Rough paths, regularity structures and renormalisation

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Integration and Multiplication

Let $f, g : [0, T] \to \mathbb{R}$ two continuous functions.

What does it mean to define the integral

$$\int_0^T f_r \, \dot{g}_r \, \mathrm{d}r$$

when f, g are not differentiable ?

Important example: $g = B$ with $(B_t)_{t \geq 0}$ a Brownian motion.

Starting point of the Rough Paths theory (Terry Lyons, Massimiliano Gubinelli).

Example of a more general problem: given a distribution ($\dot{g}$) and a non-smooth function (f), how can we define their product? Namely a distribution $f \dot{g}$.

Local approximation

If g is of class C^1, then we define

$$I_t := \int_0^t f_r \, \dot{g}_r \, \mathrm{d}r, \qquad t \in [0,T].$$

Then we have $I_0 = 0$ and for $0 \leq s \leq t \leq T$

$$I_t - I_s - f_s(g_t - g_s) = \int_s^t (f_r - f_s) \, \dot{g}_r \, \mathrm{d}r = o(|t-s|).$$

We write

$$I_0 = 0, \qquad I_t - I_s = f_s(g_t - g_s) + R_{st}, \qquad R_{st} = o(|t-s|).$$

These properties characterise $(I_t)_{t \in [0,T]}$, since if we have I^1 and I^2 then setting $I^{12} := I^1 - I^2$

$$|I^{12}_t - I^{12}_s| = o(|t-s|)$$

which implies I^{12} constant.

Local approximation

Let us still study the formula

$$I_0 = 0, \qquad I_t - I_s = f_s(g_t - g_s) + R_{st}, \qquad R_{st} = o(|t-s|).$$

If we compute for $0 \le s \le u \le t \le T$

$$R_{st} - R_{su} - R_{ut} = (f_u - f_s)(g_t - g_u)$$

which does not depend on I.

Therefore the existence of I is equivalent to the existence of R such that the above formula holds.

A cochain complex

Les us define for $n \geq 1$

$$\Delta_n := \{(t_1, \dots, t_n) \in [0,T]^n : t_1 \leq \cdots \leq t_n\},$$

$$C_n := \{f : \Delta_n \to \mathbb{R} \quad \text{continuous}\},$$

$$\delta_n : C_n \to C_{n+1}, \qquad (\delta_n f)_{t_1 \dots t_{n+1}} = \sum_{k=1}^{n+1} (-1)^{n+2-k} f_{t_1 \dots \hat{t}_k \dots t_{n+1}}.$$

Then we have

- $\delta_{n+1} \circ \delta_n \equiv 0$ (exercise!)
- if $g \in C_{n+1}$ and $\delta_{n+1} g = 0$, then $g = \delta_n f$ with $f \in C_n$ (exercise!).

In particular we have an exact cochain complex

$$\mathbb{R} \to C_1 \xrightarrow{\delta_1} C_2 \xrightarrow{\delta_2} C_3 \xrightarrow{\delta_3} \cdots$$

Local approximation

Therefore, existence of $I \in C_1$ such that

- $I_0 = 0$,
- $(\delta_1 I)_{st} = f_s(g_t - g_s) + o(|t-s|)$, where $(\delta_1 I)_{st} = I_t - I_s$,

is equivalent to the existence of $R \in C_2$ such that

- $(\delta_2 R)_{sut} = (f_u - f_s)(g_t - g_u)$, where $(\delta_2 R)_{sut} = R_{st} - R_{su} - R_{ut}$,
- $R_{st} = o(|t-s|)$.

Gubinelli calls I the integral, $A_{st} := f_s(g_t - g_s)$ the germ, and R_{st} the remainder.

The sewing lemma

For $\alpha > 0$ and $h \in C_n$ we set

$$\|h\|_\alpha := \sup_{(t_1,\dots,t_n)\in\Delta_n} \frac{|h(t_1,\dots,t_n)|}{|t_n - t_1|^\alpha}$$

and we say that $h \in C_n^\alpha$ if $\|h\|_\alpha < +\infty$. We also set $C_n^{\alpha+} := \cup_{\beta>\alpha} C_n^\beta$.

Theorem (Gubinelli)

There exists a unique map $\Lambda : C_3^{1+} \cap \delta_2 C_2 \to C_2^{1+}$ *such that* $\delta_2 \Lambda = \mathrm{id}_{C_3^{1+}\cap\delta_2 C_2}$. *Moreover* Λ *satifies for all* $\alpha > 1$

$$\|\Lambda B\|_\alpha \le K_\alpha \|B\|_\alpha, \qquad B \in C_3^{1+} \cap \delta_2 C_2.$$

Proof.

See the first lecture sheet of ▸ MG □

A first application: Young integration

Theorem

If $f \in C^\alpha$, $g \in C^\beta$ *(standard Hölder spaces) with* $\alpha + \beta > 1$ *then there exists a unique pair* $(I, R) \in C^\beta \times C_2^{\alpha+\beta}$ *such that*

$$I_0 = 0, \qquad I_t - I_s = f_s(g_t - g_s) + R_{st}.$$

The map

$$C^\alpha \times C^\beta \ni (f, g) \to I \in C^\beta$$

is the unique continuous extension of

$$C^1 \times C^1 \ni (f, g) \to \int_0^\bullet f\,\dot{g}\,\mathrm{d}u \in C^1.$$

Proof

- Existence. Setting $A_{st} := f_s(g_t - g_s) \in C_2^\beta$, we already know that $(\delta_2 A)_{sut} = -(f_u - f_s)(g_t - g_u)$, $0 \le s \le t \le T$, so that

$$|(\delta_2 A)_{sut}| \le C\,|u-s|^\alpha |t-u|^\beta \le C\,|t-s|^{\alpha+\beta}.$$

  Setting $R := -\Lambda\delta_2 A \in C_2^{\alpha+\beta}$ then $A + R \in C_2^\beta$ and $\delta_2(A+R) = \delta_2 A - \delta_2\Lambda\delta_2 A = 0$, so that $A + R = \delta_1 I$ with $I \in C^\beta$.
- Uniqueness. If I^1, I^2 then $|I_t^{12} - I_s^{12}| = o(|t-s|)$.
- Continuity. The estimate

$$\|I\|_{C^\beta} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta}$$

  follows from

$$\|\Lambda\delta_2 A\|_{\alpha+\beta} \le K_{\alpha+\beta} \|\delta_2 A\|_{\alpha+\beta}, \qquad \delta_2 A \in C_3^{\alpha+\beta} \cap \delta_2 C_2.$$

  in the Sewing Lemma.

Dyadic approximation

Let us consider for $t_i^n := i2^{-n}T$ and $n \geq 0$

$$I_t^n = \sum_{i=1}^{2^n} \mathbb{1}_{(t_i^n \leq t)} A_{t_{i-1}^n t_i^n}.$$

Then, since $t_{2i}^{n+1} = t_i^n$,

$$\begin{aligned} |I_t^n - I_t^{n+1}| &= \left| \sum_{i=1}^{2^n} \mathbb{1}_{(t_i^n \leq t)} \left(A_{t_{i-1}^n t_i^n} - A_{t_{2i-2}^{n+1} t_{2i-1}^{n+1}} - A_{t_{2i-1}^{n+1} t_{2i}^{n+1}} \right) \right| \\ &\leq \sum_{i=1}^{2^n} \left| (\delta_2 A)_{t_{2i-2}^{n+1} t_{2i-1}^{n+1} t_{2i}^{n+1}} \right| \lesssim 2^{-n(\alpha+\beta-1)} \end{aligned}$$

which is summable. Then we obtain that $I_t^n \to I_t$ as $n \to +\infty$ (see again MG)

If $\alpha = \beta > 1/2$

Theorem

If $f, g \in C^\alpha$, with $\alpha > 1/2$ then there exists a unique pair $(I, R) \in C^\alpha \times C_2^{2\alpha}$ such that

$$I_0 = 0, \qquad I_t - I_s = f_s(g_t - g_s) + R_{st}.$$

In the above situation, we write

$$I_t =: I_{[0,t]}(f,g) =: \int_0^t f \, \mathrm{d}g.$$

Then uniqueness yields the Integration by parts formula

$$I_{[0,t]}(f,g) + I_{[0,t]}(g,f) = f_t g_t - f_0 g_0,$$

since

$$\underbrace{f_t g_t - f_s g_s}_{I_t - I_s} = \underbrace{f_s(g_t - g_s) + g_s(f_t - f_s)}_{A_{st}} + \underbrace{(f_t - f_s)(g_t - g_s)}_{R_{st}}.$$

If $\alpha = \beta \leq 1/2$

However, if $\alpha = \beta \leq 1/2$ then neither existence nor uniqueness.

This problem is revelant for stochastic integration and SDEs:

$$X_t = X_0 + \int_0^t \sigma(X_s)\,\mathrm{d}B_s$$

with $(B_t)_{t\geq 0}$ a standard Brownian motion.

In particular, we can not apply the Sewing Lemma to the germ $A_{st} := f_s(g_t - g_s)$ since $2\alpha \leq 1$ and therefore in general $\delta_2 A \notin C_3^{1+}$.

We need to change the germ A in such a way that $\delta_2 A \in C_3^{1+}$.

Modifying the germ

Note that the result of the integration map is supposed to satisfy

$$I_t - I_s = f_s(g_t - g_s) + R_{st}, \qquad R \in C_2^{2\alpha}.$$

Then we could assume that also f satisfies

$$f_t - f_s = f'_s(g_t - g_s) + R'_{st}, \qquad R' \in C_2^{2\alpha}.$$

If $Y \in C_2$ is such that $(\delta_2 Y)_{sut} = (g_u - g_s)(g_t - g_u)$, setting

$$A_{st} := f_s(g_t - g_s) + f'_s Y_{st},$$

then

$$(\delta_2 A)_{sut} = -\underbrace{\left(f_u - f_s - f'_s(g_u - g_s)\right)}_{R'_{su}}(g_t - g_u) \in C_3^{3\alpha}.$$

If $1/3 < \alpha \leq 1/2$ we are in the setting of the Sewing Lemma.

Rough paths

For $g \in C^\alpha$, we want $Y \in C_2$ such that $(\delta_2 Y)_{sut} = (g_u - g_s)(g_t - g_u)$.

In fact, for $g : [0,T] \to \mathbb{R}$ it is enough to set $Y_{st} := \frac{1}{2}(g_t - g_s)^2$, since $(a+b)^2 - a^2 - b^2 = 2ab$.

This is a natural choice, which moreover shows how much all this is related to generalised Taylor expansions.

However it is not the only possible choice, nor necessarily the most desirable. As we'll see below, Itô integration is not covered by this setting.

In fact, for any such Y we can set $Y' := Y + \delta_1 h$ and Y' still has the desired property.

Note that $Y_{st} = \frac{1}{2}(g_t - g_s)^2$ belongs to $C_2^{2\alpha}$. For reasons which will be clear later, we require this property for all Y.

Rough and controlled paths

Let us summarise: given $\alpha \in]1/3, 1/2]$ and $g \in C^\alpha$, we call a pair $(g, Y) \in C^\alpha \times C_2^{2\alpha}$ a Rough Path if

$$(\delta_2 Y)_{sut} = (g_u - g_s)(g_t - g_u), \qquad 0 \le s \le u \le t \le T.$$

A pair $(f, f') \in C^\alpha \times C^\alpha$ is controlled by g if

$$|f_t - f_s - f'_s(g_t - g_s)| \lesssim |t - s|^{2\alpha}.$$

We denote by $\mathscr{D}_g^{2\alpha}$ the space of paths controlled by g.

Integration of controlled paths

In this setting, we can apply the Sewing Lemma to the germ $A_{st} := f_s(g_t - g_s) + f'_s Y_{st}$ and define the integral $I \in C^\alpha$ such that

$$\delta_1 I = A - \Lambda\delta_2 A, \qquad I_0 = 0.$$

Then the integration map acts (continuously) on controlled paths

$$\mathscr{D}_g^{2\alpha} \ni (f, f') \mapsto (I, f) \in \mathscr{D}_g^{2\alpha}.$$

Brownian motion in $\mathbb{R}$

Let us suppose that $g \equiv B$, a standard Brownian motion in $\mathbb{R}$. Then for all $\alpha < 1/2$, a.s. $B \in C^\alpha$. We fix $\alpha \in]1/3, 1/2]$.

We set $Y_{st} = \frac{1}{2}(B_t - B_s)^2$. For all $\alpha < 1/2$, a.s. $Y \in C_2^\alpha$.

A path controlled by B is $(f, f') \in C^\alpha \times C^\alpha$ such that

$$|f_t - f_s - f'_s(B_t - B_s)| \lesssim |t-s|^{2\alpha}, \qquad 0 \leq s \leq t \leq T.$$

For all such (f, f') there exists a unique $I \in C^\alpha$ such that $I_0 = 0$ and

$$|I_t - I_s - f_s(B_t - B_s) - f'_s Y_{st}| \lesssim |t-s|^{3\alpha}, \qquad 0 \leq s \leq t \leq T.$$

Moreover

$$|I_t - I_s - f_s(B_t - B_s)| \lesssim |t-s|^{2\alpha}, \qquad 0 \leq s \leq t \leq T.$$

If the Stratonovich integral $\int_0^\bullet f_s \circ \mathrm{d}B_s$ is well defined, it is equal to I.

Brownian motion in $\mathbb{R}$

Let us suppose that $g \equiv B$, a standard Brownian motion in $\mathbb{R}$. Then for all $\alpha < 1/2$, a.s. $B \in C^\alpha$. We fix $\alpha \in]1/3, 1/2]$.

We set $Y_{st} = \frac{1}{2}[(B_t - B_s)^2 - (t - s)]$. For all $\alpha < 1/2$, a.s. $Y \in C_2^\alpha$.

A path controlled by B is $(f, f') \in C^\alpha \times C^\alpha$ such that

$$|f_t - f_s - f'_s(B_t - B_s)| \lesssim |t - s|^{2\alpha}, \qquad 0 \le s \le t \le T.$$

For all such (f, f') there exists a unique $I \in C^\alpha$ such that $I_0 = 0$ and

$$|I_t - I_s - f_s(B_t - B_s) - f'_s Y_{st}| \lesssim |t - s|^{3\alpha}, \qquad 0 \le s \le t \le T.$$

Moreover

$$|I_t - I_s - f_s(B_t - B_s)| \lesssim |t - s|^{2\alpha}, \qquad 0 \le s \le t \le T.$$

If the Itô integral $\int_0^\bullet f_s \, \mathrm{d}B_s$ is well defined, it is equal to I.

Multi-dimensional (rough) paths

It is important to extend the above setting to functions $g : [0,T] \to \mathbb{R}^d$.

If $\alpha \in]1/3, 1/2]$ and $g \in C^\alpha$, we call $(g^i, Y^{ij}, 1 \le i,j \le d)$, with $(g^i, Y^{ij}) \in C^\alpha \times C_2^{2\alpha}$ a Rough Path if for all i,j

$$(\delta_2 Y^{ij})_{sut} = (g^i_u - g^i_s)(g^j_t - g^j_u), \qquad 0 \le s \le u \le t \le T.$$

We say that $(f, f'^i) \in C^\alpha \times (C^\alpha)^d$ is controlled by g if

$$|f_t - f_s - \sum_i f'^i_s (g^i_t - g^i_s)| \lesssim |t-s|^{2\alpha}.$$

We denote by $\mathscr{D}_g^{2\alpha}$ the space of paths controlled by g.

In this setting, we can apply the Sewing Lemma to the germ $A^j_{st} := f_s(g^j_t - g^j_s) + \sum_i f'^i_s Y^{ij}_{st}$ and define the integral $I^i \in C^\alpha$ such that

$$\delta_1 I^j = A^j - \Lambda \delta_2 A^j, \qquad I^j_0 = 0.$$

Multi-dimensional (rough) paths

First, this allows to cover SDEs in $\mathbb{R}^d$

$$X_t = X_0 + \int_0^t \sigma(X_s)\, \mathrm{d}B_s, \quad X, B \in C([0,T];\mathbb{R}^d), \;\; \sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d.$$

Furthermore, the situation is more insteresting and complicated, since there is no canonical choice for the off-diagonal terms

$$(\delta_2 Y^{ij})_{sut} = (g^i_u - g^i_s)(g^j_t - g^j_u), \qquad i \neq j.$$

It is always possible to find $Y^{ij} \in C_2$ satisfying this, take e.g. $Y^{ij}_{st} = -g^i_s(g^j_t - g^j_s)$. However in general this choice does not satisfy the analytical requirement $Y^{ij} \in C_2^{2\alpha}$.

Therefore existence of Rough Paths over a path $g : [0,T] \to \mathbb{R}^d$ is not obvious.

Brownian motion in $\mathbb{R}^d$

Let us suppose that $g^i \equiv B^i$, with $B = (B^1, \ldots, B^d)$ a standard Brownian motion in $\mathbb{R}^d$. We fix $\alpha \in]1/3, 1/2]$.

We set $Y^{ij}_{st} = \int_s^t (B^i_u - B^i_s) \circ \, \mathrm{d}B^j_u$. For all $\alpha < 1/2$, a.s. $Y \in C_2^{2\alpha}$ (not obvious).

A path controlled by B is $(f, f') \in C^\alpha \times (C^\alpha)^d$ such that

$$|f_t - f_s - \sum_i f'^i_s (B^i_t - B^i_s)| \lesssim |t-s|^{2\alpha}, \qquad 0 \le s \le t \le T.$$

For all such (f, f') there exists a unique $I \in (C^\alpha)^d$ such that $I_0 = 0$ and

$$|I^j_t - I^j_s - f_s(B^j_t - B^j_s) - \sum_i f'^i_s Y^{ij}_{st}| \lesssim |t-s|^{3\alpha}, \qquad 0 \le s \le t \le T.$$

If the Stratonovich integral $\int_0^\bullet f_s \circ \, \mathrm{d}B_s$ is well defined, it is equal to I.

Brownian motion in $\mathbb{R}^d$

Let us suppose that $g^i \equiv B^i$, with $B = (B^1, \dots, B^d)$ a standard Brownian motion in $\mathbb{R}^d$. We fix $\alpha \in]1/3, 1/2]$.

We set $Y^{ij}_{st} = \int_s^t (B^i_u - B^i_s)\, \mathrm{d}B^j_u$. For all $\alpha < 1/2$, a.s. $Y \in C_2^{2\alpha}$ (not obvious).

A path controlled by B is $(f, f') \in C^\alpha \times (C^\alpha)^d$ such that

$$|f_t - f_s - \sum_i f'^i_s (B^i_t - B^i_s)| \lesssim |t-s|^{2\alpha}, \qquad 0 \le s \le t \le T.$$

For all such (f, f') there exists a unique $I \in (C^\alpha)^d$ such that $I_0 = 0$ and

$$|I^j_t - I^j_s - f_s(B^j_t - B^j_s) - \sum_i f'^i_s Y^{ij}_{st}| \lesssim |t-s|^{3\alpha}, \qquad 0 \le s \le t \le T.$$

If the Itô integral $\int_0^\bullet f_s \, \mathrm{d}B_s$ is well defined, it is equal to I.

Remarks

- In the Young situation ($\alpha > 1/2$), f and g play symmetric rôles. The integral is a bilinear functional
- If $\alpha \leq 1/2$, the pair (g, Y) is a non-linear object by the constraint on $\delta_2 Y$.
- In particular, rough paths are non-linear objects. This is where algebra gets into the picture.
- On the other hand, for a fixed rough path, controlled paths form a linear space and the integral is a linear functional.
- The off-diagonal terms $Y^{ij}_{st} = \int_s^t (B^i_u - B^i_s) \, \mathrm{d}B^j_u$, $i \neq j$, are defined using Stochastic calculus. Since $\delta Y^{ij} \in C_3^{1-}$, the Sewing Lemma can not be used to define them.

Remarks

Another fundamental remark:

- the analytical bound in the Sewing Lemma implies that the integral is continuous w.r.t. (f, g, Y).
- This implies that solutions to a Rough Differential Equation are continuous w.r.t. the underlying rough path.
- This was the motivation of Terry Lyons when he introduced Rough Paths in the first place, and it is called the Continuity of the Itô-Lyons map.
- (Hans Föllmer wrote in the '80s a famous note conjecturing this kind of results)
- In the classical theory of stochastic calculus and SDEs, one has in general only measurability of the Itô map.

Lower regularity

If we want to consider a path $g : [0,T] \to \mathbb{R}$ with even lower regularity, say $g \in C^\alpha$ with $\alpha \in]1/4, 1/3]$, then we have to modify further the germ.

We assume that $(f, f', f'') \in (C^\alpha)^3$ satisfies

$$f_t - f_s = f'_s(g_t - g_s) + f''_s \frac{(g_t - g_s)^2}{2} + R_{st}, \qquad R \in C_2^{3\alpha}.$$

Then the germ

$$A_{st} := f_s(g_t - g_s) + f'_s \frac{(g_t - g_s)^2}{2} + f''_s \frac{(g_t - g_s)^3}{3!}$$

satisfies

$$(\delta_2 A)_{sut} = -R_{su}(g_t - g_u) - (f'_t - f'_s - f''_s(g_t - g_s)) \frac{(g_t - g_u)^2}{2}.$$

Lower regularity

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We assume that $(f, f', f'') \in (C^\alpha)^3$ satisfies

$$f_t - f_s = f'_s(g_t - g_s) + f''_s \frac{(g_t - g_s)^2}{2} + R_{st}, \qquad R \in C_2^{3\alpha},$$

$$f'_t - f'_s = f''_s(g_t - g_s) + R'_{st}, \qquad R' \in C_2^{2\alpha}.$$

Then the germ

$$A_{st} := f_s(g_t - g_s) + f'_s \frac{(g_t - g_s)^2}{2} + f''_s \frac{(g_t - g_s)^3}{3!}$$

satisfies (exercise...)

$$(\delta_2 A)_{sut} = -R_{su}(g_t - g_u) - R'_{su} \frac{(g_t - g_u)^2}{2}.$$

If $1/4 < \alpha \le 1/3$ we are in the setting of the Sewing Lemma.

Compact notations

Let $\alpha \in]0,1[$ and $g \in C^\alpha$.

We set $\mathbb{X}^n_{st} := \frac{1}{n!}(g_t - g_s)^n$, $s,t \in [0,T]$, $n \geq 0$. By Newton's binomial theorem

$$\mathbb{X}^n_{st} = \sum_{k=0}^{n} \mathbb{X}^k_{su}\, \mathbb{X}^{n-k}_{ut}, \qquad s,u,t \in [0,T]$$

(a convolution product...). Note that $\mathbb{X}^n \in C_2^{n\alpha}$ and

$$(\delta_2 \mathbb{X}^n)_{sut} = \sum_{k=1}^{n-1} \mathbb{X}^k_{su}\, \mathbb{X}^{n-k}_{ut}, \qquad s,u,t \in [0,T].$$

Now we define N as the largest integer such that $N\alpha \leq 1$, i.e. $N = \lfloor 1/\alpha \rfloor$.

We say that $Z : [0,T] \to \mathbb{R}^{\{0,\dots,N-1\}}$ is controlled by $\mathbb{X}$ if

$$Z^n_t = \sum_{k=n}^{N-1} Z^k_s\, \mathbb{X}^{k-n}_{st} + R^n_{st}, \qquad n \in \{0,\dots,N-1\},\ R^n \in C_2^{(N-n)\alpha}.$$

Compact notations

Then the germ $$A_{st} := \sum_{k=0}^{N-1} Z_s^k \, \mathbb{X}_{st}^{k+1}$$ satisfies

$$(\delta_2 A)_{sut} = \sum_{k=0}^{N-1} \left[Z_s^k \left(\mathbb{X}_{st}^{k+1} - \mathbb{X}_{su}^{k+1} \right) - Z_u^k \, \mathbb{X}_{ut}^{k+1} \right]$$

$$= \sum_{k=0}^{N-1} Z_s^k \sum_{i=1}^{k+1} \mathbb{X}_{su}^{k+1-i} \, \mathbb{X}_{ut}^{i} - \sum_{k=0}^{N-1} Z_u^k \, \mathbb{X}_{ut}^{k+1}$$

$$= \sum_{i=0}^{N-1} \mathbb{X}_{ut}^{i+1} \sum_{k=i}^{N-1} Z_s^k \, \mathbb{X}_{su}^{k-i} - \sum_{i=0}^{N-1} Z_u^i \, \mathbb{X}_{ut}^{i+1}$$

$$= \sum_{i=0}^{N-1} \mathbb{X}_{ut}^{i+1} \left[Z_u^i - R_{su}^i \right] - \sum_{i=0}^{N-1} Z_u^i \, \mathbb{X}_{ut}^{i+1}$$

$$= - \sum_{i=0}^{N-1} R_{su}^i \, \mathbb{X}_{ut}^{i+1} \in C_3^{(N-i+i+1)\alpha} \subset C_3^{1+}.$$

Compact notations

We define as above I by $I_0 = 0$ and

$$\delta_1 I = A - \Lambda\delta_2 A, \qquad \bar{R} := -\Lambda\delta_2 A.$$

If we set $\bar{Z} : [0,T] \to \mathbb{R}^{\{0,\dots,N-1\}}$ by

$$\bar{Z}^0_t = I_t, \qquad \bar{Z}^n_t := Z^{n-1}_t, \quad n \in \{1,\dots,N-1\},$$

then $\bar{Z}$ is a controlled path. Indeed

$$\bar{Z}^0_t - \sum_{k=0}^{N-1} \bar{Z}^k_s \, \mathbb{X}^k_{st} = I_t - I_s - \sum_{i=0}^{N-2} Z^i_s \, \mathbb{X}^{i+1}_{st} = [\delta_1 I - A]_{st} + Z^{N-1}_s \, \mathbb{X}^N_{st} \in C^{N\alpha}_2.$$

$$\bar{Z}^n_t = Z^{n-1}_t = \sum_{k=n-1}^{N-1} Z^k_s \, \mathbb{X}^{k-n+1}_{st} + R^n_{st} = \sum_{k=n}^{N-1} \bar{Z}^k_s \, \mathbb{X}^{k-n}_{st} + R^n_{st} + Z^{N-1}_s \mathbb{X}^{N-n}_{st}.$$