3.18, (H'), and the Appendix (4.3). Furthermore, using Fubini's Theorem and (H'),

$$\begin{aligned}
E(\left\langle D^W U(x,t) \right\rangle_{leb}) &= \int_0^T \int_0^L E\left[\left(\int_0^t \int_0^L \int_O G_{t-s}(x,y) h'_x(Y(y,s),z) D^W_{\alpha,\tau} Y(y,s) \tilde{N}(ds,dy,dz) \right)^2 \right] d\alpha d\tau \\
&\leq \int_0^T \int_0^L \int_0^t \int_0^L \int_O G_{t-s}^2(x,y) \eta^2(z) E\left(\left\{ D^W_{\alpha,\tau} Y(y,s) \right\}^2 \right) \varphi(z) dz dy ds d\alpha d\tau \\
&= \int_0^t \int_0^L \int_O G_{t-s}^2(x,y) \eta^2(z) E\left(\left\langle D^W Y(y,s) \right\rangle_{leb} \right) \varphi(z) dz dy ds
\end{aligned}$$

This is clearly bounded, because Y is $\mathcal{D}_2$-$\mathcal{PV}$, since $\eta \in L^2(O, \varphi(z)dz)$, and due to the Appendix (4.3).

3.4 Derivation of the solution.

In order to apply Theorem 3.11, we have to prove that V is in $\mathcal{D}_2$, and to compute its derivatives.

Theorem 3.21 *Assume (H'), (M) and (D). Let V be the unique weak solution of (0.1). Then V belongs to $\mathcal{D}_2$-$\mathcal{PV}$, and if V_- is a predictable weak version of V,*

$$\begin{aligned}
D^W_{\alpha,\tau} V(x,t) &= G_{t-\tau}(x,\alpha) f(V_-(\alpha,\tau)) 1_{\{t \geq \tau\}} \\
&+ \int_0^t \int_0^L G_{t-s}(x,y) f'(V(y,s)) D^W_{\alpha,\tau} V(y,s) W(dy,ds) \\
&+ \int_0^t \int_0^L G_{t-s}(x,y) g'(V(y,s)) D^W_{\alpha,\tau} V(y,s) dy ds \\
&+ \int_0^t \int_0^L \int_O G_{t-s}(x,y) h'_x(V(y,s),z) D^W_{\alpha,\tau} V(y,s) \tilde{N}(ds,dy,dz)
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
D^N_{\alpha,\tau,\zeta} V(x,t) &= G_{t-\tau}(x,\alpha) h'_z(V_-(\alpha,\tau),\zeta) 1_{\{t \geq \tau\}} \\
&+ \int_0^t \int_0^L G_{t-s}(x,y) f'(V(y,s)) D^N_{\alpha,\tau,\zeta} V(y,s) W(dy,ds) \\
&+ \int_0^t \int_0^L G_{t-s}(x,y) g'(V(y,s)) D^N_{\alpha,\tau,\zeta} V(y,s) dy ds \\
&+ \int_0^t \int_0^L \int_O G_{t-s}(x,y) h'_x(V(y,s),z) D^N_{\alpha,\tau,\zeta} V(y,s) \tilde{N}(ds,dy,dz)
\end{aligned} \tag{3.49}$$

In order to prove this Theorem, we will denote by $A_{\alpha,\tau}(x,t)$ and $B_{\alpha,\tau,\zeta}(x,t)$ the solutions of (3.48) and (3.49), then we will check that they are really the derivatives of V.
Equations (3.48) and (3.49) are in fact "systems". In particular, in the case of equation (3.49), we do not want to solve the equation for each α, τ, ζ fixed, but rather for each n, replacing α, τ, ζ with X_n, T_n, Z_n. The solution $B(x,t)$ will be considered as taking its values in $L^2(P(d\omega)\rho(\zeta)N(\omega, d\tau, d\alpha, d\zeta))$, for each x, t.

Lemma 3.22 *1. Equation (3.48) admits a unique solution*

$$\begin{array}{rcl} x,t & \mapsto & A(x,t) \\ [0,L]\times[0,T] & \mapsto & L^2(P(d\omega)d\alpha d\tau) \end{array} \tag{3.50}$$

such that for each fixed α,τ, the process $A_{\alpha,\tau}(x,t)$ admits a predictable weak version and such that

$$\sup_{x,t} E\left(\langle A(x,t)\rangle_{leb}\right) < \infty$$

The uniqueness holds in the sense that, if A' is another solution, then

$$\sup_{x,t} E\left[\langle A(x,t) - A'(x,t)\rangle_{leb}\right] = 0$$

2. Equation (3.49) admits a unique solution

$$\begin{array}{rcl} x,t & \mapsto & B(x,t) \\ [0,L]\times[0,T] & \mapsto & L^2(P(d\omega)\rho(\zeta)N(\omega,d\tau,d\alpha,d\zeta)) \end{array} \tag{3.51}$$

belonging to $\mathcal{DN}$. The solution is unique in the sense that if B' is another solution, then

$$\sup_{x,t} E\left[\langle B(x,t) - B'(x,t)\rangle_{\rho N}\right] = 0$$

Proof : Let us for example prove 2. The uniqueness follows easily from Lemma 3.18 and assumption (H'). We prove the existence by using a Picard iteration : we set

$$\begin{aligned} B^0_{\alpha,\tau,\zeta}(x,t) &= G_{t-\tau}(x,\alpha)h'_z(V_-(\alpha,\tau),\zeta)1_{\{t\geq\tau\}} \\ B^{n+1}_{\alpha,\tau,\zeta}(x,t) &= B^0_{\alpha,\tau,\zeta}(x,t) + \int_0^t\int_0^L G_{t-s}(x,y)f'(V(y,s))B^n_{\alpha,\tau,\zeta}(y,s)W(dy,ds) \\ &\quad + \int_0^t\int_0^L G_{t-s}(x,y)g'(V(y,s))B^n_{\alpha,\tau,\zeta}(y,s)dyds \\ &\quad + \int_0^t\int_0^L\int_O G_{t-s}(x,y)h'_x(V(y,s),z)B^n_{\alpha,\tau,\zeta}(y,s)\tilde{N}(ds,dy,dz) \end{aligned}$$

One can check recursively, by using Lemma 3.18, that for every n, B^n belongs to $\mathcal{DN}$. Then Lemma 3.18, assumption (H'), the Appendix (4.2), and Picard's Lemma allows us to say that the series with general term

$$\left[\sup_{x,t} E\left(\left\langle B^{n+1}(x,t) - B^n(x,t)\right\rangle_{\rho N}\right)\right]^{\frac{1}{2}}$$

does converge. We conclude easily.

Proof of Theorem 3.21 : we consider the Picard approximations of V defined in Section 2 by (2.6). It is immediate, by using recursively Proposition 3.19, that for each n, V^n belongs to $\mathcal{D}_2$-$\mathcal{PV}$, and we also have an expression of its derivatives. For example, if V^n_- is a predictable weak version of V^n,

$$\begin{aligned} D^N_{\alpha,\tau,\zeta}V^{n+1}(x,t) &= G_{t-\tau}(x,\alpha)h'_z(V^n_-(\alpha,\tau),\zeta)1_{\{t\geq\tau\}} \\ &\quad + \int_0^t\int_0^L G_{t-s}(x,y)f'(V^n(y,s))D^N_{\alpha,\tau,\zeta}V^n(y,s)W(dy,ds) \\ &\quad + \int_0^t\int_0^L G_{t-s}(x,y)g'(V^n(y,s))D^N_{\alpha,\tau,\zeta}V^n(y,s)dyds \\ &\quad + \int_0^t\int_0^L\int_O G_{t-s}(x,y)h'_x(V^n(y,s),z)D^N_{\alpha,\tau,\zeta}V^n(y,s)\tilde{N}(ds,dy,dz) \end{aligned} \tag{3.52}$$

We already know (see (2.11) in the proof of Theorem 1.4), that $V^n(x,t)$ goes to $V(x,t)$ uniformly in L^2. Thus we just have to check that $E\left(\left\langle A(x,t) - D^W V^n(x,t)\right\rangle_{leb}\right)$ and $E\left(\left\langle B(x,t) - D^N V^n(x,t)\right\rangle_{\rho N}\right)$ go to 0. We set

$$G_n(x,t) = E\left[\left\langle B(x,t) - D^N V^n(x,t)\right\rangle_{\rho N}\right] \quad \text{and} \quad \phi_n(t) = \sup_x G_n(x,t)$$

One can check that $G_{n+1}(x,t) \leq K(I_1^n(x,t) + ... + I_7^n(x,t))$, where :

$$I_1^n(x,t) = \int_0^t \int_0^L \int_O G^2_{t-\tau}(x,\alpha) E\Big(\left[h'_z(V^n(\alpha,\tau),\zeta) - h'_z(V(\alpha,\tau),\zeta)\right]^2 \Big) \rho(\zeta)\varphi(\zeta) d\zeta d\alpha d\tau$$

$$I_4^n(x,t) = E\Big[\int_0^T \int_0^L \int_O \Big(\int_0^t \int_0^L \int_O G_{t-s}(x,y)[h'_x(V(y,s),z) - h'_x(V^n(y,s),z)]$$
$$B(y,s)\tilde{N}(ds,dy,dz)\Big)^2 \rho(\zeta) N(d\tau,d\alpha,d\zeta)\Big]$$

$$I_7^n(x,t) = E\Big[\int_0^T \int_0^L \int_O \Big(\int_0^t \int_0^L \int_O G_{t-s}(x,y) h'_x(V^n(y,s),z)$$
$$\times [B(y,s) - D^N_{\alpha,\tau,\zeta} V^n(y,s)] \tilde{N}(ds,dy,dz)\Big)^2 \rho(\zeta) N(d\tau,d\alpha,d\zeta)\Big]$$

and where I_2^n and I_3^n (resp. I_5^n and I_6^n) correspond to the same term as I_4^n (resp. I_7^n) but with the white noise and the Lebesgue measure.

First, h''_{zx} is bounded, hence

$$\left[h'_z(V^n(\alpha,\tau),\zeta) - h'_z(V(\alpha,\tau),\zeta)\right]^2 \leq K(V^n(\alpha,\tau) - V(\alpha,\tau))^2$$

Since $\sup_{\alpha,\tau} E\left((V^n(\alpha,\tau) - V(\alpha,\tau))^2\right)$ tends to 0 (see (2.11), and using the Appendix (4.3), we see that $I_1^n(x,t) \leq K_n^1 \longrightarrow 0$.

Lemma 3.18 shows that $I_4^n(x,t)$ equals :

$$\int_0^t \int_0^L \int_O G^2_{t-s}(x,y) E\left[(h'_x(V(y,s),z) - h'_x(V^n(y,s),z))^2 \times \langle B(y,s)\rangle_{\rho N}\right] \varphi(z) dz dy ds$$

Applying Hölder's inequality (for the measure $dyds$, with $p = 5/4$ and $q = 5$), we upperbound $I_4^n(x,t)$ with :

$$\left[\int_0^t \int_0^L (G_{t-s}(x,y))^{5/2} dy ds \right]^{4/5}$$
$$\times \left[\int_0^T \int_0^L \left[\int_O E\left((h'_x(V(y,s),z) - h'_x(V^n(y,s),z))^2 \langle B(y,s)\rangle_{\rho N}\right) \varphi(z) dz \right]^5 dy ds \right]^{1/5}$$

The first part in the product is bounded (see (4.3) in the Appendix), and the second one does not depend any more on x,t, so we denote it by K_n^2. Then one can show by using three times the Lebesgue Theorem (for the measures P, $\varphi(z)dz$, then $dyds$), by using (H'), that K_n^2 goes to 0.

After a simple computation using Lemma 3.18 and (H'), we see that

$$I_7^n(x,t) \leq K \int_0^t \int_0^L G^2_{t-s}(x,y) G_n(y,s) dy ds$$

We finally obtain

$$G_{n+1}(x,t) \leq K_n + K \int_0^t \int_0^L G_n(y,s) G^2_{t-s}(x,y) dy ds \leq K'_n + K' \int_0^t \phi_n(s) \frac{ds}{\sqrt{t-s}}$$

where $K'_n \longrightarrow 0$. Hence

$$\phi_{n+1}(x,t) \le K'_n + K' \int_0^t \phi_n(s) \frac{ds}{\sqrt{t-s}}$$

Since ϕ_0 is easily bounded, it is standard to deduce that $\sup_{[0,T]} \phi_n(t)$ goes to 0.
One can show in the same way that $\sup_{x,t} E\left[\left\langle A(x,t) - D^W V^n(x,t)\right\rangle_{leb}\right]$ tends to 0, and Theorem 3.21 is proved.

3.5 Existence of the density.

We have now enough information to prove Theorem 1.5. We consider $(x,t) \in [0,L]\times]0,T]$, and we assume that (M), (D), and (H') hold. Using Theorems 3.11 and 3.21, we just have to show that a.s.,

$$\sigma(x,t) = \left\langle D^W V(x,t)\right\rangle_{leb} + \left\langle D^N V(x,t)\right\rangle_{\rho N} = \sigma^W(x,t) + \sigma^N(x,t)$$

is strictly positive under one of the assumptions (EW), $(EP1)$, or $(EP2)$.
We did not manage to compute explicitely $\sigma(x,t)$. That is why we have to write three proofs : we will show that under (EW), $\sigma^W(x,t) > 0$ a.s., and that under $(EP1)$ or $(EP2)$, $\sigma^N(x,t) > 0$ a.s.

3.5.1 Existence of the density under $(EP1)$.

We begin with the standard remark :

Remark 3.23 *It suffices to prove the result when $g' \ge c$, for an arbitrary $c > 0$.*

Proof : let $a \in \mathbb{R}$ be fixed. Notice that the Green kernel associated with the system

$$u'_t = u''_{xx} - au\ , \quad u'_x(0,t) = u'_x(L,t) = 0$$

is given by $H_t(x,y) = e^{-at}G_t(x,y)$. Since V is a weak solution of equation (0.1), it also is weak solution of

$$\begin{aligned}\left[\frac{\partial V}{\partial t}(x,t)dxdt - \frac{\partial^2 V}{\partial x^2}(x,t) + aV(x,t)\right] dxdt &= (g(V(x,t)) + aV(x,t))\,dxdt + f(V(x,t))W(dx,dt) \\ &\quad + \int_{\mathbb{R}} h(V(x,t),z)\tilde{N}(dt,dx,dz) \qquad (3.53)\end{aligned}$$

Hence,

$$\begin{aligned} V(x,t) &= \int_0^L \mathcal{V}_0(y)H_t(x,y) + \int_0^t\int_0^L H_{t-s}(x,y)f(V(y,s))W(dy,ds) \\ &\quad + \int_0^t\int_0^L H_{t-s}(x,y)\left[g(V(y,s)) + aV(y,s)\right]W(dy,ds) \\ &\quad + \int_0^t\int_0^L\int_{\mathbb{R}} H_{t-s}(x,y)h(V(y,s),z)\tilde{N}(ds,dy,dz)\end{aligned}$$

Since g' is bounded, since a is arbitrary, and since H behaves in the same way as G, in the sense that $0 < H_t(x,y) \le G_t(x,y)$, the Remark is proved.

Then we see that since V is in $\mathcal{D}_2$-$\mathcal{PV}$, $D^N_{\alpha,\tau,\zeta}V(x,t) = 0$ as soon as $\tau > t$. Furthermore, we know (by $(EP1)$) that $f = 0$, and that $|h'_x| \le \hat{\eta} \in L^1(O, \varphi(z)dz)$. Thus setting $G'(x) = g'(x) - \int_O h'_x(x,z)\varphi(z)dz$, we obtain :

$$\begin{cases} D^N_{\alpha,\tau,\zeta}V(x,t) &= G_{t-\tau}(x,\alpha)h'_z(V_-(\alpha,\tau),\zeta) \\ &+ \int_\tau^t\int_0^L G_{t-s}(x,y)G'(V(y,s))D^N_{\alpha,\tau,\zeta}V(y,s)dyds \\ &+ \int_\tau^t\int_0^L\int_O G_{t-s}(x,y)h'_x(V(y,s),z)D^N_{\alpha,\tau,\zeta}V(y,s)N(ds,dy,dz) \quad \text{if } \tau\le t \\ D^N_{\alpha,\tau,\zeta}V(x,t) &= 0 \quad \text{if } \tau > t \end{cases}$$

Let $S_{\alpha,\tau}(x,t)$ be the unique solution (in the sense of Lemma 3.22-2.) of the following equation :

$$\begin{cases} S_{\alpha,\tau}(x,t) &= G_{t-\tau}(x,\alpha)+\int_\tau^t\int_0^L G_{t-s}(x,y)G'(V(y,s))S_{\alpha,\tau}(y,s)dyds \\ &+ \int_\tau^t\int_0^L\int_O G_{t-s}(x,y)h'_x(V(y,s),z)S_{\alpha,\tau}(y,s)N(ds,dy,dz) \quad \text{if } \tau\le t \\ S_{\alpha,\tau}(x,t) &= 0 \quad \text{if } \tau > t \end{cases} \tag{3.54}$$

A uniqueness argument yields that $D^N_{\alpha,\tau,\zeta}V(x,t)=h'_z(V_-(\alpha,\tau),\zeta))S_{\alpha,\tau}(x,t)$ in the sense where

$$\sup_{x,t} E\left(\left\langle D^NV(x,t)-h'_z(V_-(\,.\,,\,.\,),\,.\,)S(x,t)\right\rangle_{\rho N}\right)=0$$

which of course implies that for each x,t, a.s., $\sigma^N(x,t)=\langle h'_z(V_-(\,.\,,\,.\,),\,.\,)S(x,t)\rangle_{\rho N}$. Using Remark 3.23, we can assume that $G'\ge 0$. Since $h'_x\ge 0$, it is obvious that for each x,t, $P(d\omega)N(\omega,d\tau,d\alpha,d\zeta)$-a.e.,

$$S_{\alpha,\tau}(x,t)\ge G_{t-\tau}(x,\alpha)1_{\{\tau\le t\}}$$

and we just have to check that for any $t>0$, any $x\in[0,L]$, a.s.,

$$\int_0^t\int_0^L\int_O G^2_{t-\tau}(x,\alpha)(h'_z(V_-(\alpha,\tau),\zeta))^2\rho(\zeta)N(d\tau,d\alpha,d\zeta)>0$$

Since $\rho>0$, it suffices to show that for every $t>0$, a.s.,

$$\int_0^t\int_0^L\int_O 1_{\{h'_z(V_-(\alpha,\tau),\zeta)\ne 0\}}N(d\tau,d\alpha,d\zeta)>0$$

To this aim, we consider the stopping time

$$R=\inf\left\{s>0\ \Big/\ \int_0^s\int_0^L\int_O 1_{\{h'_z(V_-(\alpha,\tau),\zeta)\ne 0\}}N(d\tau,d\alpha,d\zeta)>0\right\}$$

and we prove that $R=0$ a.s. : since V_- is predictable, and since N is a counting measure,

$$E\left(\int_0^R\int_0^L\int_O 1_{\{h'_z(V_-(\alpha,\tau),\zeta)\ne 0\}}\varphi(\zeta)d\zeta d\alpha d\tau\right)=E\left(\int_0^R\int_0^L\int_O 1_{\{h'_z(V_-(\alpha,\tau),\zeta)\ne 0\}}N(d\tau,d\alpha,d\zeta)\right)\le 1$$

which implies that a.s.,

$$\int_0^R\int_0^L\int_O 1_{\{h'_z(V_-(\alpha,\tau),\zeta)\ne 0\}}\varphi(\zeta)d\zeta d\alpha d\tau<\infty$$

This contradicts $(EP1)$, except if $R=0$ a.s., and Theorem 1.5 is proved under $(EP1)$.

3.5.2 Existence of the density under $(EP2)$.

As under $(EP1)$, we write $D^N_{\alpha,\tau,\zeta}V(x,t) = h'_z(V_-(\alpha,\tau),\zeta)S_{\alpha,\tau}(x,t)$, where $S_{\alpha,\tau}(x,t)$ is the unique solution, in the sense of Lemma 3.22-2, of

$$\begin{cases} S_{\alpha,\tau}(x,t) = G_{t-\tau}(x,\alpha) + \int_\tau^t \int_0^L \int_O G_{t-s}(x,y)f'(V(y,s))S_{\alpha,\tau}(y,s)W(dy,ds) \\ \quad + \int_\tau^t \int_0^L G_{t-s}(x,y)g'(V(y,s))S_{\alpha,\tau}(y,s)dyds \\ \quad + \int_\tau^t \int_0^L \int_O G_{t-s}(x,y)h'_x(V(y,s),z)S_{\alpha,\tau}(y,s)\tilde{N}(ds,dy,dz) & \text{if } \tau \le t \\ S_{\alpha,\tau}(x,t) = 0 & \text{if } \tau > t \end{cases} \tag{3.55}$$

A uniqueness argument shows that

$$\sigma^N(x,t) = \int_0^T \int_0^L \int_O S^2_{\alpha,\tau}(x,t)\left(h'_z(V_-(\alpha,\tau),\zeta)\right)^2 \rho(\zeta)N(d\tau,d\alpha,d\zeta) \quad \text{a.s.}$$

Using $(EP2)$, we see that $\sigma^N(x,t) > 0$ as soon as

$$\Sigma(x,t) = \int_0^T \int_0^L \int_O S^2_{\alpha,\tau}(x,t)1_{\mathcal{H}}(\zeta)\rho(\zeta)N(d\tau,d\alpha,d\zeta) > 0$$

We thus split $S_{\alpha,\tau}(x,t) = G_{t-\tau}(x,\alpha)1_{\{\tau\le t\}} + Q_{\alpha,\tau}(x,t)$, where

$$\begin{cases} Q_{\alpha,\tau}(x,t) = \int_\tau^t \int_0^L G_{t-s}(x,y)f'(V(y,s))S_{\alpha,\tau}(y,s)W(dy,ds) \\ \quad + \int_\tau^t \int_0^L G_{t-s}(x,y)g'(V(y,s))S_{\alpha,\tau}(y,s)dyds \\ \quad + \int_\tau^t \int_0^L \int_O G_{t-s}(x,y)h'_x(V(y,s),z)S_{\alpha,\tau}(y,s)\tilde{N}(ds,dy,dz) & \text{if } \tau \le t \\ Q_{\alpha,\tau}(x,t) = 0 & \text{if } \tau > t \end{cases}$$

Hence, for every $\epsilon > 0$ small enough,

$$\begin{aligned} \Sigma(x,t) &\ge \tfrac{2}{3}\int_{t-\epsilon}^t \int_0^L \int_{\mathcal{H}} G^2_{t-\tau}(x,\alpha)\rho(\zeta)N(d\tau,d\alpha,d\zeta) - 2\int_{t-\epsilon}^t \int_0^L \int_{\mathcal{H}} Q^2_{\alpha,\tau}(x,t)\rho(\zeta)N(d\tau,d\alpha,d\zeta) \\ &= \tfrac{2}{3}A_\epsilon(x,t) - 2B_\epsilon(x,t) \end{aligned}$$

The following lemma shows that $B_\epsilon(x,t)$ is small.

Lemma 3.24 *There exists $C_1 > 0$ such that for any $\epsilon > 0$,*

$$E\left(B_\epsilon(x,t)\right) \le C_1\epsilon$$

<u>Proof</u> : Using Lemma 3.18 then (H'), we easily obtain

$$E\left(B_\epsilon(x,t)\right) \le K\int_{t-\epsilon}^t \int_0^L G^2_{t-s}(x,y)E\left(\left\langle S_{\alpha,\tau}(y,s)1_{\mathcal{H}}(\zeta)1_{[t-\epsilon,s]}(\tau)\right\rangle_{\rho N}\right)dyds$$

But, for $s \in [t-\epsilon,t]$,

$$\begin{aligned} E\left(\left\langle S_{\alpha,\tau}(y,s)1_{\mathcal{H}}(\zeta)1_{[t-\epsilon,s]}(\tau)\right\rangle_{\rho N}\right) &\le E\left(\left\langle S_{\alpha,\tau}(y,s)1_{\mathcal{H}}(\zeta)1_{[s-\epsilon,s]}(\tau)\right\rangle_{\rho N}\right) \\ &\le KE\left(\int_{s-\epsilon}^s \int_0^L \int_{\mathcal{H}} G^2_{s-\tau}(y,\alpha)\rho(\zeta)N(d\tau,d\alpha,d\zeta)\right) \\ &\quad +KE\left(\int_{s-\epsilon}^s \int_0^L \int_{\mathcal{H}} Q^2_{\alpha,\tau}(y,s)\rho(\zeta)N(d\tau,d\alpha,d\zeta)\right) \\ &= K\left[I_1^\epsilon(y,s) + I_2^\epsilon(y,s)\right] \end{aligned}$$

Lemma 3.18 and (H') yield

$$I_2^\epsilon(y,s) \le K \int_{s-\epsilon}^{s} \int_0^L G_{s-s'}^2(y,y') E\left(\left\langle S_{\alpha,\tau}(y',s') 1_{[s-\epsilon,s']}(\tau) 1_{\mathcal{H}}(\zeta)\right\rangle_{\rho N}\right) dy'ds' \le K\sqrt{\epsilon}$$

by using the Appendix (4.4), since S is defined as satisfying $\sup_{y,s} E\left(\langle S(y,s)\rangle_{\rho N}\right) < \infty$. Furthermore,

$$I_1^\epsilon(y,s) = \int_{s-\epsilon}^{s} \int_0^L \int_{\mathcal{H}} G_{s-\tau}^2(y,\alpha)\rho(\zeta)\varphi(\zeta) d\zeta d\alpha d\tau \le K\sqrt{\epsilon}$$

since $\rho \in L^1(O, \varphi(\zeta)d\zeta)$, and thanks to the Appendix (4.4). We thus get

$$E\left(B_\epsilon(x,t)\right) \le K\sqrt{\epsilon} \int_{t-\epsilon}^{t} \int_0^L G_{t-s}^2(x,y) dy ds$$

and it suffices to use one more time (4.4) to conclude.

The next lemma will allow to prove that $E\left(e^{-\lambda A_\epsilon(x,t)}\right)$ is small (when λ is large).

Lemma 3.25 *There exist $\lambda_0 \ge 0$, $\epsilon_0 > 0$, and $K_0 > 0$, such that for all $\lambda \ge \lambda_0$, for all $\epsilon \le \epsilon_0$,*

$$\int_0^\epsilon \int_0^L \int_{\mathcal{H}} \left(1 - e^{-\lambda G_s^2(x,y)\rho(z)}\right) \varphi(z) dz dy ds \ge K_0 \lambda^{r_0} \epsilon^{\frac{3-2r_0}{2}} \tag{3.56}$$

Proof : let us first notice, using the Appendix (4.1), that for every $s \in [\epsilon/2, \epsilon]$, $y \in [x - \sqrt{\epsilon}, x + \sqrt{\epsilon}]$, $G_s^2(x,y) \ge C/\epsilon$, where $C > 0$ is a constant. The left member of (3.56) is thus greater than

$$\int_{\mathcal{H}} \int_{\epsilon/2}^{\epsilon} \int_{x-\sqrt{\epsilon}}^{x+\sqrt{\epsilon}} \left(1 - e^{-\lambda G_s^2(x,y)\rho(z)}\right) dy ds\, \varphi(z) dz \ge K\epsilon\sqrt{\epsilon} \int_{\mathcal{H}} \left(1 - e^{-C\frac{\lambda}{\epsilon}\rho(z)}\right) \varphi(z) dz \ge K_0 \epsilon^{\frac{3}{2}} \left(\frac{\lambda}{\epsilon}\right)^{r_0}$$

where the last inequality, which holds as soon as $C\lambda/\epsilon \ge \gamma_0$, comes from assumption $(EP2)$.

Now we can check that $\Sigma(x,t) > 0$ a.s. We notice that for all $\eta > 0$, $\epsilon > 0$, and $\lambda > 0$,

$$\begin{aligned} P\left(\Sigma(x,t) > 0\right) &\ge P\left(\tfrac{2}{3} A_\epsilon(x,t) > \eta\right) + P\left(2B_\epsilon(x,t) < \eta\right) - 1 \\ &\ge 1 - e^{\lambda\eta} E\left(e^{-\frac{2}{3}\lambda A_\epsilon(x,t)}\right) - \tfrac{2}{\eta} E\left(B_\epsilon(x,t)\right) \end{aligned}$$

But $E\left(B_\epsilon(x,t)\right) \le C_1\epsilon$, and if $\epsilon < \epsilon_0$, if $\lambda \ge \frac{3}{2}\lambda_0$, Lemma 3.25 yields

$$E\left(e^{-\frac{2}{3}\lambda A_\epsilon(x,t)}\right) = \exp\left(-\int_{t-\epsilon}^{t} \int_0^L \int_{\mathcal{H}} \left(1 - e^{-\frac{2}{3}\lambda G_{t-\tau}^2(x,\alpha)\rho(\zeta)}\right) \varphi(\zeta) d\zeta d\alpha d\tau\right) \le \exp\left(-C_2 \lambda^{r_0} \epsilon^{\frac{3}{2} - r_0}\right)$$

Hence for all $\eta > 0$, $\epsilon < \epsilon_0$, $\lambda \ge \frac{3}{2}\lambda_0$,

$$P\left(\Sigma(x,t) > 0\right) \ge 1 - \exp\left(-C_2 \lambda^{r_0} \epsilon^{\frac{3}{2} - r_0} + \lambda\eta\right) - 2C_1 \frac{\epsilon}{\eta}$$

We choose $\lambda = \eta^{-1} = \epsilon^{-\alpha}$ where $\alpha > 0$. We obtain, for all $\epsilon > 0$ small enough :

$$P\left(\Sigma(x,t) > 0\right) \ge 1 - \exp\left(1 - C_2 \left(\frac{1}{\epsilon}\right)^{\alpha r_0 - \frac{3}{2} + r_0}\right) - 2C_1 \epsilon^{1-\alpha}$$

Since $r_0 > \frac{3}{4}$, we can choose $\alpha > 0$ such that $\alpha r_0 - \frac{3}{2} + r_0 > 0$ and $1 - \alpha > 0$. Letting ϵ go to 0, we deduce that $\Sigma(x,t) > 0$ a.s., and Theorem 1.5 is proved under $(EP2)$.

Comparing the proofs of Theorem 1.5 under $(EP1)$ and under $(EP2)$, we see how useful are the *local* derivatives. Under $(EP1)$, we only need to consider $\left\langle D^N V(x,t)\right\rangle_{\rho N}$, and we do not really use the expression of $D^N_{\alpha,\tau,\zeta}V(x,t)$ for each α,τ,ζ. Saint Loubert Bié works in a quite similar way. But under $(EP2)$, we need the local expressions of the derivatives, which allow us to take into account the "explosion" of the Green kernel.

In [2], Bichteler et al. do not define the local derivatives, they work directly with $\left\langle D^N X_t\right\rangle_{\rho N}$ (where X_t is a diffusion process) : since this scalar product satisfies a linear S.D.E., they can use the Doléans-Dade formula in order to study its positivity. Here, we can not use such a method, because of the Green kernel $G_t(x,y)$.

3.5.3 Existence of the density under (EW).

We show here that $\sigma^W(x,t) > 0$ a.s. The next proof is inspired by Bally and Pardoux in [1], although they use the Hölder regularity of their solution.
The proof is quite similar (but easier) to that of Subsection 3.5.2. We first use a uniqueness argument, in order to write

$$\sigma^W(x,t) = \int_0^t \int_0^L \left(S_{\alpha,\tau}(x,t)\right)^2 f^2(V_-(\alpha,\tau))d\alpha d\tau$$

where $S_{\alpha,\tau}$ satisfies equation (3.54) in the sense of Lemma 3.22-1 (this in not the same object as in the previous paragraphs). Using (EW), we just have to prove that

$$\Sigma(x,t) = \int_0^t \int_0^L \left(S_{\alpha,\tau}(x,t)\right)^2 d\alpha d\tau > 0 \quad a.s.$$

We split $S_{\alpha,\tau}(x,t) = G_{t-\tau}(x,\alpha)1_{\{\tau\le t\}} + Q_{\alpha,\tau}(x,t)$, and we obtain, for all $\epsilon > 0$:

$$\Sigma(x,t) \ge \frac{2}{3}\int_{t-\epsilon}^t \int_0^L G^2_{t-\tau}(x,\alpha)d\alpha d\tau - 2\int_{t-\epsilon}^t \int_0^L (Q_{\alpha,\tau}(x,t))^2 d\alpha d\tau \ge \frac{2}{3}J^\epsilon(x,t) - 2I^\epsilon(x,t)$$

In the Appendix (4.5), one can see that $J^\epsilon(x,t) \ge C\sqrt{\epsilon}$. Furthermore, an easy computation (as in Lemma 3.24) shows that $E(I^\epsilon(x,t)) \le K\epsilon$. The conclusion follows.

4 Appendix.

The results below are elementary properties of the Green kernel $G_t(x,y)$ defined in Section 1. In all the equations below, the inequalities remain true for $(x,t) \in [0,L]\times]0,T]$, and C_T is a constant depending only on T. The three first equations are proved in Walsh, [9], p 311-323.

$$\frac{1}{\sqrt{4\pi t}}\exp\left\{\frac{-(y-x)^2}{4t}\right\} \le G_t(x,y) \le \frac{C_T}{\sqrt{t}}\exp\left\{\frac{-(y-x)^2}{4t}\right\} \tag{4.1}$$

$$\text{If } r > 0, \quad \int_0^L G_t^r(x,y)dy \le C_T t^{\frac{1-r}{2}} \tag{4.2}$$

$$\text{If } r \in]0,3[, \quad \int_0^t \int_0^L G_s^r(x,y)dyds \le C_T \tag{4.3}$$

The two following inequalities are proved by Bally and Pardoux in the Appendix of [1].

$$\text{If } r \in]0,3[, \text{ then } \forall\, \epsilon > 0, \quad \int_{t-\epsilon}^{t} \int_0^L G_{t-s}^r(x,y)dyds \leq C_T \times \epsilon^{\frac{3-r}{2}} \tag{4.4}$$

$$\text{For all } \epsilon > 0, \quad \int_{t-\epsilon}^{t} \int_{x-\sqrt{\epsilon}}^{x+\sqrt{\epsilon}} G_{t-s}^2(x,y)dyds \geq K \times \sqrt{\epsilon} \tag{4.5}$$

Acknowledgements : I wish to thank Sylvie Méléard for her help and support during the preparation of this paper. I am very grateful to the anonymous referee for his careful reading and fruitful remarks.

References

[1] V. Bally, E. Pardoux, *Malliavin Calculus for white noise driven SPDEs*, Potential Analysis, vol. 9, no 1, p 27-64, 1998.

[2] K. Bichteler, J.B. Gravereaux, J. Jacod, *Malliavin calculus for processes with jumps*, Number 2 in Stochastic monographs, Gordon and Breach, 1987.

[3] K. Bichteler, J. Jacod, *Calcul de Malliavin pour les diffusions avec sauts, existence d'une densité dans le cas unidimensionel*, Séminaire de Probabilités XVII, L.N.M. 986, p 132-157, Springer, 1983.

[4] J. Jacod, A.N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, 1987.

[5] D. Nualart, *Malliavin Calculus and related topics*, Springer, 1995.

[6] E. Pardoux, T. Zhang, *Absolute continuity for the law of the solution of a parabolic S.P.D.E.*, J. of Funct. Anal. 112, 447-458, 1993.

[7] E. Saint Loubert Bié, *Etude d'une EDPS conduite par un bruit Poissonnien*, Manuscrit de thèse, 1998.

[8] J.B. Walsh, *A stochastic model for neural response*, Advances in applied probability, vol. 13, 231-281, 1981.

[9] J.B. Walsh, *An introduction to stochastic partial differential equations*, Ecole d'été de Probabilité de Saint Flour 14, L.N.M 1180 p 265-439, Springer, 1986.